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Expansion in Eigenfunctions for Systems of Ordinary Differential Equations Without the Unique Continuation Property

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EXPANSION IN EIGENFUNCTIONS FOR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS WITHOUT THE UNIQUE CONTINUATION PROPERTY

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ABSTRACT

In this document we consider differential equations where the coefficients are matrices whose entries are distributions of order zero satisfying certain criteria. The derivative which occurs in the equation is the distributional derivative.

Chapter 1, sections 1-7 contain information necessary to understand for the reading of Chapters 2 and 3. Section 8 gives a brief introduction to what has been done so far and our goal.

Thereafter, in Chapter 2, it is shown that self-adjoint restrictions of the maximal relation to the differential equation are still given by boundary conditions. With that done, given a self-adjoint restriction of the maximal relation, it is shown that the resolvent operator at a fixed point in the resolvent set is an integral operator with an integral kernel, i.e. a Green's function exists.

In Chapter 3, we investigate the form of this Green's function under certain further assumptions on the coefficients of the equation and establish a generalized Fourier transform. One deviation from the classical case is that our Fourier transform is *not* injective, but many properties do carry over to this setting.

In the course of establishing this generalized Fourier transform, interesting facts as well as a Fatou-type convergence were found which are also recorded.

DEDICATION

To my parents, Rita and Steve, who have encouraged me to be the best that I can and supported me when I was not. To Rudi, whose mentorship has meant so much to me.

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CHAPTER 1.

INTRODUCTION

Many historical remarks can be found in the introductions of Chapter 2 and Chapter 3. This introduction is devoted to concepts necessary in understanding the mathematical content of Chapters 2 and 3.

Throughout this document, $(a, b) \subset \mathbb{R}$ is a real, nonempty interval which is possibly all of $\mathbb{R}$.

1. Functions of Locally Bounded Variation

Given a function $f : (a, b) \to \mathbb{C}^n$, we define the *variation* of f over the interval $I \subset (a, b)$ to be the supremum over the sums $\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|$ generated by elements $x_0 < x_1 < ... < x_n$ of the interval I, and we denote this number by $\mathrm{Var}_f(I)$. The function f is of *bounded variation* if $\mathrm{Var}_f((a, b))$ is finite. We denote the space of all functions of bounded variation on (a, b) by $\mathrm{BV}_{\mathrm{loc}}((a, b))$. If a function f is such that for every closed interval $K \subset (a, b)$, we have $\mathrm{Var}_f(K)$ is finite, then f is said to be *locally of bounded variation.* We denote the set of all functions of locally bounded variation on (a, b) by $\mathrm{BV}_{\mathrm{loc}}((a, b))$.

Given a function f of locally bounded variation and a point $c \in (a, b)$, a *variation function* of f is defined by

$$V_f^c : (a, b) \to \mathbb{R} : x \mapsto \begin{cases} \mathrm{Var}_f([c, x]) & x \geq c \\ -\mathrm{Var}_f([x, c]) & x < c \end{cases}.$$

Notice, when f is real-valued, $V_f^c \pm f$ are both nondecreasing functions, and

$$f = \frac{1}{2}(V_f^c + f) - \frac{1}{2}(V_f^c - f),$$

so that *any* function of locally bounded variation may be written as a difference of nondecreasing functions. From this fact, for each $x \in (a,b)$,

$$f^-(x) := \lim_{y\uparrow x} f(y) \text{ and } f^+(x) := \lim_{y\downarrow x} f(y)$$

both exist and define functions f^- and f^+ which are left continuous and right continuous respectively. If $f : (a,b) \to \mathbb{C}$ is any function such that $f = (f^+ + f^-)/2$, then we shall say f is *balanced*. The set of all balanced functions of (locally) bounded variation will be denoted $(\mathrm{BV}_{\mathrm{loc}}((a,b))^{\#})$ $\mathrm{BV}((a,b))^{\#}$.

Anyone who desires more detail and some motivation may refer to [**10**], chapter 3. A quick review of this material may be found in [**8**], chapter 7, which seems to be inspired by [**9**].

2. Measures, Integrable Functions, and Differentiation

We will assume the concepts of measure theory which can be found in Chapters 1 and 3 of [**7**], specifically, how measures are defined and how to integrate using them, as well as basic L^p space concepts.

Instead of saying that (X, M, μ) is a positive measure space whenever $\mu(E) \geq 0$ for all $E \in M$, we will call this a nonnegative measure space and μ a nonnegative measure.

Whenever (X, M, μ) is a nonnegative measure space where X has some topology and M is the smallest $\sigma-$algebra containing the open sets, then M is called the *Borel* $\sigma-$*algebra*, which we will denote by $\mathcal{B}(X)$, and μ is a Borel measure. In Hausdorff spaces, any compact set K is closed so that $K \in \mathcal{B}(X)$. Given any nonnegative Borel measure μ, if f is a function such that for any compact set $K \in B(X)$, $\int_K |f|^p \mu < \infty$, we we say that f is locally $p-$integrable with respect to μ, and we denote the space of all such functions by $L^p_{\mathrm{loc}}(\mu)$, or $L^p_{\mathrm{loc}}(X)$ when no confusion is expected.

If (X, M, μ) is a nonnegative measure space such that $\mathcal{B}(X) \subset M$ and for every $E \in M$,

$$\mu(E) = \inf\{\mu(U) | E \subset U \text{ and } U \text{ open}\} = \sup\{\mu(K) | E \supset K \text{ and } K \text{ compact}\},$$

then μ will be called a *regular measure.*

If (X, M, μ) is a nonnegative measure space and $X = \bigcup_{k=1}^{\infty} X_k$ where $X_k \in M$ and $\mu(X_k) < \infty$, then we say μ is a $\sigma-$*finite* measure. If λ is a measure on M such that for every $E \in M$ with $\mu(E) = 0$, we have $\lambda(E) = 0$, then we say that λ is *absolutely continuous* with respect to μ.

One theorem which is very far reaching, in part because of how truly little it assumes, is the following, known as the Lebesgue-Radon-Nikodym Theorem:

THEOREM 2.1. Let (X, M, μ) be a nonnegative measure space where μ is $\sigma-$finite and let λ be some complex-valued measure also defined on M. If λ is absolutely continuous with respect to μ, then there exists a unique $h \in L^1(\mu)$ such that for every $E \in M$, $\lambda(E) = \int_E h\mu$. Any such function h is called a Radon-Nikodym derivative of λ with respect to μ, and we may at times denote this by (λ/μ).

Given any function $f : (a, b) \to \mathbb{R}$ which is nondecreasing we can define an outer measure

$$df^*(E) = \inf\left\{\sum_{k=1}^{\infty}(f^-(b_k) - f^+(a_k)) | E \subset \bigcup_{k=1}^{\infty}(a_k, b_k)\right\}.$$

Using Carathéodory's construction process for measures, we may restrict this outer measure to a regular measure df on the Borel $\sigma-$algebra $\mathcal{B}((a, b))$ such that $df((c, d)) = f^-(d) - f^+(c)$, $df(\{c\}) = f^+(c) - f^-(c)$ for any $c, d \in (a, b)$. Of course, in the case of $f(x) = x$, we have the normal Lebesgue measure $m := df$, which is $\sigma-$finite on any (a, b). Measures of the form df in general are called *Lebesgue-Stieltjes measures.*

Interestingly enough, for *any* nonnegative Borel measure μ on (a, b) which is finite on compact sets, if we define $f(x) = \mu([c, x))$ for $x \geq c$ and $f(x) = -\mu([x, c))$

for $x < c$, then f is nondecreasing, left-continuous, and $df = \mu$ on Borel sets. In particular, μ restricted to the Borel sets is a regular measure. This function f is a *cumulative distribution function* of μ.

To bring the Radon-Nikodym derivative closer to the realm of the classical derivative, let us make the following statement, which relates a few theorems from Chapter 7 of [**7**]:

THEOREM 2.2. Suppose μ is a nonnegative Borel measure on (a, b) which is finite on compact sets, and hence it is a Lebesgue-Stieltjes measure $\mu = df$, and df is absolutely continuous with respect to m. It is then the case that for Lebesgue points x of f (hence m-a.e.),

$$\lim_{r \to 0} \frac{df((x-r, x+r))}{m((x-r, x+r))}$$

exists, and by choosing $x \pm r$ points of continuity of f we see this limit becomes $f'(x)$.

So, by the Lebesgue-Radon-Nikodym theorem, we have for any Borel set E, $df(E) = \int_E (df/m)\, m$. Furthermore, if x is a Lebesgue point of both (df/m) and f', then we have

$$\begin{aligned} f'(x) &= \lim_{r \downarrow 0} \frac{df((x-r, x+r))}{m((x-r, x+r))} \\ &= \lim_{r \downarrow 0} \frac{1}{m((x-r, x+r))} \int_{((x-r, x+r))} (df/m)\, m \\ &= (df/m)\,(x). \end{aligned}$$

Again, the Radon-Nikodym derivative exists under very mild assumptions, and this conforms to the classical derivative a.e. whenever a classical derivative exists, at least for nondecreasing functions so far. In particular, if ϕ is nondecreasing and continuously differentiable, $d\phi = \phi' m$.

To extend these concepts, we first make the observation that if f is of locally bounded variation, it can be written as a linear combination of nondecreasing functions.

As an example, if $c \in (a,b)$,

$$f = (V^c_{\operatorname{Re} f} + \operatorname{Re} f) - V^c_{\operatorname{Re} f} + i(V^c_{\operatorname{Im} f} + \operatorname{Im} f) - iV^c_{\operatorname{Im} f} =: \sum_{k=0}^{3} i^k f_k.$$

Now, each df_k exist as Lebesgue-Stieltjes measures, and as long as $E \in \mathcal{B}((a,b))$ has closure contained in (a,b), i.e. E is *precompact*, we may define $df(E) = \sum_{k=0}^{3} i^k f_k(E)$. It is a quick argument to show that df so defined is independent of the decomposition of f into a combination of nondecreasing functions, at least on Borel sets. Sometimes df may be referred to as a *local* measure, since df restricted to subsets of any given compact interval in (a,b) is a measure.

Furthermore, if $f, g \in \mathrm{BV}_{\mathrm{loc}}((a,b))$ and $\alpha \in \mathbb{C}$, then $\alpha f + g \in \mathrm{BV}_{\mathrm{loc}}((a,b))$ and $d(\alpha f + g) = \alpha df + dg$, where the equality is on precompact Borel sets.

To integrate against such complex valued measures, we define the *total variation* of a measure λ, denoted $|\lambda|$, by setting $|\lambda|(E)$ to be the supremum over all sums $\sum_{k=0}^{\infty} |\lambda(E_k)|$ such that $k \mapsto E_k$ is a collection of measurable, precompact sets which are pairwise disjoint and whose union is E.

It is then the case that λ is absolutely continuous with respect to $|\lambda|$, and a small extension of the Lebesgue-Radon-Nikodym theorem shows that $(\lambda/|\lambda|)$ exists in this case: i.e. for any precompact Borel E, $\lambda(E) = \int_E (\lambda/|\lambda|)|\lambda|$. Furthermore, $|(\lambda/|\lambda|)| = 1$ $|\lambda|$−a.e.

With this representation theorem, we can define

$$\int f\lambda := \int f \cdot (\lambda/|\lambda|)\,|\lambda|,$$

at least whenever $f \in L^1(|\lambda|) =: L^1(\lambda)$.

In the next section, we will introduce the last foundational object for our study.

3. Distributions of Order Zero and Test Functions

The notation of this section is motivated by Appendix A of [**5**].

Given an interval $(a,b) \subset \mathbb{R}$, we denote by $C_c^\infty((a,b))$ the set of infinitely differentiable $\phi : (a,b) \to \mathbb{C}$ which are compactly supported. We will denote the support of ϕ by $\operatorname{supp}\phi$. Such ϕ will be called *test functions*. The space of test functions is a normed vector space with respect to the supremum norm, which we will denote by $\|\cdot\|_\infty$.

Definition 3.1. A linear mapping $f : C_c^\infty((a,b)) \to \mathbb{C}$ is called a *distribution* if for each compact set $K \subset (a,b)$ there exists a number $C \geq 0$ and natural number $n \geq 0$ such that for each test function ϕ supported in K, we have the bound

$$|f(\phi)| \leq C \sum_{k=0}^{n} \left\|\phi^{(k)}\right\|_\infty .$$

If the number n may be chosen independently of K and it is the smallest such number, then f is a *distribution of order* n.

One of the simplest examples of a distribution of order zero is the map

$$d : C_c^\infty((a,b)) \to \mathbb{C} : \phi \mapsto \int_{(a,b)} \phi m,$$

where the integration is against the Lebesgue measure, denoted by m here. This is indeed a distribution of order zero, since for any compact K and ϕ with support in K,

$$|d(\phi)| = \left|\int_{(a,b)} \phi m\right| \leq m(\operatorname{supp}\phi)\, \|\phi\|_\infty \leq m(K)\, \|\phi\|_\infty .$$

Examples which are not much more complicated are the following: If $\psi \in L^1_{\text{loc}}((a,b))$, the space of locally integrable complex valued functions defined on (a,b), then $\phi \mapsto \int_{(a,b)} \phi\psi m$ is also a distribution of order 0. Again, for any ϕ supported in K we have $\left|\int_{(a,b)} \phi\psi m\right| \leq \left(\int_K |\psi|\right) \|\phi\|_\infty$. Let us denote this map by d_ψ, i.e. $d_\psi(\phi) := \int_{(a,b)} \phi\psi m$.

To motivate the following definition, we note that if $\psi : (a,b) \to \mathbb{C}$ is continuously differentiable, then for any test function ϕ, integration by parts informs us

$$d_{\psi'}(\phi) = \int_{(a,b)} \psi' \phi m = \psi\phi|_a^b - \int_{(a,b)} \psi\phi' m = \int_{(a,b)} \psi(-\phi')m = -d_\psi(\phi').$$

Definition 3.2. Given a distribution f, we define the *distributional derivative* of f, denoted by f', to be $f' : C_c^\infty((a,b)) \to \mathbb{C} : \phi \mapsto -f(\phi')$

If f is a distribution of order n, K is a compact set, and $C \geq 0$ the bounding constant for this set, then for any ϕ supported in K,

$$|f'(\phi)| = |f(\phi')| \leq C \sum_{k=0}^{n} \left\|(\phi')^{(k)}\right\|_\infty = C \sum_{k=1}^{n+1} \left\|\phi^{(k)}\right\|_\infty,$$

i.e. f' is a distribution of order at most $n+1$.

By the observations before the definition, we see if ψ is a continuously differentiable function on (a,b), then $d'_\psi = d_{\psi'}$. This implies that some distributions of order zero have derivatives which are also order zero.

To give examples where this is not the case, we first give an existence claim for test functions: In the next section we will show that for any $[c,d] \subset (c',d') \subset (a,b)$, there exists a $\phi \in C_c^\infty((a,b))$ such that $\phi = 1$ on $[c,d]$ and $\phi = 0$ outside of (c',d'), with $\|\phi\|_\infty = 1$.

A simple example of a distribution of order zero where the derivative is not a distribution of order zero is the map $\delta(\phi) = \phi(0)$. Its derivative is $\delta'(\phi) = -\phi'(0)$. With our previous existence claim, with $\epsilon \in (0, 1/2)$, we see there is a test function ϕ_ϵ such that ϕ_ϵ is 1 on $[\epsilon, 1]$ and zero outside of $(-\epsilon, 1+\epsilon)$, so by the mean value theorem, there is a $\xi \in (-\epsilon, \epsilon)$ such that

$$1 = \phi_\epsilon(\epsilon) - \phi_\epsilon(-\epsilon) = \phi'_\epsilon(\xi)2\epsilon.$$

That is, $\|\phi_\epsilon\|_\infty = 1$, and $\phi'_\epsilon(\xi) = \frac{1}{2\epsilon}$ it is thus the case that $|\delta'(\phi_\epsilon(\cdot + \xi))| = \frac{1}{2\epsilon}$, which may be made arbitrarily large while $\|\phi_\epsilon(\cdot + \xi)\|_\infty = 1$. The fact that the support of any of these ϕ_ϵ is contained in $[-2, 2]$ informs us δ' is not a distribution of order zero.

So, we have examples of distributions of order zero that have derivatives which are also distributions of order zero, and some distributions of order zero whose derivatives are not distributions of order zero.

The next section characterizes which distributions of order zero have derivatives which are also of order zero, and such distributions are the solutions to the differential equations we solve in this document.

4. The Riesz Representation Theorem, Connecting Distributions of Order Zero and Measure Theory

We already have distributions of order zero defined on $C_c^\infty((a,b))$. With the use of this space, we were able to define derivatives of distributions of order zero. In some sense (to be explained after the theorem below) this space is too restrictive.

So, we introduce the space $C_0((a,b))$ of functions $f:(a,b)\to\mathbb{C}$ such that for any $\epsilon>0$, there exists a compact $K\subset(a,b)$ such that for all $x\notin K$, $|f(x)|<\epsilon$. These functions are said to *vanish at infinity.* Notice, with respect to the supremum norm, $C_0((a,b))$ *is* a Banach space whereas $C_c^\infty((a,b))$ is not. We can now introduce a slightly modified Riesz Representation theorem, from [**7**, p. 130].

Theorem 4.1. To every bounded linear functional $f:C_0((a,b))\to\mathbb{C}$, there exists a unique nonnegative Borel measure μ and unique $\psi\in L^1(\mu)$ with $|\psi|=1$ $\mu-$a.e., such that for any $\phi\in C_0((a,b))$, $f(\phi)=\int\phi\psi\mu$.

To relate this formulation of the Riesz Representation Theorem to that in [**7**], (a,b) is our locally compact Hausdorff (LCH) space and the complex measure in [**7**] is $\psi\mu$.

The proof of this theorem in general relies upon the use of Urysohn's lemma for LCH spaces. The use of the space $C_0((a,b))$ above as the domain of f is an artifact of the fact that Urysohn's lemma fashions *continuous* functions from an arbitrary LCH space to $\mathbb{C}$, which are then used in the construction of μ.

Notice, for every bounded linear functional $f : C_0((a,b)) \to \mathbb{C}$, $f|_{C_c^\infty((a,b))}$ is a distribution of order zero, where the bounding constant in the definition is independent of the compact set wherein the corresponding test function is supported.

Conversely, we have the following:

THEOREM 4.2. To each distribution of order zero $f : C_c^\infty((a,b)) \to \mathbb{C}$, there exists a nonnegative Borel measure μ, finite on compact sets, and $\psi \in L^1_{\text{loc}}(\mu)$ such that $f(\phi) = \int \phi\psi\mu$.

To show this, the following claim is necessary: for each $\phi \in C_c((a,b))$, there exists a sequence $k \mapsto \phi_k \in C_c^\infty((a,b))$ and compact interval $K \subset (a,b)$ such that $\operatorname{supp}\phi_k, \operatorname{supp}\phi \subset K$ and $\|\phi_k - \phi\|_\infty \to 0$ as $k \to \infty$. For any such sequence ϕ_k, we say ϕ_k *converges nicely* to ϕ, writing $\phi_k \xrightarrow{n} \phi$ as $k \to \infty$.

Since $\phi \in C_c$, there is an interval $[c,d] \subset (a,b)$ such that $\operatorname{supp}\phi \subset [c,d]$. Let $a < c' < c$ and $d < d' < b$ Approximate ϕ uniformly by a sequence of polynomials p_k on $[c'-\epsilon, d'+\epsilon]$ for some $\epsilon > 0$ small enough so that this last interval is still contained in (a,b), and extend the polynomials by 0 to the rest of (a,b).

Define for $s < t$ in (a,b), $h_{s,t}(x) = \begin{cases} \exp\left(\frac{-1}{(x-s)(t-x)}\right) & x \in (s,t) \\ 0 & x \notin (s,t) \end{cases}$ so that $h_{s,t} \in C_c^\infty((a,b))$ and $\operatorname{supp} h_{s,t} = [s,t]$. The differentiability at s and t stems from $\frac{e^{-1/x}}{x^n} \to 0$ as $x \to 0$ for any $n \in \mathbb{N}$. Now, define

$$\tilde{h}_{s,t}(x) = \frac{1}{\int_{(a,b)} h_{s,t}} \int_{(a,x)} h_{s,t},$$

and finally $g(x) = \tilde{h}_{c',c}(x)\tilde{h}_{-d',-d}(-x)$, so that $g = 1$ on $[c,d]$, $g = 0$ outside (c',d'), and $g \in C_c^\infty((a,b))$. It is then the case that $k \mapsto gp_k$ is a sequence of functions in $C_c^\infty((a,b))$ such that $gp_k \xrightarrow{n} \phi$ as $k \to \infty$.

With this claim shown, let $K_n := [c_n, d_n]$ satisfy $K_n \subset K_{n+1}$ and $\cup_n K_n = (a,b)$.

If $f : C_c^\infty((a,b)) \to \mathbb{C}$ is a distribution of order zero, then $f_n := f|_{C_c^\infty((c_n,d_n))}$ is also a distribution of order zero and there exists a C_n such that for any $\phi \in C_c^\infty((c_n,d_n))$, $|f_n(\phi)| \leq C_n \|\phi\|_\infty$.

Now we may extend f_n to $C_c((c_n,d_n))$, and finally to $C_0((c_n,d_n))$ by the following: for any $\phi \in C_c((c_n,d_n))$, let $\phi_k \in C_c^\infty((c_n,d_n))$ satisfy $\phi_k \xrightarrow{n} \phi$, and define $f_n(\phi) := \lim_{k\to\infty} f_n(\phi_k)$. We claim that this limit is independent of the chosen sequence $k \mapsto \phi_k$ and that the extension is a bounded linear functional. The first claim can be done by interweaving sequences and the second claim is clear. Using the fact that $C_c((c_n,d_n))$ is dense in $C_0((c_n,d_n))$ and interweaving sequences again will extend f_n to be a bounded linear functional on $C_0((c_n,d_n))$. Thus, by the Riesz Representation theorem, there is a unique nonnegative Borel measure μ_n and $\psi_n \in L^1(\mu_n)$ such that $f_n(\phi) = \int_{(c_n,d_n)} \phi\psi_n\mu_n$ for all $\phi \in C_c((c_n,d_n))$.

If $E \in \mathcal{B}((c_n,d_n))$, then for any $m \geq n$, $\mu_n(E) = \mu_m(E)$ by the uniqueness claim above; furthermore, if $E \in \mathcal{B}((a,b))$ then for any n, $E \cap (c_n,d_n) \in \mathcal{B}((c_n,d_n))$, whence if $m \geq n$ we have

$$\mu_n(E \cap (c_n,d_n)) = \mu_m(E \cap (c_n,d_n)) \leq \mu_m(E \cap (c_m,d_m))$$

so that $n \mapsto \mu_n(E \cap (c_n,d_n))$ is a nondecreasing sequence.

Defining $\mu(E) = \lim_{n\to\infty} \mu_n(E \cap (c_n,d_n))$ for any $E \in (a,b)$ gives us a Borel measure which is finite on compact sets in (a,b). Furthermore, if we extend ψ_n to all of (a,b) for each n by zero, then for any $m \geq n$, $\psi_n = \psi_m\chi_{(c_n,d_n)}$ $\mu-$a.e.. This can show us quickly that $\psi_n \to \psi$ pointwise $\mu-$a.e, and $\int_{(c_n,d_n)} |\psi|\mu = \int_{(c_n,d_n)} |\psi_n|\mu_n < \infty$ so $\psi \in L^1_{\text{loc}}((a,b))$. Notice further $|\psi| = 1$ μ-a.e. since $|\psi_n| = 1$ μ_n-a.e.

By considering the compact set where any given $\phi \in C_c^\infty((a,b))$ is supported, one can take a limit to see $f(\phi) = \int \phi\psi\mu$. With this representation, if we define F to be the cumulative distribution function of $\psi\mu$, then $dF = \psi\mu$ on precompact Borel sets, and we write the previous integral as $f(\phi) = \int \phi dF$. It is quick to see $|dF| = \mu$ and $(dF/|dF|) = \psi$.

Definition 4.3. If f is a distribution of order zero with representation $f(\phi) = \int \phi dF$, then for any $\alpha \in L^1_{\text{loc}}(\mu)$ we define $(\alpha f)(\phi) = \int \phi \alpha dF$.

Finally, recall from [**6**, p. 419], that for any $f, g \in \text{BV}_{\text{loc}}((a,b))$, we have

$$\int_{[x_1,x_2)} f^+ dg + g^- df = (fg)^-(x_2) - (fg)^-(x_1),$$

the *integration by parts* formula for Lebesgue-Stieltjes measures. Setting $g = \phi \in C_c^\infty((a,b))$, the fact that $f = f^+$ $m-$a.e., the compactness of the support of ϕ, and $d\phi = \phi' m$ gives

$$\int_{(a,b)} \phi df = -\int_{(a,b)} \phi' fm = d_f'(\phi).$$

That is to say, for any function $f \in BV_{\text{loc}}((a,b))$, $d_f'(\phi) = \int \phi df$.

One thing to note is that the distributions of order zero whose derivatives are also of order zero are exactly those distributions which can be written as fm for some $f \in \text{BV}_{\text{loc}}((a,b))$.

5. Summary of Distributions of Order Zero and Measures

Given any function $f \in \text{BV}_{\text{loc}}((a,b))$, we can consider the distribution of order zero $d_f(\phi) = \int \phi fm$ and the local measure df. These are related in the sense that $d_f'(\phi) = \int \phi df$. Because of this equality, we will identify $d_f' = df$. The prime notation used in subsequent chapters will be used for both the distributional derivative and the local measure generated by the function of locally bounded variation. By the Riesz representation theorem, this will not be a logical issue. As an example of what to expect, we will allow for q to be a distribution of order zero and consider $q(E)$ for some precompact Borel E.

Given a distribution of order zero q, it has an associated local measure dQ, where $Q : (a,b) \to \mathbb{C}$ is of locally bounded variation. Then $Q' := d_Q' = q$, so we will call Q an *anti-derivative* of q.

6. Matrix-Valued Distributions of Order Zero

This section is mainly a restatement of Appendix A.3 in [**5**]

If f is a distribution of order zero, we define the *conjugate* of f to be $\overline{f}(\phi) = \overline{f(\overline{\phi})}$. We say that f is *real* if $f = \overline{f}$, and f is *nonnegative* if for any $\phi \geq 0$, we have $f(\phi) \geq 0$. If f is real with antiderivative F, then dF is a real local measure. If f is nonnegative, then dF is a nonnegative measure.

Given any function $Q : (a,b) \to \mathbb{C}^{n\times n}$, we can define the variation of Q over the interval I to be $\mathrm{Var}_Q(I) := \sum_{i,j=1}^{n} \mathrm{Var}_{Q_{i,j}}(I)$. If Q is of locally bounded variation, then we can consider the matrix Q' of distributions of order zero, the differentiation here is entry-wise. Similarly, given an $n \times n$ matrix of distributions of order zero, q, we see that an antiderivative matrix Q exists, which is of locally bounded variation. We say that such a matrix q is *hermitian* if $q = q^*$, where the star here is defined to be $(q^*)_{i,j} = \overline{q_{j,i}}$. We say that q is nonnegative if for every $z \in \mathbb{C}^n$, z^*qz is a nonnegative distribution of order zero. A nonnegative distribution must be hermitian. Furthermore, if Q is an antiderivative of the nonnegative distribution q, then $z^*qz = d(z^*Qz)$ so that z^*Qz is a nondecreasing function. If we let $t = \mathrm{tr}\, Q$, the trace of Q, so that t is nondecreasing and so is $tI - Q$, where I is the $n \times n$ identity matrix. It is thus the case that dQ is absolutely continuous with respect to dt, and so a Radon-Nikodym derivative, $\tilde{Q} \in L^1_{\mathrm{loc}}(dt)^{n\times n}$ exists so that $q = dQ = \tilde{Q}dt$. For any vector z, notice $0 \leq z^*qz = z^*\tilde{Q}zdt \leq |z|^2 \,\mathrm{tr}\, \tilde{Q}dt = |z|^2$ at least $dt-$a.e. Since the vectors with rational entries are dense in $\mathbb{C}^n$ and countable, call them $n \mapsto z_n$, and if we set B_n to be the set of those $x \in (a,b)$ such that either $0 > z_n^*\tilde{Q}(x)z_n$ or $|z_n|^2 \,\mathrm{tr}\, \tilde{Q}(x) < z_n^*\tilde{Q}(x)z_n$, then noting that $dt(B_n) = 0$ for all n and taking limits for $x \notin \cup_n B_n$ will inform us $0 \leq \tilde{Q}(x) \leq \mathrm{tr}\, \tilde{Q}(x) = 1$ for $dt-$a.e. x.

This representation for a nonnegative q helps us define the space $\mathcal{L}^2(q)$ of those functions $f : (a,b) \to \mathbb{C}^n$ such that each entry is $dt-$measurable, and $\int f^*\tilde{Q}f dt < \infty$. On this space, we get the natural semi-inner product $\langle f, g\rangle = \int f^*\tilde{Q}g dt$, and after identifying those f such that $\int f^*\tilde{Q}f dt = 0$, we get a Hilbert space $L^2(q)$.

Last but not least, for any q which is an $n \times n$ matrix of distributions of order zero, we define for any f, a $n-$vector valued function of locally bounded variation, the product qf which is a vector of distributions of order zero whose k^{th} entry is $\sum_{j=1}^{n} f_j q_{k,j}$. Whenever such q is nonnegative, we have $qf = \tilde{Q} f dt$.

7. Linear Relations and the Spectral Theorem

All of the content covered in this section can be found in [**4**], Chapters 2 and 3 in more detail. It is assumed the reader is familiar with elementary Hilbert space theory, which can be found in [**4**] Chapter 1, and sections 2.1,2.2.

Given a linear space V, a linear subspace $R \subset V \times V$ is a *linear relation*. For instance, if $T : V \to V$ is a linear operator, then $\{(v, Tv) | v \in V\}$ is a linear relation, sometimes referred to as the *graph* of T. In the following chapters, we identify a linear function T and its graph. Not all linear relations are linear maps, e.g. $\{0\} \times V$.

We define the *range*, $\operatorname{ran} R = \{y \in V | \exists x : (x, y) \in R\}$, *domain*, $\operatorname{dom} R = \{x \in V | \exists y : (x, y) \in R\}$, *kernel*, $\ker R = \{x \in V | (x, 0) \in R\}$, and *multi-valued part*, $\operatorname{mul} R = \{y \in V | (0, y) \in R\}$. Notice R is an operator if and only if $\operatorname{mul} R = \{0\}$. We also define

$$R^{-1} = \{(y, x) \in V \times V | (x, y) \in R\},$$

and, given another linear relation S, the sum

$$R + S := \{(u, f + g) \in V \times V | (u, f) \in R \text{ and } (u, g) \in S\}$$

and product, for $\lambda \in \mathbb{C}$,

$$\lambda R = \{(u, \lambda f) \in V \times V | (u, f) \in R\}.$$

Notice that R^{-1} is an operator if and only if $\operatorname{mul} R^{-1} = \ker R = \{0\}$.

If it happens that V is a Hilbert space and $T : V \to V$ is a bounded linear map, then we know the adjoint $T^* : V \to V$ exists as a bounded linear map. Observe that for any $(u, Tu) \in T$, that $(v, T^*v) \in T^*$ if and only if $\langle Tu, v\rangle = \langle u, T^*v\rangle$, so (while this seems absurd) we may think about

$$T^* = \{(v, T^*v) \in V \times V | \forall u \in V, \langle Tu, v\rangle = \langle u, T^*v\rangle\}.$$

This helps motivate the definition of adjoint for any linear *relation* R. Given R, we define the *adjoint* of R to be

$$R^* = \{(v, g) \in V \times V | \forall (u, f) \in R, \langle f, v\rangle = \langle u, g\rangle\}.$$

This definition of adjoint coincides with the classical one when R happens to be a bounded, everywhere defined linear map.

Much interest is found in those linear relations which are *symmetric*, i.e. $R \subset R^*$. If we define $D_\lambda(R)$ to be those elements $(u, \lambda u) \in R^*$, called the *deficiency space of R at λ*, then there is a decomposition

$$R^* = \overline{R} \oplus D_i(R) \oplus D_{-i}(R),,$$

where $\overline{R}$ is the topological closure of R in $V \times V$ and the sums orthogonal with respect to $V \times V$ as a Hilbert space with inner product

$$\langle (v, g), (u, f)\rangle_{V\times V} = \langle v, u\rangle + \langle g, f\rangle.$$

This is referred to as von Neumann's decomposition of a symmetric (linear) relation.

For any linear relation R, let us define

$$\rho(R) = \{\lambda \in \mathbb{C} : (R - \lambda)^{-1} \text{ is an everywhere defined bounded linear map}\},$$

called the *resolvent set* of R and $\sigma(R) = \mathbb{C}\backslash\rho(R)$ the *spectrum* of R. Then the *resolvent* of R is the mapping

$$r_R : \rho(R) \to V^V : \lambda \mapsto (R - \lambda)^{-1},$$

and $r_R(\lambda)$ is the *resolvent of R at λ*.

If $R \subset S$, each of these linear relations, the we say R is a *restriction* of S and S an *extension* of R. In our research, much emphasis is placed on extensions S which are *self-adjoint*, i.e. $S = S^*$. Notice such relations are closed. For such an S, if we define $H := (\operatorname{mul} S)^\perp$, then $S_o := S \cap (H \times H)$ is a densely defined self adjoint operator, $\operatorname{dom} S_o = \operatorname{dom} S$, $\rho(S) = \rho(S_o)$, and $S = S_0 \oplus (\{0\} \times \operatorname{mul} S)$. In essence, a self-adjoint relation S is *almost* a densely defined self-adjoint operator with the same spectrum and resolvent, but on different spaces and with a purely multi-valued part. Notice, $r_{S_o}(\lambda) : H \to H$, but we may extend this operator to a bounded operator on $V = H \oplus H^\perp$ by defining $r_{S_o}(\lambda)u = 0$ if $u \in H^\perp = \operatorname{mul} S$. With that definition, notice $r_{S_o} = r_S$. Furthermore, the following properties hold for the resolvent of a self-adjoint relation S.

(1) $\mathbb{C}\backslash\mathbb{R} \subset \rho(S)$, and if $\lambda \in \mathbb{C}\backslash\mathbb{R}$, then $\|r_S(\lambda)\| \leq 1/|\operatorname{Im}\lambda|$.
(2) for any $\lambda, \mu \in \rho(S)$, $\ker r_S(\lambda) = \ker r_S(\mu) = \operatorname{mul} S$
(3) for any $\lambda \in \rho(S)$, $r_S(\lambda)^* = r_S(\bar{\lambda})$
(4) for any $\lambda, \mu \in \rho(S)$,

$$r_S(\lambda) - r_S(\mu) = (\lambda - \mu) r_S(\lambda) r_S(\mu) = (\lambda - \mu) r_S(\mu) r_S(\lambda),$$

and the first of these equalities is called the *first resolvent relation*.
(5) r_S is analytic in the sense that for any $\mu \in \rho(S)$, there exists an open ball centered at μ, $B \subset \rho(S)$ such that for any $\lambda \in B$,

$$r_S(\lambda) = \sum_{k=1}^{\infty} (\lambda - \mu)^{k-1} r_S(\mu)^k,$$

where the convergence is uniform.

In particular, properties 3 and 4 tell us, for any $u \in V$, the mapping

$$\rho(S) \to \mathbb{C} : \lambda \mapsto \langle r_S(\lambda)u, u \rangle$$

is differentiable and maps the upper half plane into itself and the lower half plane into itself. Hence, this mapping is *Nevanlinna* (see [**4**, p. 331]) so there exists a nondecreasing function $\eta_{u,u} : \mathbb{R} \to \mathbb{R}$ and constants $A_{u,u} \in \mathbb{R}$, $B_{u,u} \geq 0$ such that $\int_{\mathbb{R}} \frac{1}{1+t^2} d\eta_{u,u}(t) < \infty$ and

$$\langle r_S(\lambda)u, u\rangle = A_{u,u} + B_{u,u}\lambda + \int_{\mathbb{R}} \frac{1+t\lambda}{t-\lambda} \frac{d\eta_{u,u}(t)}{1+t^2},$$

whenever $\lambda \notin \mathbb{R}$. It then turns out that $A_{u,u} = B_{u,u} = 0$ for this particular Nevanlinna function. This equality is the cornerstone of the proof of the spectral theorem:

THEOREM 7.1. Suppose $T : H \to H$ is densely defined and self-adjoint. Then there exists a unique, nondecreasing, and left-continuous function $E : \mathbb{R} \to H^H$ such that each $E(t)$ is an orthogonal projection with the following properties:

(1) $\|E(t)\| \to 0$ as $t \to -\infty$ and $\|E(t) - I\| \to 0$ as $t \to \infty$;
(2) for any $u \in H$, $u \in \operatorname{dom} T$ if and only if $\int_{\mathbb{R}} t^2 d\langle E(t)u, u\rangle < \infty$, and in which case this integral is equal to $\|Tu\|^2$;
(3) it is the case that for any $u \in \operatorname{dom} T$ and $v \in H$, $t \mapsto t \in L^1(|d\langle E(\cdot)u, v\rangle|)$, and further
$$\langle Tu, v\rangle = \int_{\mathbb{R}} td\langle E(t)u, v\rangle.$$
(4) for any $t \in \mathbb{R}$, $TE(t) = \overline{E(t)T}$.

The mapping E is called the *resolution of identity* for the operator T. For a detailed proof of this theorem, see [**4**, p. 60].

Beyond the theory for linear relations introduced in [**4**], Chapter 1 of [**2**] is another great resource for these topics.

8. Differential Equations and Our Goals

All distributions will be defined on $C_c^\infty((a, b))$.

Given an $n \times n$ matrix r of distributions of order zero, g an $n-$vector of distributions of order zero, [**3**] proved the existence and uniqueness of left continuous solutions u of locally bounded variation to initial value problems $u' = ru+g$, $u(c) = u_0$, at least to the right of the point c. The proof uses a Picard iteration technique and an extension of Gronwall's Lemma, also proven in the same paper.

A modification of this argument will show the existence of right continuous solutions u to the left of c.

The differential equation which we study in this document is of the form $Ju' + qu = wf$, where J is a constant, skew-hermitian, invertible matrix, q is a hermitian $n \times n$ matrix of distributions of order zero, and $w \geq 0$. We can rewrite this equation as $u' = J^{-1}qu + J^{-1}wf$ so that we are ensured left continuous and right continuous solutions to this differential equation on their respective intervals.

Motivated by (at least) the following consideration, [**5**] proves the existence of *balanced* solutions to the IVP on the whole (a, b), given that both the matrices $I \pm r(\{x\})/2$, or equivalently $J \pm q(\{x\})/2$, are invertible for all $x \in (a, b)$. Suppose that there is some $t \in [0, 1]$ such that, given $f, g \in \mathrm{BV}_{\mathrm{loc}}((a, b))^n$, $f = tf^- + (1-t)f^-$ and $g = tg^- + (1-t)g^+$. With this, $t = 0$ will yield right continuous functions, $t = 1$ left continuous, and $t = 1/2$ balanced. We still have an integration by parts formula:

$$\int_{[x,y)} (f^+)^* dg + (df)^* g^- = (f^* g)^-(y) - (f^* g)^-(x).$$

We will have need of such a formula in the subsequent pages, but instead of the right continuous or the left continuous version of f, g, we would like for the functions themselves to be plugged in to this formula. Doing so yields

$$\int_{[x,y)} f^* dg + (df)^* g = (f^* g)^-(y) - (f^* g)^-(x) + (2t-1) \int_{[x,y)} (df)^*(g^+ - g^-).$$

Now the formula has just f and just g, but the right hand side depends on more than f and g near x and y, which is undesirable. Notice, if we set $t = 1/2$, giving balanced solutions, we obtain a nice generalization of the classical integration by parts formula. This desirable property of balanced functions was also recorded in [**6**, p. 419].

The proof that there exist unique balanced solutions to such initial value problems consists of modifying the differential equation so that left continuous solutions to the modified differential equation, after balancing them, become solutions to the original differential equation.

Throughout the work in [**5**], the assumption that $J \pm p(\{x\})/2$ are invertible for every $x \in (a,b)$ is in force. Beyond the existence and uniqueness theorems above, [**5**] also constructs associated minimal and maximal relations to the differential equation.

Given that $w \geq 0$, there is a natural Hilbert space in $L^2(w)$. To relate our equation to this, let us first define the linear relation $\mathcal{T}_{\max}$ as those pairs (u,f) such that $Ju' + qu = wf$, u is a balanced function of locally bounded variation, and $u, f \in \mathcal{L}^2(w)$. We also define $\mathcal{T}_{\min}$ as those $(u,f) \in \mathcal{T}_{\max}$ such that $\operatorname{supp} u$ is compact. Finally, $T_{\max} = \{([u],[f]) \in L^2(w) \times L^2(w) | (u,f) \in \mathcal{T}_{\max}\}$ and similarly for $T_{\min}$. Part of the work in [**5**] was showing that $T_{\min}^* = T_{\max}$ so that, since $T_{\min} \subset T_{\max}$, we see $T_{\min}$ is a symmetric relation, and hence may have self-adjoint extensions which we study.

The invertibility of the matrices $J \pm \frac{1}{2}q(\{x\})$ was a major assumption in [**5**], and our following work is part of an ongoing attempt to drop this assumption. Already, in [**1**], without this assumption, it was shown that $T_{\min}^* = T_{\max}$. We pick up in Chapter 2 just after these findings and push further.

REFERENCES

[1] Kevin Campbell, Minh Nguyen, and Rudi Weikard. On the spectral theory for first-order systems without the unique continuation property. *Linear Multilinear Algebra*, 69(12):2315–2323, 2021. Published online: 04 Oct 2019.

[2] Jussi Behrndt and Seppo Hassi and Henk de Snoo. *Boundary Value Problems, Weyl Functions, and Differential Operators*. Birkhäuser ©2020

[3] Christer Bennewitz. Spectral Asymptotics for Sturm-Liouville Equations. *Proc. London Math. Soc. (3)* 59 (2) (1989) 294-338

[4] Christer Bennewitz, Malcolm Brown, and Rudi Weikard. *Spectral and scattering theory for ordinary differential equations. Vol. I: Sturm-Liouville equations.* Universitext. Springer, Cham, [2020] ©2020.

[5] Ahmed Ghatasheh and Rudi Weikard. Spectral theory for systems of ordinary differential equations with distributional coefficients. *J. Differential Equations*, 268(6):2752–2801, 2020.

[6] Edwin Hewitt and Karl Stromberg. *Real and Abstract Analysis.* Springer-Verlag Berlin Heidelberg ©1965

[7] Walter Rudin. *Real and Complex Analysis.* McGraw-Hill, Inc. ©1987.

[8] Rudi Weikard. *Real Analysis.* lecture notes, 2022.

[9] Gerald Folland. *Real Analysis.* John Wiley & Sons, Inc. New York ©1999

[10] Gregory Convertito and David Cruz-Uribe. *The Stieltjes Integral.* Taylor & Francis Group, LLC ©2023

CHAPTER 2.

GREEN'S FUNCTIONS FOR FIRST-ORDER SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS WITHOUT THE UNIQUE CONTINUATION PROPERTY

by

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GREEN'S FUNCTIONS FOR FIRST-ORDER SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS WITHOUT THE UNIQUE CONTINUATION PROPERTY

ABSTRACT. This paper is a contribution to the spectral theory associated with the differential equation $Ju' + qu = wf$ on the real interval (a, b) when J is a constant, invertible skew-Hermitian matrix and q and w are matrices whose entries are distributions of order zero with q Hermitian and w non-negative. Under these hypotheses it may not be possible to uniquely continue a solution from one point to another, thus blunting the standard tools of spectral theory. Despite this fact we are able to describe symmetric restrictions of the maximal relation associated with $Ju' + qu = wf$ and show the existence of Green's functions for self-adjoint relations even if unique continuation of solutions fails.

1. Introduction

This paper is a contribution to the spectral theory for the differential equation

$$Ju' + qu = wf$$

posed on the real interval (a, b) when J is a constant, invertible, and skew-Hermitian $n \times n$-matrix while the entries of the matrices q and w are distributions of order zero[1] with q Hermitian and w non-negative. Ghatasheh and Weikard [**7**] studied this equation under the additional hypothesis that initial value problems have unique balanced[2] solutions in the space of functions of locally bounded variation.

[1]Recall that distributions of order 0 are distributional derivatives of functions of locally bounded variation and hence may be thought of, on compact subintervals of (a, b), as measures. For simplicity we might use the word measure instead of distribution of order 0 below.

[2]A function of locally bounded variation is called balanced, if its values at any given point are averages of its left- and right-hand limits at that point.

The equation $Ju' + qu = wf$ has, of course, been investigated by many people when the coefficients q and w are locally integrable. In that situation initial value problems always have unique solutions. This is not necessarily the case when the measures induced by q or w have discrete components. It appears that an equation with measure coefficients was first considered in 1952, when Krein [**8**] modelled a vibrating string. In 1964 Atkinson [**2**] suggested to unify the treatment of differential and difference equations by writing them as systems of integral equation where integrals were to be viewed as matrix-valued Riemann-Stieltjes integrals. Atkinson explained that the presence of point masses may prevent the continuation of solutions across such points and posed a condition avoiding that problem but more restrictive than the one posed in [**7**]. In 1999 Savchuk and Shkalikov [**10**] treated Schrödinger equations with potentials in the Sobolev space $W^{-1,2}_{\text{loc}}$. Their paper was very influential and spurred many further developments. Nevertheless, Eckhardt et al. [**5**] showed in 2013, with the help of quasi-derivatives or, equivalently, by writing the equation as a system, that a treatment without leaving the realm of locally integrable coefficients is possible. In the same year Eckhardt and Teschl [**6**] investigated 2×2-systems with diagonal measure-valued matrices q and w requiring essentially Atkinson's condition.

A more thorough account of the subject's history is given in [**7**]. The papers [**5**] and [**6**], mentioned above, may also serve as excellent sources, with perhaps different emphases, of this history.

One feature of systems of first-order equations is that, generally, they are represented by linear relations rather than linear operators. There is a well-developed spectral theory for linear relations initiated by Arens [**1**], see also Orcutt [**9**], and Bennewitz [**3**]. The most important results (for our purposes) are also surveyed in Appendix B of [**7**].

Existence or uniqueness of solutions of an initial value problem for $Ju'+qu = wf$ fails when, for some $x \in (a, b)$, the matrices

$$B_{\pm}(x, 0) = J \pm \frac{1}{2}\Delta_q(x)$$

are not invertible. Here $\Delta_q(x) = Q^+(x) - Q^-(x)$ when Q denotes an anti-derivative of q. Equivalently, $\Delta_q(x) = dQ(\{x\})$ where dQ is the measure (locally) generated by q. Assuming the unique continuation property for solutions of $Ju' + qu = wf$ Ghatasheh and Weikard defined maximal and minimal relations $T_{\max}$ and $T_{\min}$ associated with the differential equation $Ju' + qu = wf$ and showed that $T_{\max}$ is the adjoint of $T_{\min}$. They characterized the self-adjoint restrictions of $T_{\max}$, if any, with the aid of boundary conditions and proved that resolvents are given as integral operators, i.e., the existence of a Green's function for any such self-adjoint relation T. Under even more restrictive conditions they also showed the existence of a Fourier transform diagonalizing T.

Campbell, Nguyen, and Weikard [**4**] defined maximal and minimal relations and showed that $T_{\max} = T_{\min}^*$ without the hypothesis of unique continuation of solutions. Our goal here is to advance their ideas. In particular, even though the equation $Ju' + qu = w(\lambda u + f)$ may have infinitely many linearly independent solutions the deficiency indices, i.e., the number of linearly independent solutions of $Ju'+qu = \pm iwu$ of finite positive norm, is still bounded by n, the size of the system. We show that symmetric restrictions of $T_{\max}$, in particular the self-adjoint ones, are still given by posing boundary conditions and we show that the resolvents of self-adjoint restrictions are integral operators by proving the existence of Green's functions.

We will not approach the problem of Fourier transforms and eigenfunction expansions but hope to return to it in future work.

The material in this paper is arranged as follows. In Section 2 we recall the circumstances under which existence and uniqueness of solutions to initial value problems does hold and investigate the sets of those $x \in (a, b)$ and $\lambda \in \mathbb{C}$ giving rise to trouble. Then, in Section 3 we discuss the manifold of solutions of our differential equation in the special case when a and b are regular endpoints. These results are instrumental in Section 4 where we investigate the deficiency indices of the minimal relation and its symmetric extensions but without the assumption that a and b are regular. Before we prove the existence of Green's functions for self-adjoint restrictions of the maximal relation in Section 6 we discuss the role played by non-trivial solutions of zero norm in Section 5.

Let us add a few words about notation. $\mathcal{D}'^0((a,b))$ is the space of distributions of order 0, i.e., the space of distributional derivatives of functions of locally bounded variation. Any function u of locally bounded variation has left- and right-hand limits denoted by u^- and u^+, respectively. Also, u is called balanced if $u = u^\# = (u^+ + u^-)/2$. The space of balanced functions of bounded variation defined on (a,b) is denoted by $\mathrm{BV}^\#((a,b))$ while $\mathrm{BV}^\#_{\mathrm{loc}}((a,b))$ stands for the space of balanced functions of *locally* bounded variation. We use $\mathbb{1}$ to denote an identity matrix of appropriate size and superscripts $^\top$ and * indicate transposition and adjoint, respectively. The sum of two closed only trivially intersecting subspaces S and T of some Hilbert space (i.e., their direct sum) is denoted by $S \uplus T$; if S and T are even orthogonal we may use $\oplus$ instead of $\uplus$. The orthogonal complement of a subspace S of a Hilbert space H is denoted by $H \ominus S$ or by $S^\perp$. For $c_1, ..., c_N \in \mathbb{C}^n$ we abbreviate the column vector $(c_1^\top, ..., c_N^\top)^\top \in \mathbb{C}^{nN}$ by $(c_1, ..., c_N)^\diamond$.

2. Preliminaries

Throughout this paper we assume the following hypothesis to be in force.

HYPOTHESIS 2.1. J is a constant, invertible and skew-Hermitian $n \times n$-matrix. Both q and w are in $\mathcal{D}'^0((a,b))^{n\times n}$, w is non-negative and q Hermitian.

Given that w is non-negative it gives rise to a positive measure on (a,b) and we denote the space of functions f which satisfy $\int f^* w f < \infty$ by $\mathcal{L}^2(w)$. This space permits the semi-inner product $\langle f, g\rangle = \int f^* w g$ (note that $\langle f, f\rangle$ may be 0 without f being 0).

Consider the differential equation

$$Ju' + (q - \lambda w)u = wf \tag{2.1}$$

where λ is a complex parameter and f an element of $\mathcal{L}^2(w)$. The latter condition guarantees that wf is in $\mathcal{D}'^0((a,b))^n$. We will search for solutions in $\mathrm{BV}^\#_{\mathrm{loc}}((a,b))^n$. In this case each term in (2.1) is a distribution of order 0 so that it makes sense to pose the equation.

The point a is called a regular endpoint for $Ju' + qu = wf$, if there is a point $c \in (a,b)$ such that the left-continuous anti-derivatives Q and W of q and w are of bounded variation on (a,c). In this case q and w may be thought of as finite measures on (a,c). Similarly, b is called regular, if Q and W are of bounded variation on (c,b). If an endpoint is not regular, it is called singular. Not surprisingly, the study of our problem is less complicated when the endpoints are regular and we will use this fact to our advantage.

Despite our earlier denigration of the existence and uniqueness theorem of solutions of initial value problems it continues to play a crucial role. The following theorem was proved in [**7**].

Theorem 2.2. Suppose $r \in \mathcal{D}'^0((a,b))^{n\times n}$, $g \in \mathcal{D}'^0((a,b))^n$ and that the matrices $\mathbb{1} \pm \Delta_r(x)/2$ are invertible for all $x \in (a,b)$. Let x_0 be a point in (a,b). Then the initial value problem $u' = ru + g$, $u(x_0) = u_0 \in \mathbb{C}^n$ has a unique balanced solution $u \in \mathrm{BV}^\#_{\mathrm{loc}}((a,b))^n$.

If a is a regular endpoint we may pose an initial condition (for u^+) at a. Similarly, if b is regular we may prescribe $u^-(b)$ as the initial condition.

Suppose now that u is a solution of (2.1). Treating either side of this equation as a measure (restricted to a compact subset of (a,b)) evaluation at a singleton $\{x\}$ shows that

$$J(u^+(x) - u^-(x)) + \Delta_{q-\lambda w}(x)u^\#(x) = \Delta_w(x)f(x)$$

or, equivalently,

$$B_+(x,\lambda)u^+(x) - B_-(x,\lambda)u^-(x) = \Delta_w(x)f(x) \tag{2.2}$$

when we define

$$B_\pm(x,\lambda) = J \pm \frac{1}{2}\big(\Delta_q(x) - \lambda\Delta_w(x)\big).$$

Note that, if $B_+(x,\lambda)$ is not invertible, we could be in one of the following two situations: (i) a solution given on (a,x) may fail to exist on (x,b) or (ii) there are infinitely many ways to continue a solution on (a,x) to (x,b). An analogous statement holds, of course, if $B_-(x,\lambda)$ is not invertible.

Let us now investigate the circumstances when a pair (x,λ) gives such trouble. Define the sets $\Lambda_x = \{\lambda \in \mathbb{C} : \det(B_+(x,\lambda))\det(B_-(x,\lambda)) = 0\}$ and $\Xi_\lambda = \{x \in (a,b) : \det(B_+(x,\lambda))\det(B_-(x,\lambda)) = 0\}$. First note, since $B_-(x,\lambda) = -B_+(x,\overline{\lambda})^*$, we have that $\Xi_\lambda = \Xi_{\overline{\lambda}}$ and that each Λ_x is symmetric with respect to the real axis. Also, Λ_x is empty unless at least one of $\Delta_q(x)$ and $\Delta_w(x)$ is different from 0 and hence for all but countably many x. Next, we claim that Λ_x is finite as soon as it misses one point. To see this suppose that $B_+(x,\lambda_0)$ is invertible and that $\lambda \neq \lambda_0$. Since

$$B_+(x,\lambda) = (\lambda_0 - \lambda)B_+(x,\lambda_0)\big(\frac{1}{2}B_+(x,\lambda_0)^{-1}\Delta_w(x) - 1/(\lambda-\lambda_0)\big)$$

we see that $B_+(x,\lambda)$ fails to be invertible only if $1/(\lambda-\lambda_0)$ is an eigenvalue of some $n \times n$-matrix. A similar statement holds, of course, for B_- proving our claim.

The really bad points x, namely those where $\Lambda_x = \mathbb{C}$, are thus contained in Ξ_0. Here we wish to remove the hypothesis $\Xi_0 = \emptyset$ posed in [**7**]. On any subinterval of (a,b) on which q gives rise to a finite measure we find that $\sum_{k=1}^\infty \|\Delta_q(x_k)\|$ must be finite, when $k \mapsto x_k$ is a sequence of distinct points in that interval. It follows now that Ξ_0 is a discrete set. One shows similarly that, for any fixed complex number λ the set Ξ_λ is discrete.

Lemma 2.3. Suppose $[s,t] \subset (a,b)$ and $(s,t) \cap \Xi_0 = \emptyset$. Then we have that $\Lambda_{(s,t)} = \bigcup_{x \in (s,t)} \Lambda_x$ is a discrete subset of $\mathbb{C}$.

Proof. There are only finitely many points x in (s,t) where $\|J^{-1}\Delta_q(x)\| > 1$. Using a Neumann series one sees that only at such points the norm of $B_+(x,0)^{-1}$ can be larger than $2\|J^{-1}\|$. Thus there is a positive number C such that $\|B_+(x,0)^{-1}\| \le C$ for all $x \in (s,t)$. Now suppose that $B_+(x,\lambda)$ is not invertible and that $|\lambda| \le R$. Then $1/\lambda$ is an eigenvalue of $\frac{1}{2}B_+(x,0)^{-1}\Delta_w(x)$. This requires that $\|\Delta_w(x)\| \ge 2/(RC)$ and thus can happen only for finitely many $x \in (s,t)$. Since similar arguments work for B_- the number of points in $\bigcup_{x \in (s,t)} \Lambda_x$ which lie in a disk of radius R centered at 0 must be finite. $\square$

We remark that, when one of the anti-derivatives of q and w is only locally of bounded variation, the set $\bigcup_{x \in (a,b)} \Lambda_x$ need not be discrete even if every Λ_x is finite.

Theorem 2.4. Suppose $[s,t] \subset (a,b)$ and $(s,t) \cap \Xi_0 = \emptyset$. If $u_0 \in \mathbb{C}^n$ and $\lambda \in \mathbb{C} \setminus \Lambda_{(s,t)}$, then the initial value problem $Ju' + qu = \lambda wu$, $u^+(s) = u_0$ has a unique balanced solution in (s,t). Moreover, $u(x,\cdot)$ for $x \in (s,t)$ as well as $u^-(t,\cdot)$ are analytic in $\mathbb{C} \setminus \Lambda_{(s,t)}$ and meromorphic on $\mathbb{C}$. An analogous statement holds when the initial condition is posed at t.

Proof. The first claim is simply a consequence of Theorem 2.2. When $x \in (s,t)$ the analyticity of $u(x,\cdot)$ in $\mathbb{C} \setminus \Lambda_{(s,t)}$, which is an open set, was proved in Section 2.3 of [**7**]. If we modify q and w by setting them 0 on $[t,b)$ we do not change the solution on (s,t). The solution for the modified problem evaluated at t is analytic and coincides with $u^-(t,\cdot)$ proving its analyticity. It remains to show that a point $\lambda_0 \in \Lambda_{(s,t)}$ can merely give rise to poles.

We know already that there are only finitely many points x in (s,t) where one of $B_\pm(x,\lambda_0)$ fails to be invertible. Suppose x' and x'' are two consecutive such points. If we know the solution on (s,x') and that $u^-(x',\cdot)$ has, at worst, a pole at λ_0, then the solution in (x', x'') is determined by the initial value

$$u^+(x', \lambda) = B_+(x', \lambda)^{-1} B_-(x', \lambda) u^-(x', \lambda)$$

which also has, at worst, a pole at λ_0 since this is true for $B_+(x', \lambda)^{-1}$. For $x \in (s, t)$ the claim follows now by induction. To prove that $u^-(t, \cdot)$ is also meromorphic we proceed as before and modify q and w on $[t, b)$. □

3. Solving the differential equation

Our goal in this section is to investigate the set of solutions of the differential equation $Ju' + (q - \lambda w)u = wf$ on (a, b) under a strengthened hypothesis.

HYPOTHESIS 3.1. In addition to Hypothesis 2.1 we ask that a and b are regular endpoints for $Ju' + qu = wf$.

Moreover, given the partition

$$a = x_0 < x_1 < x_2 < ... < x_N < x_{N+1} = b \tag{3.1}$$

of (a, b) we require that $\Xi_0 \subset \{x_1, ..., x_N\}$. We then consider only λ for which both $B_+(x, \lambda)$ and $B_-(x, \lambda)$ are invertible unless x is in $\{x_1, ..., x_N\}$.

This hypothesis is in force throughout this section but later only if explicitly mentioned. We emphasize that Ξ_0 is finite when a and b are regular. Also, the set of permissible λ, which we call Ω_0, is symmetric with respect to the real axis and avoids only a discrete set.

On each interval (x_j, x_{j+1}) we let $U_j(\cdot, \lambda)$ be a fundamental matrix of balanced solutions of the homogeneous differential equation $Ju' + (q - \lambda w)u = 0$ such that $\lim_{x \downarrow x_j} U_j(x, \lambda) = \mathbb{1}$. The existence of these fundamental matrices is guaranteed by Theorem 2.2. The general balanced solution u of the non-homogeneous equation $Ju' + (q - \lambda w)u = wf$ on (x_j, x_{j+1}) satisfies, according to Lemma 3.3 in [**7**],

$$u^-(x) = U_j^-(x, \lambda)\big(c_j + J^{-1} \int_{(x_j, x)} U_j(\cdot, \overline{\lambda})^* wf\big)$$

for any $c_j \in \mathbb{C}^n$. Define

$$U_j(x_{j+1}, \lambda) = \lim_{x \uparrow x_{j+1}} U_j(x, \lambda) \quad \text{and} \quad I_j(f, \lambda) = \int_{(x_j, x_{j+1})} U_j(\cdot, \overline{\lambda})^* w f.$$

Using $u^+(x_j) = c_j$ and $u^-(x_j) = U_{j-1}(x_j, \lambda)(c_{j-1} + J^{-1} I_{j-1}(f, \lambda))$ in equation (2.2) gives

$$(-B_-(x_j, \lambda) U_{j-1}(x_j, \lambda), B_+(x_j, \lambda)) \begin{pmatrix} c_{j-1} \\ c_j \end{pmatrix}$$
$$= \Delta_w(x_j) f(x_j) + B_-(x_j, \lambda) U_{j-1}(x_j, \lambda) J^{-1} I_{j-1}(f, \lambda).$$

We need to consider these equations for $j = 1, ..., N$ simultaneously. This gives rise to the system

$$\mathbb{B}(\lambda)\tilde{u} = \mathcal{F}_0(f, \lambda) \tag{3.2}$$

where $\tilde{u} = (c_0, ..., c_N)^\diamond$, $\mathbb{B}(\lambda)$, to be specified presently, is in $\mathbb{C}^{nN \times n(N+1)}$, and $\mathcal{F}_0(f, \lambda)$ is in $\mathbb{C}^{nN}$. The two-diagonal block-matrix structure of $\mathbb{B}$ suggests the introduction of matrices $E_\top$ and $E_\perp$, which, respectively, strip the first and last n components off a vector in their domain $\mathbb{C}^{n(N+1)}$. If we also define the block-matrices

$$\mathcal{B}(\lambda) = \operatorname{diag}(B_+(x_1, \lambda), ..., B_+(x_N, \lambda)),$$

$$\mathcal{U}(\lambda) = \operatorname{diag}(U_0(x_1, \lambda), ..., U_{N-1}(x_N, \lambda)),$$

and $\mathcal{J} = \operatorname{diag}(J, ..., J)$ and when we note that

$$\mathcal{B}(\overline{\lambda})^* = \operatorname{diag}(-B_-(x_1, \lambda), ..., -B_-(x_N, \lambda)),$$

we obtain

$$\mathbb{B}(\lambda) = \mathcal{B}(\overline{\lambda})^* \mathcal{U}(\lambda) E_\perp + \mathcal{B}(\lambda) E_\top. \tag{3.3}$$

The vector $\mathcal{F}_0(f,\lambda)$ is given by

$$\mathcal{F}_0(f,\lambda) = \mathcal{R}(f) - \mathcal{B}(\overline{\lambda})^*\mathcal{U}(\lambda)\mathcal{J}^{-1}\mathcal{I}(f,\lambda)$$

with $\mathcal{R}(f) = ((\Delta_w f)(x_1), ..., (\Delta_w f)(x_N))^\diamond$ and $\mathcal{I}(f,\lambda) = (I_0(f,\lambda), ..., I_{N-1}(f,\lambda))^\diamond$.

We now have the following theorem.

Theorem 3.2. The differential equation $Ju' + (q - \lambda w)u = wf$ has a solution u on (a,b) if and only if $\tilde{u} = (u^+(x_0), ..., u^+(x_N))^\diamond$ is a solution of equation (3.2). In particular, in the homogeneous case, where $f = 0$, the space of solutions has dimension $n(N+1) - \operatorname{rk}\mathbb{B}(\lambda) \geq n$.

We note that $\operatorname{rk}\mathbb{B}(\lambda) = n$ when $N = 1$ so that the space of solutions of $Ju' + (q - \lambda w)u = 0$ is then exactly n-dimensional. For $N = 2$, however, consider the example $(a,b) = \mathbb{R}$, $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $q = \left(\begin{smallmatrix} 0 & 2 \\ 2 & 0 \end{smallmatrix}\right)(\delta_1 - \delta_2)$, $w = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 0 \end{smallmatrix}\right)(\delta_1 + \delta_2)$, where the δ_k are Dirac point measures concentrated on $\{k\}$. It shows that the dimension of the space of solutions of $Ju' + (q - \lambda w)u = 0$ may be strictly larger than n.

Next we investigate the connection between the right-hand limits of a solution u of the homogeneous equation $Ju' + (q - \lambda w)u = 0$ at the points x_0, ..., x_N (given by the vector $\tilde{u}$) and the vector $\hat{u} = (u(x_1), ..., u(x_N))^\diamond$. We have $\hat{u} = \mathbb{D}(\lambda)\tilde{u}$ where

$$\mathbb{D}(\lambda) = \frac{1}{2}(\mathcal{U}(\lambda)E_\perp + E_\top) \tag{3.4}$$

is again a two-diagonal block-matrix. If $N \geq 2$ we will also introduce the matrices $\mathbb{B}_m(\lambda)$ and $\mathbb{D}_m(\lambda)$ which are obtained by deleting the first and last n columns from $\mathbb{B}(\lambda)$ and $\mathbb{D}(\lambda)$, respectively. If $N = 1$ we should think of $B_m(\lambda)$ and $D_m(\lambda)$ as maps from the trivial vector space to $\mathbb{C}^n$. Their adjoints are the map from $\mathbb{C}^n$ to $\{0\}$. With this understanding the following results hold also for $N = 1$ even though they then involve "matrices" with no rows or columns.

Lemma 3.3. $\mathbb{D}(\overline{\lambda})^*\mathbb{B}(\lambda) - \mathbb{B}(\overline{\lambda})^*\mathbb{D}(\lambda) = \operatorname{diag}(-J, 0, ..., 0, J)$ and $\mathbb{D}_m(\overline{\lambda})^*\mathbb{B}(\lambda) - \mathbb{B}_m(\overline{\lambda})^*\mathbb{D}(\lambda) = 0$.

Proof. This follows since $\mathcal{U}(\overline{\lambda})^*\mathcal{J}\mathcal{U}(\lambda)=\mathcal{J}$ which, in turn, follows from Lemma 3.2 in [**7**]. □

Lemma 3.4. The map $v\mapsto\mathbb{B}(\lambda)v$, restricted to $\ker\mathbb{D}(\lambda)$, is a bijection onto $\ker\mathbb{D}_m(\overline{\lambda})^*$. Similarly, the map $v\mapsto\mathbb{D}(\lambda)v$, restricted to $\ker\mathbb{B}(\lambda)$, is a bijection onto $\ker\mathbb{B}_m(\overline{\lambda})^*$. In particular, $\dim\ker\mathbb{D}(\lambda)=\dim\ker\mathbb{D}_m(\overline{\lambda})^*$ and $\dim\ker\mathbb{B}(\lambda)=\dim\ker\mathbb{B}_m(\overline{\lambda})^*$.

Proof. The identity $\mathbb{D}_m(\overline{\lambda})^*\mathbb{B}(\lambda)-\mathbb{B}_m(\overline{\lambda})^*\mathbb{D}(\lambda)=0$ shows that $\mathbb{B}(\lambda)$ maps $\ker\mathbb{D}(\lambda)$ to $\ker\mathbb{D}_m(\overline{\lambda})^*$ as well as that $\mathbb{D}(\lambda)$ maps $\ker\mathbb{B}(\lambda)$ to $\ker\mathbb{B}_m(\overline{\lambda})^*$.

If $v\in\ker\mathbb{B}(\lambda)\cap\ker\mathbb{D}(\lambda)$ one shows that $E_\perp v=E_\top v=0$ using the definitions (3.3) and (3.4) of $\mathbb{B}$ and $\mathbb{D}$ and the fact that $\mathcal{B}(\lambda)-\mathcal{B}(\overline{\lambda})^*=2\mathcal{J}$. This, of course, implies that $v=0$ and hence the injectivity of both $\mathbb{B}(\lambda)|_{\ker\mathbb{D}(\lambda)}$ and $\mathbb{D}(\lambda)|_{\ker\mathbb{B}(\lambda)}$.

Clearly, both $\mathbb{D}(\lambda)$ and $\mathbb{D}_m(\overline{\lambda})^*$, having invertible matrices along their main diagonal, are of full rank. The rank-nullity theorem shows therefore that their kernels both have dimension n. This proves surjectivity of $\mathbb{B}(\lambda)|_{\ker\mathbb{D}(\lambda)}$.

Finally, assume that $v\in\ker\mathbb{B}_m(\overline{\lambda})^*$. Then $v=\mathbb{D}(\lambda)x$ for some $x\in\mathbb{C}^{n(N+1)}$ which implies that $0=\mathbb{B}_m(\overline{\lambda})^*\mathbb{D}(\lambda)x=\mathbb{D}_m(\overline{\lambda})^*\mathbb{B}(\lambda)x$. The first part of the proof shows that there is a $y\in\ker\mathbb{D}(\lambda)$ such that $\mathbb{B}(\lambda)y=\mathbb{B}(\lambda)x$. Hence $v=\mathbb{D}(\lambda)(x-y)$ where $x-y\in\ker\mathbb{B}(\lambda)$. □

The following theorem establishes a connection between solutions of the differential equation $Ju'+(q-\lambda w)u=0$ and elements of $\ker\mathbb{B}_m(\overline{\lambda})^*$.

Theorem 3.5. If u is a solution of $Ju'+(q-\lambda w)u=0$ on (a,b), then $\hat{u}=(u(x_1),...,u(x_N))^\diamond$ is in $\ker\mathbb{B}_m(\overline{\lambda})^*$. If, in addition, $u^+(a)=u^-(b)=0$, then $\hat{u}\in\ker\mathbb{B}(\overline{\lambda})^*$ (a subspace of $\ker\mathbb{B}_m(\overline{\lambda})^*$).

Conversely, if $\hat{u}\in\ker\mathbb{B}_m(\overline{\lambda})^*$, then $Ju'+(q-\lambda w)u=0$ has a unique solution u on (a,b) such that $(u(x_1),...,u(x_N))^\diamond=\hat{u}$. If, indeed, $\hat{u}\in\ker\mathbb{B}(\overline{\lambda})^*$, we further have $u^+(a)=u^-(b)=0$.

Let us emphasize that $\operatorname{supp}u\subset[x_1,x_N]$ when $u^+(a)=u^-(b)=0$.

Proof. If u solves $Ju' + (q - \lambda w)u = 0$, then, by Theorem 3.2, $\tilde{u} \in \ker \mathbb{B}(\lambda)$. Lemma 3.4 shows then that $\hat{u} = \mathbb{D}(\lambda)\tilde{u}$ is in $\ker \mathbb{B}_m(\overline{\lambda})^*$. If $u^+(a) = u^-(b) = 0$, then Lemma 3.3 gives $0 = \mathbb{B}(\overline{\lambda})^*\mathbb{D}(\lambda)\tilde{u} = \mathbb{B}(\overline{\lambda})^*\hat{u}$.

Conversely, assume that $\hat{u} \in \ker \mathbb{B}_m(\overline{\lambda})^* = \mathbb{D}(\lambda)(\ker \mathbb{B}(\lambda))$. Then there is a unique vector $\tilde{u} \in \ker \mathbb{B}(\lambda)$ such that $\hat{u} = \mathbb{D}(\lambda)\tilde{u}$, which, in turn, defines a unique solution u of $Ju' + (q - \lambda w)u = 0$ such that $(u(x_1), ..., u(x_N))^\diamond = \hat{u}$. If $\hat{u} \in \ker \mathbb{B}(\overline{\lambda})^*$, then, according to Lemma 3.3, $\operatorname{diag}(-J, 0, ..., 0, J)\tilde{u} = 0$ which shows that $u^+(a) = u^-(b) = 0$. $\square$

Given an algebraic system $Ax = b$ we know that there exist solutions only if $b \in \operatorname{ran} A = (\ker A^*)^\perp$. For the differential equation $Ju' + (q - \lambda w)u = wf$ with integrable coefficients q and w the unique continuation property for the solutions gives rise to the variation of constants formula, which then guarantees the existence of solutions for any non-homogeneity f (within reason). In the present situation, however, the problem of existence raises its head and we now set out to give necessary and sufficient conditions for f guaranteeing the existence of a solution in the spirit of Linear Algebra.

Lemma 3.6. If $\tilde{v} \in \ker \mathbb{B}(\overline{\lambda})$ and $\hat{v} = \mathbb{D}(\overline{\lambda})\tilde{v}$, then

$$\tilde{v}^* E_\perp^* = -\hat{v}^* \mathcal{B}(\overline{\lambda})^* \mathcal{U}(\lambda) \mathcal{J}^{-1} \quad \text{and} \quad \tilde{v}^* E_\top^* = \hat{v}^* \mathcal{B}(\lambda) \mathcal{J}^{-1}.$$

Moreover, if $f \in \mathcal{L}^2(w)$ and $Jv' + (q - \overline{\lambda} w)v = 0$, then

$$\int v^* w f = \hat{v}^* \mathcal{F}_0(f, \lambda) + \hat{v}^* \mathcal{B}(\lambda) \mathcal{J}^{-1} \tilde{\mathcal{I}}(f, \lambda) = \hat{v}^* \mathcal{F}_0(f, \lambda) + \tilde{v}^* E_\top^* \tilde{\mathcal{I}}(f, \lambda)$$

where $(v(x_1), ..., v(x_N))^\diamond = \hat{v} = \mathbb{D}(\overline{\lambda})\tilde{v}$ and $\tilde{\mathcal{I}}(f, \lambda) = (0, ..., 0, I_N(f, \lambda))^\diamond \in \mathbb{C}^{nN}$.

Proof. Using the definitions (3.3) and (3.4) of $\mathbb{B}$ and $\mathbb{D}$ and the identities $\mathcal{B}(\lambda) - \mathcal{B}(\overline{\lambda})^* = 2\mathcal{J}$ and $\mathcal{U}(\lambda)^* \mathcal{J} \mathcal{U}(\overline{\lambda}) = \mathcal{J}$ we obtain that $\mathbb{B}(\overline{\lambda})\tilde{v} = 0$ implies

$$\mathcal{B}(\overline{\lambda})\mathbb{D}(\overline{\lambda})\tilde{v} = \mathcal{U}(\lambda)^{*-1} \mathcal{J} E_\perp \tilde{v} \quad \text{and} \quad \mathcal{B}(\lambda)^* \mathbb{D}(\overline{\lambda})\tilde{v} = -\mathcal{J} E_\top \tilde{v}.$$

Taking adjoints gives the first claim since $\hat{v} = \mathbb{D}(\overline{\lambda})\tilde{v}$.

The second claim is an immediate consequence of this, since

$$\begin{aligned}
\int v^* wf &= \hat{v}^* \mathcal{R}(f) + \tilde{v}^*(I_0(f,\lambda), ..., I_N(f,\lambda))^\diamond \\
&= \hat{v}^* \mathcal{R}(f) + \tilde{v}^* E_\perp^* \mathcal{I}(f,\lambda) + \tilde{v}^* E_\top^* \tilde{\mathcal{I}}(f,\lambda) \\
&= \hat{v}^* \mathcal{R}(f) - \hat{v}^* \mathcal{B}(\overline{\lambda})^* \mathcal{U}(\lambda) \mathcal{J}^{-1} \mathcal{I}(f,\lambda) + \hat{v}^* \mathcal{B}(\lambda) \mathcal{J}^{-1} \tilde{\mathcal{I}}(f,\lambda) \\
&= \hat{v}^* \mathcal{F}_0(f,\lambda) + \hat{v}^* \mathcal{B}(\lambda) \mathcal{J}^{-1} \tilde{\mathcal{I}}(f,\lambda).
\end{aligned}$$

□

THEOREM 3.7. The differential equation $Ju' + (q - \lambda w)u = wf$ has a solution on (a, b) if and only if $\int v^* wf = 0$ for every solution v of $Jv' + (q - \overline{\lambda} w)v = 0$ which vanishes at a and b.

PROOF. By Theorem 3.2 the solution u exists if and only if the system (3.2) has a solution $\tilde{u} = (u^+(x_0), ..., u^+(x_N))^\diamond$. This, in turn, happens if and only if $\mathcal{F}_0(f,\lambda) \in \operatorname{ran} \mathbb{B}(\lambda) = (\ker \mathbb{B}(\lambda)^*)^\perp$.

By Theorem 3.5 the solutions of $Jv' + (q - \overline{\lambda} w)v = 0$ which vanish at a and b are in one-to-one correspondence with elements of $\ker \mathbb{B}(\lambda)^*$. Since $v^+(x_N) = 0$ we have $\tilde{v}^* E_\top^* \tilde{\mathcal{I}}(f,\lambda) = 0$ and then, from Lemma 3.6, we obtain $\hat{v}^* \mathcal{F}_0(f,\lambda) = \int v^* wf$. □

In the case of unique continuation of solutions the condition that v vanishes at a or b implies, of course, that $v = 0$. Consequently, $Ju' + (q - \lambda w)u = wf$ has then a solution for any $f \in \mathcal{L}^2(w)$. The set of all solutions is thus obtained by adding the general solution of $Ju' + (q - \lambda w)u = 0$ whose dimension is $n(N + 1) - \operatorname{rk} \mathbb{B}(\lambda) \geq n$.

THEOREM 3.8. The differential equation $Ju' + (q - \lambda w)u = wf$ has a solution on (a, b) which vanishes at a and b if and only if $\int v^* wf = 0$ for every solution v of $Jv' + (q - \overline{\lambda} w)v = 0$.

PROOF. For u to vanish at a and b it is required that $u^+(x_0) = 0$ and $u^+(x_N) = -J^{-1}I_N(f,\lambda)$. The system (3.2) is therefore equivalent to

$$\mathbb{B}_m(\lambda)(c_1, ..., c_{N-1})^\diamond = \mathcal{F}_0(f,\lambda) + \mathcal{B}(\lambda)\mathcal{J}^{-1}\tilde{\mathcal{I}}(f,\lambda).$$

The proof is now analogous to the one for Theorem 3.7. □

We conclude this section by "counting" the solutions of $Ju' + qu = \lambda wu$ which are not compactly supported. More precisely, we will determine the dimension of the quotient space of all solutions of $Ju' + qu = \lambda wu$ modulo the space of compactly supported solutions. Theorem 3.5 shows that the space of all solutions of $Ju' + qu = \lambda wu$ is in one-to-one correspondence with $\ker \mathbb{B}_m(\overline{\lambda})^*$ and that the space of compactly supported solutions of $Ju' + qu = \lambda wu$ is in one-to-one correspondence with $\ker \mathbb{B}(\overline{\lambda})^*$. We therefore define

$$\tilde{n}(\lambda) = \dim(\ker \mathbb{B}_m(\overline{\lambda})^* / \ker \mathbb{B}(\overline{\lambda})^*) = \dim\ker \mathbb{B}_m(\overline{\lambda})^* - \dim\ker \mathbb{B}(\overline{\lambda})^*.$$

LEMMA 3.9. $\tilde{n}(\lambda) + \tilde{n}(\overline{\lambda}) = 2n$.

PROOF. Since $\operatorname{rk}\mathbb{B}(\lambda) = \operatorname{rk}\mathbb{B}(\lambda)^*$, the rank-nullity theorem implies

$$\dim\ker \mathbb{B}(\lambda) = n(N+1) - \operatorname{rk}\mathbb{B}(\lambda)^* = n + \dim\ker \mathbb{B}(\lambda)^*.$$

Hence, using also the analogous equation for $\overline{\lambda}$,

$$\dim\ker \mathbb{B}(\lambda) - \dim\ker \mathbb{B}(\lambda)^* + \dim\ker \mathbb{B}(\overline{\lambda}) - \dim\ker \mathbb{B}(\overline{\lambda})^* = 2n.$$

Lemma 3.4 gives that $\dim\ker \mathbb{B}(\lambda) = \dim\ker \mathbb{B}_m(\overline{\lambda})^*$ yielding the claim. □

From Theorem 2.4 we know that the matrices $U_j(x_{j+1}, \cdot)$ are meromorphic on $\mathbb{C}$ with poles at most at points in the complement of Ω_0. It follows that the entries of $\mathbb{B}$ are also meromorphic. Since the meromorphic functions on $\mathbb{C}$ form a field there is a row-echelon matrix $\tilde{\mathbb{B}}$ with meromorphic entries such that $\mathbb{B}\tilde{u} = 0$ has the same solutions as $\tilde{\mathbb{B}}\tilde{u} = 0$. Now define a set Ω as Ω_0 without the set of all poles of $\tilde{\mathbb{B}}$ as well as their complex conjugates, and the set of zeros and their conjugates of any of the pivots of $\tilde{\mathbb{B}}$.

THEOREM 3.10. If $\lambda \in \Omega$, then $\dim\ker \mathbb{B}(\lambda) = \dim\ker \mathbb{B}(\overline{\lambda})$ and $\tilde{n}(\lambda) = n$.

PROOF. The construction of Ω entails that $\operatorname{rk} \mathbb{B}(\lambda) = \operatorname{rk} \tilde{\mathbb{B}}(\lambda) = \operatorname{rk} \mathbb{B}(\overline{\lambda})$ if $\lambda \in \Omega$. Since $\tilde{n}(\lambda) = \dim\ker \mathbb{B}(\lambda) - \dim\ker \mathbb{B}(\overline{\lambda})^* = \dim\ker \mathbb{B}(\lambda) + n - \dim\ker \mathbb{B}(\overline{\lambda})$ we obtain $\tilde{n}(\lambda) = n$. □

4. Symmetric restrictions of $T_{\max}$

Given a differential equation $Ju' + qu = wf$ we now define associated minimal and maximal relations. Recall that $\mathcal{L}^2(w)$ is the space of functions f such that $\int f^* w f < \infty$. First we define

$$\mathcal{T}_{\max} = \{(u, f) \in \mathcal{L}^2(w) \times \mathcal{L}^2(w) : u \in \mathrm{BV}^{\#}_{\mathrm{loc}}((a, b))^n, Ju' + qu = wf\}.$$

Subsequently we will always tacitly assume that $u \in \mathrm{BV}^{\#}_{\mathrm{loc}}((a, b))^n$, when we use u'. Next, let

$$\mathcal{T}_{\min} = \{(u, f) \in \mathcal{T}_{\max} : \operatorname{supp} u \text{ is compact in } (a, b)\}.$$

Note that these are spaces of pairs of functions. To employ the power of functional analysis we need to realize these relations in Hilbert spaces. Therefore we introduce, as usual, the space $L^2(w)$ as the quotient of $\mathcal{L}^2(w)$ modulo the subspace of all $u \in \mathcal{L}^2(w)$ for which $\|u\|^2 = \int u^* w u = 0$. Denoting the equivalence class corresponding to u by $[u]$ we now set

$$T_{\max} = \{([u], [f]) \in L^2(w) \times L^2(w) : (u, f) \in \mathcal{T}_{\max}\}$$

and

$$T_{\min} = \{([u], [f]) \in T_{\max} : (u, f) \in \mathcal{T}_{\min}\}.$$

Here (and elsewhere) we choose brevity over precision: whenever we have a pair $([u], [f])$ in $T_{\max}$ we choose u and f such that $(u, f) \in \mathcal{T}_{\max}$.

Define the vector space

$$\mathcal{L}_0 = \{u \in \mathrm{BV}^{\#}_{\mathrm{loc}}((a,b))^n : Ju' + qu = 0 \text{ and } \|u\| = 0\}.$$

In many cases this space is trivial and some authors restrict their attention to the case where it is; this is then called the definiteness condition. However, we will not do so here. Note that $\|u\| = 0$ if and only if wu is the zero distribution. The significance of $\mathcal{L}_0$ stems from the following fact. Suppose $([u],[f]) \in T_{\max}$ and that there are $u, v \in [u]$ and $f, g \in [f]$ such that $Ju' + qu = wf$ and $Jv' + qv = wg$. Then $J(u-v)' + q(u-v) = w(f-g) = 0$ as well as $w(u-v) = 0$, i.e., $u - v \in \mathcal{L}_0$. In other words, in the presence of a non-trivial space $\mathcal{L}_0$, the class $[u]$ has many representatives of locally bounded variation satisfying the differential equation for a given class $[f]$ (the choice of a representative of $[f]$, on the other hand, is irrelevant). In Section 5 we will describe a procedure to choose a representative of $[u]$ in a distinctive way.

In [**4**] it was proved that $T_{\min}$ is symmetric, indeed that $T_{\min}^* = T_{\max}$. In this case it is well-known that von Neumann's theorem holds. Setting $D_\lambda = \{([u], \lambda[u]) \in T_{\max}\}$ it states that

$$T_{\max} = \overline{T_{\min}} \uplus D_\lambda \uplus D_{\overline{\lambda}}$$

when $\operatorname{Im}\lambda \neq 0$. Moreover, when $\lambda = \pm i$, these direct sums are even orthogonal. It is also known that the dimension of D_λ does not change as λ varies in either the upper or the lower half plane. The numbers $n_\pm = \dim D_{\pm i}$ are called deficiency indices of $T_{\min}$ and we are now setting out to investigate these.

If u is a solution of $Ju' + qu = \lambda wu$ which is compactly supported then $(u, \lambda u) \in \mathcal{T}_{\min}$ and $([u], \lambda[u]) \in T_{\min} \cap D_\lambda$. If λ is not real, then $T_{\min} \cap D_\lambda$ is trivial and it follows that compactly supported solutions of $Ju' + qu = \lambda wu$ do not contribute to the corresponding deficiency index. We now have, as a corollary of Theorem 3.10, that the deficiency indices of $T_{\min}$ cannot be more than n if a and b are regular endpoints. We do not state this result separately since it is included in the next theorem about the general case.

Thus, to emphasize, we allow in the following a and b to be either regular or singular endpoints. Let τ_k, $k \in \mathbb{Z}$, be a strictly increasing sequence in (a, b) having a and b as its only limit points and such that all points in Ξ_0 are among the τ_k. Considering now only the interval $I_k = (\tau_{-k}, \tau_k)$ we set $x_j = \tau_{-k+j}$ for $j = 0, ..., N+1 = 2k$. We can then introduce the objects from Section 3. To emphasize their dependence on k we will add a superscript (k) to those objects. We have then, in particular, the matrices $\mathbb{B}^{(k)}$, $\mathbb{B}_m^{(k)}$ and the sets $\Omega^{(k)}$ of permissible values of λ. We now define $\Omega = \bigcap_{k=1}^{\infty} \Omega^{(k)}$ and note that Ω is symmetric with respect to the real axis and misses only countably many values from $\mathbb{C}$.

Now fix a non-real $\lambda \in \Omega$. If u is a solution of $Ju' + qu = \lambda wu$ on (a, b) we denote its restriction to the interval I_k by $u^{(k)}$. We are interested in the quotient space X_k of all solutions of $Ju' + qu = \lambda wu$ on I_k modulo the compactly supported solutions. If u is a solution of $Ju' + qu = \lambda wu$ on I_k we denote the associated equivalence class in X_k by $\lfloor u \rfloor_k$. A compactly supported solution u of $Ju' + qu = \lambda wu$ on I_k can be extended by 0 to all of (a, b) yielding an element in $T_{\min} \cap D_\lambda$. This implies, since $\operatorname{Im} \lambda \neq 0$, that $\|u\|^2 = \int_{I_k} u^* wu = 0$ and shows that X_k is a normed space with the norm given by $\|u\|_k^2 = \int_{I_k} u^* wu$. According to Theorem 3.5 the quotient space X_k is isomorphic to $\ker \mathbb{B}_m^{(k)}(\overline{\lambda})^* / \ker \mathbb{B}^{(k)}(\overline{\lambda})^*$ and, by Theorem 3.10, its dimension is equal to n since $\lambda \in \Omega \subset \Omega^{(k)}$.

Theorem 4.1. The deficiency indices of $T_{\min}$ are less than or equal to n.

Proof. Fix a non-real $\lambda \in \Omega$. Suppose u_1, ..., u_m are solutions of $Ju' + qu = \lambda wu$ such that $[u_1]$, ..., $[u_m]$ are linearly independent elements of D_λ. We will show below that there is an interval $I_p = (\tau_{-p}, \tau_p)$ such that $\lfloor u_1^{(p)} \rfloor_p$, ..., $\lfloor u_m^{(p)} \rfloor_p$ are linearly independent elements of X_p. Hence $m \leq n$, the dimension of X_p. Since deficiency indices are constant in either half-plane they cannot be larger than n.

We will now prove the existence of I_p by induction. That is we prove that, for every $k \in \{1, ..., m\}$, there is an interval I_{ℓ_k} such that the restrictions of u_1, ..., u_k to

I_{ℓ_k} generate linearly independent elements $\lfloor u_1^{(\ell_k)} \rfloor_{\ell_k}$, ..., $\lfloor u_k^{(\ell_k)} \rfloor_{\ell_k}$ of X_{ℓ_k}. Once this is achieved we set $p = \ell_m$.

Suppose $k = 1$ and let I_{ℓ_1} be an interval such that $\|u_1^{(\ell_1)}\| > 0$. By what we argued above we know that $u_1^{(\ell_1)}$ is not compactly supported in I_{ℓ_1} and thus gives rise to a non-zero (and hence linearly independent) element of X_{ℓ_1}.

Now suppose we had already shown our claim for some $k < m$. If $\lfloor u_1^{(\ell_k)} \rfloor_{\ell_k}$, ..., $\lfloor u_{k+1}^{(\ell_k)} \rfloor_{\ell_k}$ are already linearly independent as elements of X_{ℓ_k} we choose $\ell_{k+1} = \ell_k$ and our induction step is complete. Otherwise, there are unique complex numbers α_1, ..., α_k such that

$$\|(\alpha_1 u_1 + ... + \alpha_k u_k + u_{k+1})^{(\ell_k)}\|_{\ell_k} = 0.$$

However, there must be an interval $I_{\ell_{k+1}} \supset I_{\ell_k}$ where

$$\|(\alpha_1 u_1 + ... + \alpha_k u_k + u_{k+1})^{(\ell_{k+1})}\|_{\ell_{k+1}} > 0$$

on account that $[u_1]$, ..., $[u_{k+1}]$ are linearly independent. It follows now that, as elements of $X_{\ell_{k+1}}$ the vectors $\lfloor u_1^{(\ell_{k+1})} \rfloor_{\ell_{k+1}}$, ..., $\lfloor u_{k+1}^{(\ell_{k+1})} \rfloor_{\ell_{k+1}}$ are linearly independent. This completes our induction step also in this case. □

Corollary 4.2. If a and b are regular, then $n_+ = n_-$.

Proof. Fix a non-real λ in Ω. Since a and b are regular, the set $\Xi_\lambda = \Xi_{\overline{\lambda}}$ is finite. Thus we may assume that it is contained in $I_k = (\tau_{-k}, \tau_k)$ for some appropriate k. Then $\dim \ker \mathbb{B}^{(k)}(\lambda)$ is the number of linearly independent solutions of $Ju' + qu = \lambda wu$. Theorem 3.10 shows that $Ju' + qu = \overline{\lambda} wu$ has the same number of linearly independent solutions. Any of these solutions has finite norm but some may have norm 0. Now note, that if u is a solution of $Ju' + qu = \lambda wu$ of norm 0, then we have $wu = 0$, so that u is also a solution of $Ju' + qu = \overline{\lambda} wu$. Therefore $n_+ = n_-$. □

As mentioned above, it is well-known, even in the case of relations, that von Neumann's theorem $E^* = E \oplus D_i \oplus D_{-i}$ holds when E is a closed symmetric relation in $\mathcal{H} \times \mathcal{H}$ when $\mathcal{H}$ is a Hilbert space. In our case, when $d = \dim D_i \oplus D_{-i}$ is finite, as we just showed, we can use Theorem B.5 in [**7**] to characterize the symmetric restriction of $T_{\max}$ in terms of boundary conditions. We state that theorem here for easy reference. The operator $\mathcal{J}$ appearing there is defined by $\mathcal{J}(u,f)=(f,-u)$ for $u,f\in\mathcal{H}$.

THEOREM 4.3. Suppose E is a closed symmetric relation in $\mathcal{H}\times\mathcal{H}$ with $d=\dim D_i\oplus D_{-i}<\infty$ and that $m\le d/2$ is a natural number or 0. If $A:E^*\to\mathbb{C}^{d-m}$ is a surjective linear operator such that $E\subset\ker A$ and $A\mathcal{J}A^*$ has rank $d-2m$ then $\ker A$ is a closed symmetric restriction of E^* for which the dimension of $(\ker A)\ominus E$ is m. Conversely, every closed symmetric restriction of E^* is the kernel of such a linear operator A. Finally, $\ker A$ is self-adjoint if and only if $A\mathcal{J}A^*=0$ (entailing $m=d/2$).

A second ingredient for our next considerations is Lagrange's identity (or Green's formula). If (u,f) and (v,g) are in $\mathcal{T}_{\max}$, then v^*wf and g^*wu are finite measures. Therefore $v^*Ju'+v'^*Ju=v^*wf-g^*wu$ is also a finite measure. Its antiderivative v^*Ju is of bounded variation and thus has limits at a and b. Integration now gives Lagrange's identity

$$(v^*Ju)^-(b)-(v^*Ju)^+(a)=\langle v,f\rangle-\langle g,u\rangle. \tag{4.1}$$

Note the right-hand side, and hence the left-hand side, does not change upon choosing different representatives in place of u,f,v, or g.

Now, if (v,g) is an element of $D_i\oplus D_{-i}$, then $(u,f)\mapsto\langle(v,g),(u,f)\rangle$ is a bounded linear functional on $T_{\max}$. Conversely, since $T_{\max}$ is a Hilbert space, a bounded linear functional on $T_{\max}$ is given by $(u,f)\mapsto\langle(v,g),(u,f)\rangle$ for some $(v,g)\in T_{\max}$. When it is also known that $\overline{T_{\min}}$ is in the kernel of this functional, (v,g) may be chosen in $D_i\oplus D_{-i}$. Hence, in our situation, the operator A from Theorem 4.3 is given by $d-m$ linearly independent elements in $D_i\oplus D_{-i}$. Lagrange's identity implies that the entries of the matrix $A\mathcal{J}A^*$ are then given by

$$(A\mathcal{J}A^*)_{k,\ell}=\langle(v_k,g_k),(g_\ell,-v_\ell)\rangle=(g_k^*Jg_\ell)^-(b)-(g_k^*Jg_\ell)^+(a). \tag{4.2}$$

Therefore we arrive at the following theorem.

THEOREM 4.4. Let $d = n_+ + n_-$ and suppose that $m \leq \min\{n_+, n_-\}$. If (v_1, g_1), ..., (v_{d-m}, g_{d-m}) are linearly independent elements of $D_i \oplus D_{-i}$ such that the matrix defined in (4.2) has rank $d - 2m$, then

$$T = \{(u, f) \in T_{\max} : (g_j^* Ju)^-(b) - (g_j^* Ju)^+(a) = 0 \text{ for } j = 1, ..., d - m\} \tag{4.3}$$

is a closed symmetric restriction of $T_{\max}$.

Conversely, if T is a closed symmetric restriction of $T_{\max}$ and m is the dimension of $T \ominus \overline{T_{\min}}$, then T is given by (4.3) for appropriate elements (v_1, g_1), ..., (v_{d-m}, g_{d-m}) of $D_i \oplus D_{-i}$ for which the matrix defined in (4.2) has rank $d - 2m$.

For self-adjoint restrictions of $T_{\max}$ it is hence necessary and sufficient that $n_+ = n_- = m = d - m$ and that $(g_k^* J g_\ell)^-(b) - (g_k^* J g_\ell)^+(a) = 0$ for all $1 \leq k, \ell \leq m = d/2$.

5. The space $\mathcal{L}_0$

We mentioned earlier that the class $[u]$ does not have a unique balanced representative when $([u], [f]) \in T_{\max}$, if the space $\mathcal{L}_0$ has non-trivial elements. In this section we describe a procedure to choose a representative in a distinctive way.

To this end we assume, without loss of generality, that $B_+(\tau_0, 0) = B_-(\tau_0, 0) = J$ so that solutions of our differential equations are continuous at τ_0. Define $N_0 = \{h(\tau_0) : h \in \mathcal{L}_0\}$ and for each $k \in \mathbb{N}$ both $N_k = \{h^+(\tau_k) : h \in \mathcal{L}_0, \operatorname{supp} h \subset [\tau_k, b)\}$ and $N_{-k} = \{h^-(\tau_{-k}) : h \in \mathcal{L}_0, \operatorname{supp} h \subset (a, \tau_{-k}]\}$. Then, for $k \in \mathbb{N}_0$, we say that a function $u \in \mathrm{BV}^\#_{\mathrm{loc}}((a, b))^n$ satisfies condition $(\pm k)$, if $u^\pm(\tau_{\pm k})$ is perpendicular to $N_{\pm k}$ (using always the upper sign or always the lower sign).

LEMMA 5.1. Suppose $([u], [f]) \in T_{\max}$. Then there is a unique balanced $v \in [u]$ such that $(v, f) \in \mathcal{T}_{\max}$ and v satisfies condition (k) for every $k \in \mathbb{Z}$.

Proof. First consider uniqueness. Suppose u and v are two functions satisfying the given conditions. Then $u - v \in \mathcal{L}_0$ and hence $(u - v)(\tau_0)^* t(\tau_0) = 0$ for $t = u$ and $t = v$. Subtract these equations to find $(u - v)(\tau_0) = 0$, and thus $u = v$ on (τ_{-1}, τ_1). Moreover, $h_1 = (u - v)\chi_{[\tau_1,b)}$ and $h_{-1} = (u - v)\chi_{(a,\tau_{-1}]}$ are in $\mathcal{L}_0$. Conditions (1) and (−1) show therefore that $(u - v)^+(\tau_1)$ and $(u - v)^-(\tau_{-1})$ are also 0 which proves that $u = v$ on (τ_{-2}, τ_2). Induction informs us now that $u = v$ everywhere.

We now turn to existence. Pick a balanced representative $u \in [u]$ such that $(u, f) \in \mathcal{T}_{\max}$. There is an element $h_0 \in \mathcal{L}_0$ such that the orthogonal projection of $u(\tau_0)$ onto N_0 equals $h_0(\tau_0)$. Thus $v_0 = u - h_0$ satisfies $(v_0, f) \in \mathcal{T}_{\max}$, $v_0 \in [u]$, and condition (0).

Next, there is an element $h_1 \in \mathcal{L}_0$ with support in $[\tau_1, b)$ such that the orthogonal projection of $v_0^+(\tau_1)$ onto N_1 equals $h_1^+(\tau_1)$. We now define $v_1 = v_0 - h_1$. Then $(v_1, f) \in \mathcal{T}_{\max}$, $v_1 \in [u]$, and v_1 satisfies condition (1). Notice that $v_1 = v_0$ on (a, τ_1) implying that v_1 also satisfies condition (0).

Proceeding recursively, we may define, for each $k \in \mathbb{N}$, functions $h_k \in \mathcal{L}_0$ supported in $[\tau_k, b)$ such that $v_k = u - \sum_{j=0}^{k} h_j$ satisfies conditions (0), ..., (k), $v_k \in [u]$, and $(v_k, f) \in \mathcal{T}_{\max}$.

Since, for a fixed $x \in [\tau_0, b)$, only finitely many of the numbers $h_k(x)$ are different from 0, we find that the sequence $k \mapsto v_k$ converges pointwise to a function $\tilde{v} \in [u]$ satisfying conditions (k) for all $k \in \mathbb{N}_0$ and $(\tilde{v}, f) \in \mathcal{T}_{\max}$. We can now repeat this process for negative integers starting from the function $\tilde{v}$ instead of u arriving eventually at a function $v \in [u]$ satisfying conditions (k) for all $k \in \mathbb{Z}$ and $(v, f) \in \mathcal{T}_{\max}$. □

We denote the operator which assigns the function v just constructed to a given element $([u], [f]) \in T_{\max}$ by E. If $I_m = (\tau_{-m}, \tau_m)$ we also define $E_m : T_{\max} \to \mathrm{BV}^\#(I_m)^n$ by composing E with the restriction to the interval I_m. Note that $\mathrm{BV}^\#(I_m)^n$ is a Banach space with the norm $|||u|||_m$ defined as the sum of the variation of u over I_m and the norm of $u(\tau_0)$.

Theorem 5.2. The operator $E_m : T_{\max} \to \mathrm{BV}^{\#}(I_m)^n$ is bounded.

Proof. Due to the closed graph theorem we merely have to show that E_m is a closed operator. Thus assume that the sequence $([u_j],[f_j])$ converges to $([u],[f])$ in $T_{\max}$ and that $E_m([u_j],[f_j])$ converges to v in $\mathrm{BV}^{\#}(I_m)^n$ and hence pointwise. To simplify notation we assume that $E_m([u_j],[f_j]))$ and $E_m([u],[f])$ are the restrictions of u_j and u, respectively, to the interval I_m. We need to show that $u = v$ on I_m.

First note that $u_j^{\pm}(\tau_{\pm k}) \in N_{\pm k}^{\perp}$ and $\left|u_j^{\pm}(\tau_{\pm k}) - v^{\pm}(\tau_{\pm k})\right| \to 0$ imply that v satisfies conditions $(\pm k)$ for each $k \in \{0,...,m-1\}$. For $\ell \in \{-m, m-1\}$ and $x \in (\tau_\ell, \tau_{\ell+1})$ we have

$$u_j^-(x) = U_\ell^-(x)\Big(u_j^+(\tau_\ell) + J^{-1}\int_{(\tau_\ell,x)} U_\ell^* w f_j\Big)$$

when U_ℓ denotes the fundamental matrix of $Ju' + qu = 0$ on the interval $(\tau_\ell, \tau_{\ell+1})$ satisfying $U_\ell^+(\tau_\ell) = \mathbb{1}$. Taking the limit as $j \to \infty$ gives

$$v^-(x) = U_\ell^-(x)\Big(v^+(\tau_\ell) + J^{-1}\int_{(\tau_\ell,x)} U_\ell^* w f\Big)$$

since the integral may be considered as a vector of scalar products which are, of course, continuous. The variation of constants formula shows that v is a balanced solution for $Jv' + qv = wf$ on $(\tau_\ell, \tau_{\ell+1})$. We also have

$$J(u_j^+(\tau_\ell) - u_j^-(\tau_\ell)) + \Delta_q(\tau_\ell)u_j(\tau_\ell) = \Delta_w(\tau_\ell)f_j(\tau_\ell). \tag{5.1}$$

The fact that $[f_j]$ converges to $[f]$ in $L^2(w)$ implies, on account of the positivity of w, that $\Delta_w(\tau_\ell)f_j(\tau_\ell)$ converges to $\Delta_w(\tau_\ell)f(\tau_\ell)$. Therefore taking a limit in (5.1) shows, in conjunction with the previous observations, that $Jv' + qv = wf$ on the interval I_m. Since u satisfies the same equation we have that $u - v$ satisfies $J(u-v)' + q(u-v) = 0$ on I_m.

Next we show $w(u-v)=0$ on I_m. Fatou's lemma implies

$$0 \leq \int_{I_m} (u-v)^* w(u-v) \leq \liminf_{j\to\infty} \int_{I_m} (u-u_j)^* w(u-u_j) = 0.$$

It follows that $w(u-v)=0$ on I_m.

Finally, a variant of Lemma 5.1 shows now that $u=v$. $\square$

6. Green's function

Now suppose that we have a self-adjoint restriction T of $T_{\max}$. The resolvent set of T is the set of those λ for which $T-\lambda : \mathrm{dom}(T) \to L^2(w)$ is bijective, i.e.,

$$\varrho(T) = \{\lambda \in \mathbb{C} : \ker(T-\lambda) = \{0\}, \mathrm{ran}(T-\lambda) = L^2(w)\}$$

which is an open set. We denote its complement, the spectrum of T, by $\sigma(T)$. Since T is self-adjoint, $\sigma(T)$ is a subset of $\mathbb{R}$. If $\lambda \in \varrho(T)$, then the resolvent $R_\lambda = (T-\lambda)^{-1}$ is a bounded linear operator from $L^2(w)$ to $\mathrm{dom}(T)$. We now define $\mathcal{R}_\lambda : L^2(w) \to \mathrm{BV}^{\#}_{\mathrm{loc}}((a,b))^n$ by

$$\mathcal{R}_\lambda[f] = E((R_\lambda[f], \lambda R_\lambda[f] + [f])).$$

Thus $\mathcal{R}_\lambda[f]$ is the unique solution of $Ju' + qu = w(\lambda u + f)$ in $\mathcal{L}^2(w)$ satisfying condition (k) for every $k \in \mathbb{Z}$.

We will now show that $\mathcal{R}_\lambda$ is an integral operator. Its kernel G is called a Green's function for T.

Theorem 6.1. If T is a self-adjoint restriction of $T_{\max}$, then there exists, for given $x \in (a,b)$ and $\lambda \in \varrho(T)$, a matrix $G(x,\cdot,\lambda)$ such that the columns of $G(x,\cdot,\lambda)^*$ are in $L^2(w)$ and

$$(\mathcal{R}_\lambda[f])(x) = \int G(x,\cdot,\lambda) w f. \tag{6.1}$$

PROOF. Fix $x \in I_m$ and $\lambda \in \varrho(T)$. Consider the restriction of $\mathcal{R}_\lambda[f]$ to the interval I_m. Since E_m and R_λ are bounded operators the map $[f] \mapsto (\mathcal{R}_\lambda[f])(x)$ is a bounded linear map from $L^2(w)$ to $\mathbb{C}^n$. Hence there are elements $[g_1], ..., [g_n] \in L^2(w)$ such that the k-th component of $(\mathcal{R}_\lambda[f])(x)$ equals $\langle [g_k], [f] \rangle$. Let these be the columns of the matrix-valued function $G(x, \cdot, \lambda)^*$. Then we obtain (6.1). □

One wishes to complement this fairly abstract existence result by a more concrete one where Green's function is given in terms of solutions of the differential equation as is done in the classical case, see, for instance, Zettl [**11**]. This was also achieved in [**7**] under the assumption that Ξ_0 is empty and minor generalizations of this are certainly possible. Such an explicit construction of Green's function, where possible, is the cornerstone of many other results in spectral theory, in particular the development of a spectral transformation and more detailed information about the resolvent, e.g., the compactness of the resolvent in the regular case. Due to the difficulties posed by the absence of an existence and uniqueness theorem for initial value problems we have, so far, not been able to obtain such a construction in general. However, we hope to return to this issue in the future.

7. Example

In this section we treat an example where the matrices $B_\pm(x, \lambda)$ fail to be invertible for infinitely many x and all λ, in other words where Ξ_0 is infinite and $\Lambda_x = \mathbb{C}$ for all $x \in \Xi_0$ (recall that in [**7**] the hypothesis $\Xi_0 = \emptyset$ was made causing each Λ_x to be finite). The example is $Ju' + qu = wf$ on $(a, b) = \mathbb{R}$ where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ q = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \sum_{k \in \mathbb{Z}} (\delta_{2k} - \delta_{2k+1}), \text{ and, } w = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \sum_{k \in \mathbb{Z}} \delta_k$$

with δ_k denoting the Dirac point measure concentrated on $\{k\}$. Since we are seeking balanced solutions we need the matrices

$$B_-(2k-1,\lambda) = \begin{pmatrix} \lambda & 0 \\ 2 & 0 \end{pmatrix} \quad \text{and} \quad B_+(2k-1,\lambda) = \begin{pmatrix} -\lambda & -2 \\ 0 & 0 \end{pmatrix}$$

as well as

$$B_-(2k,\lambda) = \begin{pmatrix} \lambda & -2 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B_+(2k,\lambda) = \begin{pmatrix} -\lambda & 0 \\ 2 & 0 \end{pmatrix}.$$

If x is not an integer we have $B_\pm(x,\lambda) = J$. Note that $f \in \mathcal{L}^2(w)$ if and only if $k \mapsto f_1(k)$ is in $\ell^2(\mathbb{Z})$ and any element in $L^2(w)$ is uniquely determined by these values (here f_1 denotes the first component of f).

In any interval $(k, k+1)$ solutions of $Ju' + qu = w(\lambda u + f)$ are constant, say $(\alpha_k, \beta_k)^\top$. At $x = 2k-1$ the equation

$$B_+(2k-1,\lambda)u^+(2k-1) - B_-(2k-1,\lambda)u^-(2k-1) = (2f_1(2k-1), 0)^\top$$

implies $\alpha_{2k-2} = 0$ and

$$-\lambda\alpha_{2k-1} - 2\beta_{2k-1} = 2f_1(2k-1). \tag{7.1}$$

Similarly, at $x = 2k$ we get $\alpha_{2k} = 0$ and

$$-\lambda\alpha_{2k-1} + 2\beta_{2k-1} = 2f_1(2k). \tag{7.2}$$

We can now describe the space $\mathcal{T}_{\max}$. A pair (u, f) is in $\mathcal{T}_{\max}$ if and only if the sequences $k \mapsto f_1(k)$ and $k \mapsto u_1(k)$ are in $\ell^2(\mathbb{Z})$, $f_1(2k) = -f_1(2k-1)$, $u_1(2k) = u_1(2k-1)$, and

$$u = \sum_{k\in\mathbb{Z}} \Big(\begin{pmatrix} 2u_1(2k) \\ f_1(2k) \end{pmatrix} \chi^\#_{(2k-1,2k)} + \begin{pmatrix} 0 \\ \beta_{2k} \end{pmatrix} \chi^\#_{(2k,2k+1)} \Big)$$

with arbitrary numbers β_{2k}. Note that $\|u\|^2 = 4\sum_{k\in\mathbb{Z}} |u_1(2k)|^2$.

Choosing here $f = 0$ shows that 0 is an eigenvalue of $T_{\max}$ with infinite multiplicity. Choosing $f = 0$ and requiring $\|u\| = 0$ determines the space $\mathcal{L}_0$. Indeed,

$$\mathcal{L}_0 = \Big\{ \sum_{k\in\mathbb{Z}} \begin{pmatrix} 0 \\ \beta_{2k} \end{pmatrix} \chi^{\#}_{(2k,2k+1)} : \beta_{2k} \in \mathbb{C} \Big\}$$

which is infinite-dimensional. We now define the sequence τ setting $\tau_0 = 1/2$ and, for $k \in \mathbb{N}$, $\tau_k = k$ and $\tau_{-k} = 1 - k$. A solution u of $Ju' + qu = w(\lambda u + f)$ always satisfies condition $(2k + 1)$ and it satisfies condition $(2k)$ exactly when $\beta_{2k} = 0$.

For $f = 0$ equations (7.1) and (7.2) show that no non-zero λ can be an eigenvalue of $T_{\max}$. In particular, the deficiency indices $n_\pm$ are 0, i.e., $T_{\max}$ is self-adjoint. Now choose $\lambda \neq 0$ and f arbitrary in $L^2(w)$. Then

$$(\mathcal{R}_\lambda f)(x) = -\frac{1}{2\lambda} \sum_{k\in\mathbb{Z}} \begin{pmatrix} 2f_1(2k-1) + 2f_1(2k) \\ \lambda f_1(2k-1) - \lambda f_1(2k) \end{pmatrix} \chi^{\#}_{(2k-1,2k)}(x) \tag{7.3}$$

is the unique solution of $Ju' + qu = w(\lambda u + f)$ satisfying condition (k) for any $k \in \mathbb{Z}$. Since

$$\|\mathcal{R}_\lambda f\|^2 = \sum_{k\in\mathbb{Z}} 2|(\mathcal{R}_\lambda f)_1(k)|^2 = \frac{1}{|\lambda|^2} \sum_{k\in\mathbb{Z}} |f_1(2k-1) + f_1(2k)|^2 \tag{7.4}$$

is finite we have that $\mathbb{C} \setminus \{0\}$ is the resolvent set of $T_{\max}$.

We now define $\mathcal{H} = \{u \in L^2(w) : u_1(2k-1) = u_1(2k)\}$ and $\mathcal{H}_\infty = \{f \in L^2(w) : f_1(2k-1) = -f_1(2k)\}$. These spaces are orthogonal to each other and their direct sum is $L^2(w)$. Equation (7.4) shows that $\ker R_\lambda = \mathcal{H}_\infty$. Moreover, we have

$$T_{\max} = (\mathcal{H} \times \{0\}) \oplus (\{0\} \times \mathcal{H}_\infty).$$

This is an instance of a general feature for a self-adjoint linear relation T: if $\mathcal{H}$ is the closure of the domain of T, $\mathcal{H}_\infty$ the orthogonal complement of $\mathcal{H}$, and $T_0 = T \cap (\mathcal{H} \times \mathcal{H})$, then $T = T_0 \oplus (\{0\} \times \mathcal{H}_\infty)$. The former summand is then a linear operator densely defined in $\mathcal{H}$ called the operator part of T. The latter summand is called the multi-valued part of T.

We end this example by identifying Green's function for our example. It may be guessed by looking at equation (7.3). In any case one can check directly that $(\mathcal{R}_\lambda f)(x) = \int G(x,\cdot,\lambda)wf$. Note that the second column of G is irrelevant since the second row of w is 0. When x is not integer $G(x,y,\lambda)$ is given by

$$\sum_{k\in\mathbb{Z}} \left[-\frac{1}{\lambda}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \operatorname{sgn}(x-y) \right] \chi^{\#}_{(2k-1,2k)}(x)\chi^{\#}_{(2k-1,2k)}(y).$$

If x is an integer we have instead

$$G(2k-1,y,\lambda) = \frac{1}{2}\lim_{x\downarrow 2k-1} G(x,y,\lambda) \quad \text{and} \quad G(2k,y,\lambda) = \frac{1}{2}\lim_{x\uparrow 2k} G(x,y,\lambda).$$

REFERENCES

[1] Richard Arens. Operational calculus of linear relations. *Pacific J. Math.*, 11:9–23, 1961.

[2] F. V. Atkinson. *Discrete and continuous boundary problems.* Mathematics in Science and Engineering, Vol. 8. Academic Press, New York-London, 1964.

[3] Christer Bennewitz. Symmetric relations on a Hilbert space. Pages 212–218. Lecture Notes in Math., Vol. 280, 1972.

[4] Kevin Campbell, Minh Nguyen, and Rudi Weikard. On the spectral theory for first-order systems without the unique continuation property. *Linear Multilinear Algebra*, 69(12):2315–2323, 2021. Published online: 04 Oct 2019.

[5] Jonathan Eckhardt, Fritz Gesztesy, Roger Nichols, and Gerald Teschl. Weyl-Titchmarsh theory for Sturm-Liouville operators with distributional potentials. *Opuscula Math.*, 33(3):467–563, 2013.

[6] Jonathan Eckhardt and Gerald Teschl. Sturm-Liouville operators with measure-valued coefficients. *J. Anal. Math.*, 120:151–224, 2013.

[7] Ahmed Ghatasheh and Rudi Weikard. Spectral theory for systems of ordinary differential equations with distributional coefficients. *J. Differential Equations*, 268(6):2752–2801, 2020.

[8] M. G. Kreĭn. On a generalization of investigations of Stieltjes. *Doklady Akad. Nauk SSSR (N.S.)*, 87:881–884, 1952.

[9] Bruce Call Orcutt. *Canonical differential equations.* PhD thesis, University of Virginia, 1969.

[10] A. M. Savchuk and A. A. Shkalikov. Sturm-Liouville operators with singular potentials. *Mathematical Notes*, 66(6):741–753, 1999. Translated from Mat. Zametki, Vol. 66, pp. 897–912 (1999).

[11] Anton Zettl. *Sturm-Liouville theory*, volume 121 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2005.

CHAPTER 3.

ON FOURIER EXPANSIONS FOR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS WITH DISTRIBUTIONAL COEFFICIENTS

by

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ON FOURIER EXPANSIONS FOR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS WITH DISTRIBUTIONAL COEFFICIENTS

Abstract. We study the spectral theory for the first-order system $Ju'+qu = wf$ of differential equations on the real interval (a, b) where J is a constant, invertible, skew-hermitian matrix and q and w are matrices whose entries are distributions of order 0 with q hermitian and w non-negative. Specifically, we construct a generalized Weyl-Titchmarsh m-function with corresponding spectral measure τ and a generalized Fourier transform after imposing certain conditions on J, q, and w. Different conditions are motivated and studied in the later sections. A Fatou-type identity needed for our result is recorded in the appendix.

1. Introduction

In this paper we establish, under certain conditions, a Fourier expansion theorem for the differential equation

$$Ju' + (q - \lambda w)u = wf \tag{1.1}$$

posed on a real interval (a, b). Here J is a constant, invertible, skew-hermitian $n \times n$-matrix and λ is a complex parameter while q and w are $n \times n$-matrices whose entries are distributions of order 0, q is hermitian and w non-negative. If f is in $L^2(w)$, a Hilbert space to be defined below, solutions u of the equation are to be sought among functions of locally bounded variation. We emphasize that, as the coefficients in equation (1.1) become rougher, so do the solutions u. At some point it becomes impossible to define the products qu and wu. This point is reached when q or w are rougher than distributions of order 0.

When the entries of q and w are locally integrable functions one has an existence and uniqueness theorem (the unique continuation property) for solutions of the initial value problem $Ju' + (q - \lambda w)u = wf$, $u(x_0) = u_0$. This theorem is a central tool for the construction of Fourier expansions. However, if q or w involve δ-distributions, the existence or uniqueness of solutions of initial value problems can no longer be guaranteed. In [**10**] Ghatasheh and Weikard were able to overcome this obstacle in the case when the unique continuation property is only required for real λ. It is the goal of this paper to weaken this requirement and thus to show how to establish a Fourier expansion under milder conditions.

The first to consider a Sturm-Liouville equation with measure coefficients was (to the best of our knowledge) Krein [**12**] in 1952 when he modeled a vibrating string. Further contributions we are aware of were made by Kac [**11**], Feller [**8**], Mingarelli [**15**], Gesztesy and Holden [**9**], Kurasov [**13**], Kurasov and Boman[**14**], Savchuk and Shkalikov [**18**], Eckhardt et al. [**5**], and Eckhardt and Teschl [**6**]. We refer the reader to Ghatasheh and Weikard [**10**] and Eckhardt et al. [**5**] for more details on the history of the subject.

We end this introduction with a short overview of the content of this paper. Our main result is Theorem 5.1 in Section 5 establishing the existence of the Fourier transform associated with a self-adjoint linear relation T representing the differential equation (1.1) under appropriate boundary conditions. The space of Fourier transforms of elements in $L^2(w)$ is a Hilbert space $L^2(\tau)$ where τ is a matrix-valued non-negative measure on $\mathbb{R}$. This measure is determined through the Nevanlinna representation of a matrix-valued Weyl-Titchmarsh function $\mathbb{M}$ as we show in Section 4. The matrix $\mathbb{M}$, in turn, is obtained by constructing Green's function, the kernel of the resolvent operator $R_\lambda = (T - \lambda)^{-1}$. To this end we study the solutions of the equation $Ju' + (q - \lambda w)u = wf$ in Section 3 without the aid of the unique continuation property. Before we can do all this we gather in Section 2 some material previously established in [**10**], [**4**], and [**16**]. In the appendix we record a result privately communicated to us by B. and C. Bennewitz [**1**] concerning an extension of Fatou's theorem.

2. Preliminaries

In this section we gather some basic material on ordinary differential equations with distributional coefficients.

A distribution on the real interval (a, b) is a linear functional r on the set of infinitely often differentiable, compactly supported functions, the test functions, satisfying the following property: for any compact subinterval K of (a, b) there exist a positive constant C and a non-negative integer k such that

$$|r(\phi)| \leq C \sum_{j=0}^{k} \sup\{|\phi^{(j)}(x)| : x \in K\}$$

whenever ϕ is a test function with support in K. If k can be chosen to be 0 independently of K we say that r is a distribution of order 0.

Every distribution has a derivative and antiderivatives. The derivative of r is $\phi \mapsto r'(\phi) = -r(\phi')$. According to du Bois-Reymond's lemma the difference of two antiderivatives of r is a constant distributions, i.e., $\phi \mapsto C \int \phi \, dx$ for some $C \in \mathbb{C}$. A distribution r is called non-negative, if $r(\phi) \geq 0$ whenever ϕ is a test function assuming only non-negative values. The conjugate of a distribution r is defined by $\phi \mapsto \overline{r}(\phi) = \overline{r(\overline{\phi})}$.

Typical examples of distributions of order 0 are (i) $\phi \mapsto \int_{(a,b)} \phi f \, dx$ when f is a locally integrable function (with respect to Lebesgue measure) and (ii) $\phi \mapsto \phi(x_0)$, the δ-distribution at x_0, when x_0 is some fixed point in (a, b). More generally, if μ is a non-negative Lebesgue-Stieltjes measure on (a, b) and h a function in $L^1_{\text{loc}}(\mu)$, then $\phi \mapsto \int_{(a,b)} \phi h \mu$ is a distribution of order 0. Indeed, by Riesz's representation theorem, all distributions of order 0 are of this type. The cumulative distribution function of the (local) complex measure $h\mu$ is a function of locally bounded variation and the associated distribution is an antiderivative of $\phi \mapsto \int_{(a,b)} \phi h \mu$. Conversely, given a function of locally bounded variation on (a, b), it generates a Lebesgue-Stieltjes measure ν on each compact subset of (a, b). We may write $\nu = h|\nu|$ where $|\nu|$ is (locally) the total variation of ν and h a function of modulus 1 and thus in $L^1_{\text{loc}}(|\nu|)$. In the following we use the terms distribution of order 0 and measure interchangeably thus deviating from the traditional meaning of the term measure.

Given a measure (or distribution of order 0) r and a function $f \in L^1_{\text{loc}}(|r|)$ we define the distribution $rf = fr$ by setting $(rf)(\phi) = \int \phi f r$. Note that rf is also a distribution of order 0.

Next we turn to matrices with distributional entries. For any $n \times n$-matrix w we have its adjoint w^* defined in the canonical way as the conjugate of the transpose. If $w = w^*$ then w is called hermitian. If, for any vector $z \in \mathbb{C}^n$, the distribution $z^* w z$ is non-negative, we call w non-negative. A non-negative distribution is necessarily hermitian.

Associated with a non-negative distribution is a non-negative measure and one may define the space $\mathcal{L}^2(w)$ consisting of those functions f for which the measure $f^* w f$ is defined and finite. In $\mathcal{L}^2(w)$ one may define the semi-scalar product $\langle f, g \rangle = \int f^* w g$ and the semi-norm $\|f\| = \langle f, f \rangle^{1/2}$. Identifying functions in $\mathcal{L}^2(w)$ whose difference have semi-norm 0 we obtain the Hilbert space $L^2(w)$.

Throughout this paper we require the following canonical hypothesis to be valid. More hypotheses (which we suspect to be only of a technical nature) need to be added later.

Hypothesis 2.1. (a, b) is a real interval. J is a constant, invertible, skew-hermitian $n \times n$-matrix. q and w are $n \times n$-matrices whose entries are distributions of order 0 on (a, b); q is hermitian and w is non-negative.

Our goal is to investigate the differential equation

$$Ju' + (q - \lambda w)u = wf$$

where $\lambda \in \mathbb{C}$, $f \in L^2(w)$, and the solutions u are to be sought among the balanced (to be defined presently) functions of locally bounded variation on (a,b). It makes sense to pose the equation, since all terms occurring in it are distributions of order 0.

For each function u of locally bounded variation we denote by u^- and u^+ its left- and right-hand limits, respectively. More precisely, $u^-(x) = \lim_{t\uparrow x} u(t)$ and $u^+(x) = \lim_{t\downarrow x} u(t)$. The balanced version $u^\#$ of a function u of locally bounded variation is defined by $u^\# = (u^+ + u^-)/2$. The reason for considering only balanced solutions of the differential equation lies in the integration by parts formula which states

$$\int_{[c,d]} (udv + vdu) = (uv)^+(d) - (uv)^-(c) + (2\kappa - 1)\int_{[c,d]} (v^+ - v^-)du \tag{2.1}$$

whenever $[c,d] \subset (a,b)$, $u = \kappa u^+ + (1-\kappa)u^-$ and $v = \kappa v^+ + (1-\kappa)v^-$ for some fixed parameter κ. Thus, only when $\kappa = 1/2$, does the integration by parts formula have its familiar simple form when discontinuities of u and v are allowed.

If r is a matrix-valued measure with cumulative distribution function R, let $\Delta_r(x) = r(\{x\}) = R^+(x) - R^-(x)$. If one of $\Delta_q(x)$ and $\Delta_w(w)$ is different from zero, satisfying the differential equation $Ju' + (q - \lambda w)u = wf$ requires that

$$B_+(x,\lambda)u^+(x) - B_-(x,\lambda)u^-(x) = \Delta_w(x)f(x)$$

where

$$B_\pm(x,\lambda) = J \pm \frac{1}{2}(\Delta_q(x) - \lambda\Delta_w(x)).$$

If $B_+(x,\lambda)$ is not invertible it will not be possible to determine $u^+(x)$ uniquely even if $u^-(x)$ is given. An analogous statement holds, of course, when $B_-(x,\lambda)$ is not invertible. We emphasize that $B_-(x,\lambda) = -B_+(x,\overline{\lambda})^*$.

Still we have the following theorem on existence and uniqueness of solutions of initial value problems for $Ju' + (q - \lambda w)u = wf$.

Theorem 2.2 ([**10**], Theorem 2.2). Suppose Hypothesis 2.1 holds, that $f \in L^2(w)$ and $\lambda \in \mathbb{C}$, that the matrices $B_\pm(x,\lambda)$ are invertible for all $x \in (c,d) \subset (a,b)$, and that x_0 is a point in (c,d). Then the initial value problem $Ju' + (q - \lambda w)u = wf$, $u(x_0) = u_0 \in \mathbb{C}^n$ has a unique balanced solution u in the interval (c,d).

We may pose an initial condition at c (for u^+) or at d (for u^-) if q and w are finite measures near the point in question.

This theorem implies that we have fundamental matrices of solutions on (c,d). If U is one such fundamental matrix of solutions for which U^+ or U^- is equal to the identity matrix at some point $x_0 \in (c,d)$ then it satisfies the following Wronskian relationship

$$U^\pm(x,\overline{\lambda})^* J U^\pm(x,\lambda) = J \text{ and } U^\pm(x,\lambda) J^{-1} U^\pm(x,\overline{\lambda})^* = J^{-1} \tag{2.2}$$

which follows from Lemma 3.2 in [**10**].

Let us quickly recall from [**16**] some basic facts about the matrices $B_\pm(x,\lambda)$. Let Λ_x denote the set of those $\lambda \in \mathbb{C}$ where invertibility of either $B_+(x,\lambda)$ or $B_-(x,\lambda)$ fails. The set of those $x \in (a,b)$ where invertibility of $B_+(x,\lambda)$ or $B_-(x,\lambda)$ fails is denoted by Ξ_λ. Then Λ_x is empty for all but countably many $x \in (a,b)$. When it is not empty, it is either finite or all of $\mathbb{C}$. Moreover, it is symmetric with respect to the real axis. The intersections of $\Xi_\lambda = \Xi_{\overline{\lambda}}$ with any compact subset of (a,b) are finite.

Associated with the differential equation $Ju' + qu = wf$ are the linear relations

$$\mathcal{T}_{\max} = \{(u,f) \in \mathcal{L}^2(w) \times \mathcal{L}^2(w) : u \in \mathrm{BV}^\#_{\mathrm{loc}}((a,b))^n, Ju' + qu = wf\}$$

and

$$\mathcal{T}_{\min} = \{(u,f) \in \mathcal{T}_{\max} : \operatorname{supp} u \text{ is compact in } (a,b)\}.$$

Here $\mathrm{BV}^\#_{\mathrm{loc}}((a,b))^n$ denotes the space of $\mathbb{C}^n$-valued functions on (a,b) which are of locally bounded variation and balanced. The relations $\mathcal{T}_{\max}$ and $\mathcal{T}_{\min}$ give rise to linear relations in the Hilbert space $L^2(w) \times L^2(w)$

$$T_{\max} = \{([u],[f]) \in L^2(w) \times L^2(w) : (u,f) \in \mathcal{T}_{\max}\}$$

and

$$T_{\min} = \{([u],[f]) \in L^2(w) \times L^2(w) : (u,f) \in \mathcal{T}_{\min}\}.$$

$T_{\min}$ and $T_{\max}$ are called minimal and maximal relation, respectively.

In [**4**] it was shown that $T_{\min}^* = T_{\max}$ proving that $T_{\min}$ is a symmetric relation. Therefore we have von Neumann's relation

$$T_{\max} = \overline{T_{\min}} \oplus D_{\mathrm{i}} \oplus D_{-\mathrm{i}}$$

where $\overline{T}_{\min}$ is the closure of $T_{\min}$ and $D_\lambda = \{(u, \lambda u) \in T_{\max}\}$. The dimensions of D_λ are constant as λ varies in either the lower or upper half plane. These numbers are called deficiency indices of $T_{\min}$ and we use the notation $n_\pm = \dim D_{\pm\mathrm{i}}$. While, in general, the equation $Ju' + qu = \lambda wu$ may have infinitely many linearly independent solutions we showed in [**16**] that the deficiency indices are still no larger than n. We also established Green's formula or Lagrange's identity, i.e.,

$$(v^*Ju)^-(b) - (v^*Ju)^+(a) = \langle v, f\rangle - \langle g, u\rangle$$

provided that (v,g) and (u,f) are in $\mathcal{T}_{\max}$.

We showed in [**16**] that the closed symmetric extensions of $T_{\min}$ are given as restrictions of $T_{\max}$ by boundary conditions. In particular, this is true for the self-adjoint extensions in which case we have the following special case of Theorem 4.4 in [**16**].

Theorem 2.3. If the deficiency indices of $T_{\min}$ satisfy $n_+ = n_-$ and if (v_1, g_1), ..., $(v_{n_\pm}, g_{n_\pm})$ are linearly independent elements of $D_{\mathrm{i}} \oplus D_{-\mathrm{i}}$ such that

$$(g_k^* J g_\ell)^-(b) - (g_k^* J g_\ell)^+(a) = 0 \text{ for } 1 \le k, \ell \le n_\pm, \tag{2.3}$$

then

$$(2.4)\qquad T = \{(u,f) \in T_{\max} : (g_j^* Ju)^-(b) - (g_j^* Ju)^+(a) = 0 \text{ for } j = 1, ..., n_\pm\}$$

is a self-adjoint extension of $T_{\min}$.

Conversely, if T is a self-adjoint extension of $T_{\min}$, then T is given by (2.4) for appropriate elements (v_1, g_1), ..., $(v_{n_\pm}, g_{n_\pm})$ of $D_{\mathrm{i}} \oplus D_{-\mathrm{i}}$ for which (2.3) holds.

If T is a self-adjoint relation and λ is in $\varrho(T)$, the resolvent set of T, we showed in [**16**] that the resolvent operator $R_\lambda = (T - \lambda)^{-1}$ is an integral operator, i.e., we established the existence of a Green's function for T.

Some solutions of $Ju' + qu = \lambda wu$ may have norm zero, i.e., $\|u\|^2 = \int u^* wu = 0$. Since w is non-negative $\|u\| = 0$ if and only if the distribution $wu = 0$. It follows that such a function u satisfies $Ju' + qu = \lambda wu$ for any λ, in particular for $\lambda = 0$. The space of solutions of $Ju' + qu = 0$ satisfying $wu = 0$ is denoted by $\mathcal{L}_0$.

In particular, if $u \in \mathcal{L}_0$ (and hence $[u] = 0$ and $f = 0$), then

$$(2.5)\qquad (v^* Ju)^-(b) - (v^* Ju)^+(a) = \langle v, f\rangle - \langle g, u\rangle = 0$$

for all $(v, g) \in \mathcal{T}_{\max}$. Since $(v, g) \in D_{\mathrm{i}} \oplus D_{-\mathrm{i}}$ implies that $(g, -v) \in D_{\mathrm{i}} \oplus D_{-\mathrm{i}}$, it follows that any element of $\mathcal{L}_0$ satisfies the boundary conditions defining a self-adjoint relation T.

3. Constructing solutions

Throughout the remainder of the paper we require, in addition to Hypothesis 2.1, that $\Lambda_x \cap \mathbb{R} = \emptyset$ for all but finitely many $x \in (a, b)$. The points where $\Lambda_x \cap \mathbb{R} \neq \emptyset$ are (included among) the points x_1, ..., x_N which we consider ordered by size. We shall allow that $N = 0$ when $\Lambda_x \cap \mathbb{R}$ is empty for all $x \in (a, b)$. We also define $x_0 = a$ and $x_{N+1} = b$. Note that, while the set $\Lambda = \bigcup_{x\in(a,b)} \Lambda_x$ may be all of $\mathbb{C}$, the set

$$\tilde{\Lambda} = \bigcup_{x \notin \{x_1,\dots,x_N\}} \Lambda_x$$

does not intersect $\mathbb{R}$ and is countable.

According to Theorem 2.2 initial value problems for $Ju' + qu = \lambda wu$ have unique solutions on (x_j, x_{j+1}) for $j = 0, ..., N$. In each interval (x_j, x_{j+1}) we choose a balanced fundamental system of solutions $U_j(\cdot, \lambda)$ such that $U_j(\xi_j, \lambda) = \mathbb{1}$ for some point ξ_j which is a point of continuity for q and w, i.e., $q(\{\xi_j\}) = w(\{\xi_j\}) = 0$. These are extended to all of (a, b) as balanced functions which vanish outside $\overline{(x_j, x_{j+1})}$. The extensions are also denoted by $U_j(\cdot, \lambda)$. With these we define the $n \times n(N+1)$-matrix.

$$\mathcal{U}(x, \lambda) = (U_0(x, \lambda), ..., U_N(x, \lambda)).$$

We may now define the Fourier transform $\mathcal{F}f$ of an element $f \in L^2(w)$, at least as long as f is compactly supported, by

$$(\mathcal{F}f)(\lambda) = \int \mathcal{U}(\cdot, \overline{\lambda})^* wf,$$

a vector in $\mathbb{C}^{n(N+1)}$.

At this point let us introduce some more notation needed below. Identity matrices of various sizes are denoted by $\mathbb{1}$ while (square or rectangular) zero matrices are denoted by 0. Sometimes it may be advisable to indicate dimensions as subscripts, e.g., $\mathbb{1}_k$ is the $k \times k$ identity matrix while $0_{k,j}$ the $k \times j$ zero matrix.

In [**4**] and [**16**] the equation $Ju' + (q - \lambda w)u = wf$ was studied in the presence of points across which solutions cannot be uniquely continued. There, the description of solutions involved a two-diagonal block matrix $\mathbb{B}$ with blocks of size $n \times n$. Such a two-diagonal block matrix has the form $A\mathbb{E}_\top + B\mathbb{E}_\perp$ where A and B are $N \times N$ diagonal block matrices while $\mathbb{E}_\top$ and $\mathbb{E}_\perp$ are $N \times (N+1)$-block matrices, which, respectively, strip the first and last n components off a vector in their domain $\mathbb{C}^{n(N+1)}$. Thus $\mathbb{E}_\top = (0_{nN,n}, \mathbb{1}_{nN})$ and $\mathbb{E}_\perp = (\mathbb{1}_{nN}, 0_{nN,n})$. In our particular application we have

$$\mathbb{B}(\lambda) = \mathcal{B}(\lambda)\mathcal{U}^+(\lambda)\mathbb{E}_\top + \mathcal{B}(\overline{\lambda})^*\mathcal{U}^-(\lambda)\mathbb{E}_\perp$$

where

$$\mathcal{B}(\lambda) = \mathrm{diag}(B_+(x_1,\lambda), ..., B_+(x_N,\lambda)),$$

$$\mathcal{U}^-(\lambda) = \mathrm{diag}(U_0^-(x_1,\lambda), ..., U_{N-1}^-(x_N,\lambda)),$$

and

$$\mathcal{U}^+(\lambda) = \mathrm{diag}(U_1^+(x_1,\lambda), ..., U_N^+(x_N,\lambda)).$$

We will also need the matrix

$$\tilde{\mathbb{B}}(\lambda) = \mathcal{B}(\lambda)\mathcal{U}^+(\lambda)\mathbb{E}_\top - \mathcal{B}(\overline{\lambda})^*\mathcal{U}^-(\lambda)\mathbb{E}_\perp.$$

If $N = 0$, $\mathbb{E}_\top$ and $\mathbb{E}_\perp$ have to be considered as linear transformations from $\mathbb{C}^n$ to $\{0\}$. In this case $\mathbb{B}(\lambda)$ and $\tilde{\mathbb{B}}(\lambda)$ are "matrices" with no rows. This causes no problems below. We also let $\mathcal{J} = \mathrm{diag}(J, ..., J)$ stand for either an $N \times N$ or an $(N+1)\times(N+1)$ block matrix.

3.1. A representative of $R_\lambda f$. If $\lambda \in \varrho(T) \setminus \tilde{\Lambda}$ and $f \in L^2(w)$ is compactly supported, we shall now construct a balanced representative of $R_\lambda f$.

3.1.1. *General solution.* Using the variation of constants formula in the intervals (x_{j-1}, x_j) shows that any solution u of $Ju' + (q - \lambda w)u = wf$ satisfies

$$u^-(x) = \mathfrak{U}^-(x,\lambda)\big(\tilde{u} + \mathcal{J}^{-1}\int_{(a,x)} \mathfrak{U}(\cdot,\overline{\lambda})^* wf\big) \tag{3.1}$$

where $\tilde{u} = (\tilde{u}_0, ..., \tilde{u}_N)^\diamond$ is an appropriate vector in $\mathbb{C}^{n(N+1)}$.[1] For u to be indeed a solution it is necessary and sufficient that

$$-B_-(x_j,\lambda)u^-(x_j) + B_+(x_j,\lambda)u^+(x_j) = \Delta_w(x_j)f(x_j) \tag{3.2}$$

[1]We use $\diamond$ to indicate a block column vector as in $(c_1, ..., c_N)^\diamond = (c_1^\top, ..., c_N^\top)^\top$.

for $j = 1, ..., N$. Since, by equation (2.2), $U_j^-(x_j,\lambda)J^{-1}U_j^-(x_j,\overline{\lambda})^* = J^{-1}$ we find

$$\begin{aligned} u^-(x_j) &= U_{j-1}^-(x_j,\lambda)\big(\tilde{u}_{j-1} + J^{-1}\int_{(a,x_j)} U_{j-1}(\cdot,\overline{\lambda})^* wf\big) \\ &= U_{j-1}^-(x_j,\lambda)\big(\tilde{u}_{j-1} + J^{-1}\int U_{j-1}(\cdot,\overline{\lambda})^* wf\big) - \frac{1}{2}J^{-1}(\Delta_w f)(x_j) \end{aligned}$$

so that $(u^-(x_1), ..., u^-(x_N))^\diamond = \mathcal{U}^-(\lambda)\mathbb{E}_\perp\big(\tilde{u} + \mathcal{J}^{-1}(\mathcal{F}f)(\lambda)\big) - \frac{1}{2}\mathcal{J}^{-1}\mathcal{W}_0(f)$ where $\mathcal{W}_0(f) = ((\Delta_w f)(x_1), ..., (\Delta_w f)(x_N))^\diamond$.

Similarly, $(u^+(x_1), ..., u^+(x_N))^\diamond = \mathcal{U}^+(\lambda)\mathbb{E}_\top\tilde{u} + \frac{1}{2}\mathcal{J}^{-1}\mathcal{W}_0(f)$, since

$$\begin{aligned} u^+(x_j) &= U_j^+(x_j,\lambda)\big(\tilde{u}_j + J^{-1}\int_{(a,x_j]} U_j(\cdot,\overline{\lambda})^* wf\big) \\ &= U_j^+(x_j,\lambda)\tilde{u}_j + \frac{1}{2}J^{-1}(\Delta_w f)(x_j). \end{aligned}$$

Thus, taking into account that $\mathcal{B}(\lambda) - \mathcal{B}(\overline{\lambda})^* = 2\mathcal{J}$, equations (3.2) show that $\tilde{u}$ must solve the system

$$\mathbb{B}(\lambda)\tilde{u} = -\mathcal{B}(\overline{\lambda})^*\mathcal{U}^-(\lambda)\mathbb{E}_\perp\mathcal{J}^{-1}(\mathcal{F}f)(\lambda). \tag{3.3}$$

3.1.2. *Integrability conditions.* Next we require that u is an element of $\mathcal{L}^2(w)$. We define $N_+(\lambda) \subset \mathbb{C}^n$ to be the set of all η such that $u = U_N(\cdot,\lambda)\eta$ satisfies $\int_{(x_N,b)} u^* wu < \infty$. Similarly, $N_-(\lambda) \subset \mathbb{C}^n$ is the set of all η such that $u = U_0(\cdot,\lambda)\eta$ satisfies $\int_{(a,x_1)} u^* wu < \infty$. We also denote the orthogonal projections onto $N_\pm(\lambda)$ by $P_\pm(\lambda)$, respectively.

To pick out the solutions of the differential equation which are in $L^2(w)$ we proceed as follows. Recall that wf is compactly supported. In (a, x_1) we have

$$u^-(x) = U_0^-(x,\lambda)(\tilde{u}_0 + J^{-1}\int_{(a,x)} U_0(\cdot,\overline{\lambda})^* wf).$$

In particular, if x is below the support of wf, we have $u^-(x) = U_0^-(x,\lambda)\tilde{u}_0$. Hence we want $\tilde{u}_0 \in N_-(\lambda)$. Define $\mathcal{Q}_-(\lambda) = (\mathbb{1} - P_-(\lambda), 0, ..., 0) \in \mathbb{C}^{n\times n(N+1)}$ so that the condition becomes

$$\mathcal{Q}_-(\lambda)\tilde{u} = 0. \tag{3.4}$$

For $x \in (x_N, b)$ we get instead

$$u^-(x) = U_N^-(x,\lambda)(\tilde{u}_N + J^{-1}\int_{(x_N,x)} U_N(\cdot,\overline{\lambda})^* wf)$$

In particular, $u^-(x) = U_N^-(x,\lambda)(\tilde{u}_N + J^{-1}(\mathcal{F}f)_N(\lambda))$ if x is above the support of wf. Hence $(\mathbb{1} - P_+(\lambda))\tilde{u}_N = -(\mathbb{1} - P_+(\lambda))J^{-1}(\mathcal{F}f)_N(\lambda)$. Setting $\mathcal{Q}_+(\lambda) = (0, ..., 0, \mathbb{1} - P_+(\lambda)) \in \mathbb{C}^{n\times n(N+1)}$ the condition becomes

$$\mathcal{Q}_+(\lambda)\tilde{u} = -\mathcal{Q}_+(\lambda)\mathcal{J}^{-1}(\mathcal{F}f)(\lambda). \tag{3.5}$$

If a (or b) is regular, then $P_-(\lambda)$ (or $P_+(\lambda)$) is the identity and the corresponding condition is vacuous.

3.1.3. *Boundary conditions.* We now invoke Theorem 2.3 to deal with the boundary conditions representatives of R_λ have to satisfy. These boundary conditions are given by

$$(g_j^* Ju)^-(b) - (g_j^* Ju)^+(a) = 0, \quad \text{for } j = 1, ..., n_\pm \tag{3.6}$$

where (v_1, g_1), ..., $(v_{n_\pm}, g_{n_\pm})$ are linearly independent elements of $D_{\mathrm{i}} \oplus D_{-\mathrm{i}}$ satisfying equation (2.3). Let $g = (g_1, ..., g_{n_+})$ and introduce the matrices[2]

$$A_-(\lambda) = -(g^* JU_0(\cdot,\lambda)P_-(\lambda))^+(a),$$

$$A_+(\lambda) = (g^* JU_N(\cdot,\lambda)P_+(\lambda))^-(b),$$

[2]The definition of A_- is differs from the one in [**10**] by a factor of -1.

and

$$\mathcal{A}_-(\lambda) = (A_-(\lambda), 0, ..., 0), \quad \mathcal{A}_+(\lambda) = (0, ..., 0, A_+(\lambda))$$

with N blocks of zero-matrices. Then the boundary conditions are

$$(\mathcal{A}_+(\lambda) + \mathcal{A}_-(\lambda))\tilde{u} = -\mathcal{A}_+(\lambda)\mathcal{J}^{-1}(\mathcal{F}f)(\lambda). \tag{3.7}$$

3.1.4. *Solutions of zero norm.* Finally, we take account of the presence of solutions with norm 0. We define $N_0 = \{\eta \in \mathbb{C}^{n(N+1)} : \mathcal{U}(\cdot, 0)\eta \in \mathcal{L}_0\}$. Adding any element of $\mathcal{L}_0$ to a solution u of $Ju' + (q - \lambda w)u = wf$ which is in $\mathcal{L}^2(w)$ and satisfies the boundary conditions will yield a solution with the same properties. Each of these is given by (3.1) for an appropriate choice of $\tilde{u}$ and exactly one of them will have $\tilde{u} \in N_0^\perp$. Thus, if $\mathbb{P}$ is the orthogonal projection onto $N_0^\perp$, we require

$$(\mathbb{1} - \mathbb{P})\tilde{u} = 0. \tag{3.8}$$

Note that $\mathbb{P}$ does not depend on λ.

We also emphasize that $w\mathcal{U}(\cdot, \lambda)(\mathbb{1} - \mathbb{P})\eta$ is the zero distribution for any $\eta \in \mathbb{C}^{n(N+1)}$ and any $\lambda \in \mathbb{C} \setminus \tilde{\Lambda}$. Thus $w\mathcal{U}(\cdot, \lambda) = w\mathcal{U}(\cdot, \lambda)\mathbb{P}$ or, equivalently, $\mathcal{U}(\cdot, \overline{\lambda})^* w = \mathbb{P}\mathcal{U}(\cdot, \overline{\lambda})^* w$ and, in particular,

$$(\mathcal{F}f)(\lambda) = \mathbb{P}(\mathcal{F}f)(\lambda) \tag{3.9}$$

which we record here for later use.

We shall also need the following.

LEMMA 3.1. $\mathbb{B}(\lambda)(\mathbb{1} - \mathbb{P}) = 0$, $\mathcal{Q}_\pm(\lambda)(\mathbb{1} - \mathbb{P}) = 0$, and $(\mathcal{A}_+(\lambda) + \mathcal{A}_-(\lambda))(\mathbb{1} - \mathbb{P}) = 0$.

PROOF. Suppose v is an arbitrary element of $\mathbb{C}^{n(N+1)}$. Then $(\mathbb{1} - \mathbb{P})v$ is in N_0 and $\mathcal{U}(\cdot, \lambda)(\mathbb{1} - \mathbb{P})v$ is an element of $\mathcal{L}_0$. As such it solves the differential equation $Ju' + qu = \lambda wu$ which implies that $\mathbb{B}(\lambda)(\mathbb{1} - \mathbb{P})v = 0$ according to equation (3.3) for $f = 0$. It is also (trivially) in $L^2(w)$ and satisfies the boundary conditions, as we argued in equation (2.5), implying the other two claims. □

3.1.5. *Putting it all together.* We now collect the equations (3.3), (3.4), (3.5), (3.7), and (3.8) into the system

$$\mathbb{F}(\lambda)\tilde{u} = \mathbb{H}_\ell(\lambda)\mathcal{J}^{-1}(\mathcal{F}f)(\lambda) \tag{3.10}$$

where

$$\mathbb{F}(\lambda) = \begin{pmatrix} \mathbb{B}(\lambda) \\ \mathcal{Q}_-(\lambda) \\ \mathcal{Q}_+(\lambda) \\ \mathcal{A}_+(\lambda) + \mathcal{A}_-(\lambda) \\ \mathbb{1} - \mathbb{P} \end{pmatrix} \quad \text{and} \quad \mathbb{H}_\ell(\lambda) = -\begin{pmatrix} \mathcal{B}(\overline{\lambda})^*\mathcal{U}^-(\lambda)\mathbb{E}_\perp \\ 0 \\ \mathcal{Q}_+(\lambda) \\ \mathcal{A}_+(\lambda) \\ 0 \end{pmatrix}.$$

We will show presently that $\mathbb{F}$ has trivial kernel and hence full column rank. Therefore it has a left inverse $\mathbb{F}^\dagger$ and since we know that a solution exists we may solve (3.10) for $\tilde{u}$. Using $\tilde{u} = \mathbb{P}\tilde{u}$ and equation (3.9) this gives

$$\tilde{u} = \mathbb{P}\mathbb{F}(\lambda)^\dagger\mathbb{H}_\ell(\lambda)\mathcal{J}^{-1}\mathbb{P}(\mathcal{F}f)(\lambda) = \mathbb{M}_\ell(\lambda)(\mathcal{F}f)(\lambda) \tag{3.11}$$

thereby defining the matrix $\mathbb{M}_\ell$. With this value of $\tilde{u}$ we repeat equation (3.1) to get

$$u^-(x) = \mathcal{U}^-(x,\lambda)\int(\mathbb{M}_\ell(\lambda) + \mathcal{J}^{-1}\chi_{(a,x)})\mathcal{U}(\cdot,\overline{\lambda})^*wf. \tag{3.12}$$

Lemma 3.2. If $\lambda \in \mathbb{C}\setminus\tilde{\Lambda}$ and $\operatorname{Im}(\lambda) \neq 0$, then $\ker\mathbb{F}(\lambda) = \{0\}$.

Proof. Assume $v \in \ker\mathbb{F}(\lambda)$. Since then $v \in \ker\mathbb{B}(\lambda)$ it follows that $u = \mathcal{U}(\cdot,\lambda)v$ is a solution of $Ju' + qu = \lambda wu$. Notice further that u must be in $\mathcal{L}^2(w)$ since $\mathcal{Q}_\pm(\lambda)v = 0$. Finally, $(\mathcal{A}_+(\lambda) + \mathcal{A}_-(\lambda))v = 0$ gives that u satisfies the boundary conditions. It is thus the case that $([u],\lambda[u]) \in T$. Since λ is not real we must have $[u] = 0$ so that $u \in \mathcal{L}_0$. We thus see that $v \in N_0$. However, we also have $(\mathbb{1} - \mathbb{P})v = 0$ and hence $v = 0$. □

We now obtain u^+ by taking limits from the right in equation (3.12) to get

$$u^+(x) = \mathcal{U}^+(x,\lambda) \int (\mathbb{M}_\ell(\lambda) + \mathcal{J}^{-1}\chi_{(a,x]})\mathcal{U}(\cdot,\overline{\lambda})^* wf.$$

Hence

$$u(x) = \mathcal{U}(x,\lambda) \int (\mathbb{M}_\ell(\lambda) + \mathcal{J}^{-1}\chi_{(a,x)})\mathcal{U}(\cdot,\overline{\lambda})^* wf + \frac{1}{2}\mathcal{U}^+(x,\lambda)\mathcal{J}^{-1}\mathcal{U}(x,\overline{\lambda})^*\Delta_w(x)f(x) \tag{3.13}$$

is a representative of $R_\lambda f$.

3.2. Striving for symmetry. We now construct another representative v of $R_\lambda f$ by starting from integrals over (x,b) instead of (a,x). Specifically,

$$v^+(x) = \mathcal{U}^+(x,\lambda)\big(\tilde{v} - \mathcal{J}^{-1}\int_{(x,b)} \mathcal{U}(\cdot,\overline{\lambda})^* wf\big)$$

with an appropriate vector $\tilde{v}$ in $\mathbb{C}^{n(N+1)}$.

Equation (3.10) becomes

$$\mathbb{F}(\lambda)\tilde{v} = \mathbb{H}_r(\lambda)\mathcal{J}^{-1}(\mathcal{F}f)(\lambda) \tag{3.14}$$

where $\mathbb{F}$ is as before but

$$\mathbb{H}_r(\lambda) = \begin{pmatrix} \mathcal{B}(\lambda)\mathcal{U}^+(\lambda)\mathbb{E}_\top \\ \mathcal{Q}_-(\lambda) \\ 0 \\ \mathcal{A}_-(\lambda) \\ 0 \end{pmatrix}.$$

Analogously to (3.11) we get now

$$\tilde{v} = \mathbb{P}\mathbb{F}(\lambda)^\dagger\mathbb{H}_r(\lambda)\mathcal{J}^{-1}\mathbb{P}(\mathcal{F}f)(\lambda) = \mathbb{M}_r(\lambda)(\mathcal{F}f)(\lambda) \tag{3.15}$$

defining $\mathbb{M}_r(\lambda)$. Taking limits of v^+ from the left we obtain v^- and therefore, in analogy to (3.13),

$$v(x) = \mathcal{U}(x,\lambda)\int(\mathbb{M}_r(\lambda) - \mathcal{J}^{-1}\chi_{(x,b)})\mathcal{U}(\cdot,\overline{\lambda})^* wf - \frac{1}{2}\mathcal{U}^-(x,\lambda)\mathcal{J}^{-1}\mathcal{U}(x,\overline{\lambda})^*\Delta_w(x)f(x). \tag{3.16}$$

We emphasize that, since u and v are representatives of $R_\lambda f$, their difference must be an element of $\mathcal{L}_0$.

3.3. Constructing $\mathbb{M}$. We now define $\mathbb{M}(\lambda) = \frac{1}{2}(\mathbb{M}_\ell(\lambda) + \mathbb{M}_r(\lambda))$, i.e.,

$$\mathbb{M}(\lambda) = \mathbb{P}\mathbb{F}(\lambda)^\dagger\mathbb{H}(\lambda)\mathcal{J}^{-1}\mathbb{P}$$

where

$$\mathbb{F}(\lambda) = \begin{pmatrix} \mathbb{B}(\lambda) \\ \mathcal{Q}_-(\lambda) \\ \mathcal{Q}_+(\lambda) \\ \mathcal{A}_+(\lambda) + \mathcal{A}_-(\lambda) \\ \mathbb{1} - \mathbb{P} \end{pmatrix}, \quad \mathbb{H}(\lambda) = \frac{\mathbb{H}_\ell(\lambda) + \mathbb{H}_r(\lambda)}{2} = \frac{1}{2}\begin{pmatrix} \tilde{\mathbb{B}}(\lambda) \\ \mathcal{Q}_-(\lambda) \\ -\mathcal{Q}_+(\lambda) \\ \mathcal{A}_-(\lambda) - \mathcal{A}_+(\lambda) \\ 0 \end{pmatrix}.$$

Then we find, combining (3.13) and (3.16), that

$$(\mathcal{R}_\lambda f)(x) = \frac{u(x)+v(x)}{2} = \mathcal{U}(x,\lambda)\int(\mathbb{M}(\lambda) + \frac{1}{2}\mathcal{J}^{-1}\operatorname{sgn}(x-\cdot))\mathcal{U}(\cdot,\overline{\lambda})^* wf + \frac{1}{4}(\mathcal{U}^+(x,\lambda) - \mathcal{U}^-(x,\lambda))\mathcal{J}^{-1}\mathcal{U}(x,\overline{\lambda})^*\Delta_w(x)f(x). \tag{3.17}$$

This is our final representative of $R_\lambda f$. We emphasize that the last term in (3.17) is zero for all but countably many $x \in (a,b)$.

4. Properties of $\mathbb{M}$

The function

$$\mathbb{M}(\lambda) = \mathbb{P}\mathbb{F}(\lambda)^\dagger\mathbb{H}(\lambda)\mathcal{J}^{-1}\mathbb{P},$$

occurring in equation (3.17), is a matrix-valued Nevanlinna function in a wide variety of circumstances (including all regular cases and when $N = 0$) as we will show in this section. In fact we believe this is always the case but can, at this point, not prove it.

Recall that a Nevanlinna function $\mathbb{M}$ is a function defined on $\mathbb{C} \setminus \mathbb{R}$ with the following properties: (i) $\mathbb{M}$ is symmetric in the sense that $\mathbb{M}(\lambda) = \mathbb{M}(\overline{\lambda})^*$, (ii) $\operatorname{Im} \mathbb{M}$ is non-negative in the upper half-plane, and (iii) $\mathbb{M}$ is analytic. As a Nevanlinna function $\mathbb{M}$ has the representation

$$\mathbb{M}(\lambda) = A + B\lambda + \int_{\mathbb{R}} \Big(\frac{1}{t-\lambda} - \frac{t}{t^2+1}\Big)\tau(t) \tag{4.1}$$

where A is hermitian, B is non-negative, and τ is a non-negative matrix-valued measure whose significance will become clear in the next section. The measure τ is called a spectral measure.

4.1. Symmetry of $\mathbb{M}$.

For $\lambda \in \mathbb{C} \setminus \tilde{\Lambda}$ define the set

$$\mathbf{B}(\lambda) = \{(\mathcal{F}f)(\lambda) : f \in L^2(w), \operatorname{supp} f \text{ compact}\}.$$

Lemma 4.1. If $\lambda, \mu \in \mathbb{C} \setminus \tilde{\Lambda}$ then $\mathbf{B}(\lambda) = \mathbf{B}(\mu)$.

Proof. Let $u = \mathcal{U}(\cdot, \overline{\lambda})\alpha$ and $v = \mathcal{U}(\cdot, \overline{\mu})\alpha$. Since $\alpha \in \mathbf{B}(\lambda)^\perp$ if and only if $wu = 0$ it follows for such α that u satisfies the differential equation $Ju' + qu = \overline{\lambda} wu = 0 = \overline{\mu} wu$ on (x_j, x_{j+1}). Since $u(\xi_j) = v(\xi_j) = \alpha_j$ we have actually that u and v coincide on (x_j, x_{j+1}) and hence everywhere. Therefore $wv = 0$ and $\alpha \in \mathbf{B}(\mu)^\perp$. Switching the roles played by λ and μ shows the other inclusion. □

In view of Lemma 4.1 we will write subsequently simply $\mathbf{B}$ rather than $\mathbf{B}(\lambda)$.

Because of equation (3.9) it is clear that $\mathbf{B}$ is a subspace of $N_0^\perp = \operatorname{ran} \mathbb{P}$. This inclusion may be strict[3] but when it is not we have the following lemma.

[3]Let, for instance, $(a, b) = \mathbb{R}$, $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $q = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 2 \end{smallmatrix}\right)\delta_0$, and $w = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 0 \end{smallmatrix}\right)\delta_0$. Then $\dim \mathbf{B} = 1$ while $\dim \operatorname{ran} \mathbb{P} = 3$.

LEMMA 4.2. Suppose $\lambda \in \varrho(T) \setminus \tilde{\Lambda}$ and $\mathbf{B} = \operatorname{ran} \mathbb{P}$. Then $\mathbb{M}(\overline{\lambda})^* = \mathbb{M}(\lambda)$. This is true, in particular, when $N = 0$.

PROOF. Since $R_{\overline{\lambda}} = R_\lambda^*$ we have

$$0 = \langle g, R_\lambda f \rangle - \langle R_{\overline{\lambda}} g, f \rangle = (\mathcal{F}g)(\overline{\lambda})^* (\mathbb{M}(\lambda) - \mathbb{M}(\overline{\lambda})^*)(\mathcal{F}f)(\lambda).$$

Here we used (2.2) to show that the integrated terms stemming from (3.17) cancel each other. The claim follows then because $(\operatorname{ran} \mathbb{P})^\perp = N_0$ is in the kernel of both $\mathbb{M}(\lambda)$ and $\mathbb{M}(\overline{\lambda})^*$.

Next we show that for $N = 0$ we have indeed $\mathbf{B} = \operatorname{ran} \mathbb{P}$. Let $\alpha \in \mathbf{B}^\perp$ so that $\int f^* w \mathcal{U}(\cdot, \overline{\lambda})\alpha = 0$. But here $\mathcal{U}(\cdot, \overline{\lambda}) = U_0(\cdot, \overline{\lambda})$ implying that $\mathcal{U}(\cdot, \overline{\lambda})\alpha$ is a solution of $Ju' + qu = \overline{\lambda} w u$ which is perpendicular to a dense set in $L^2(w)$. This shows that $\alpha \in N_0 = (\operatorname{ran} \mathbb{P})^\perp$. □

If $N \geq 1$ we still have that $u = \mathcal{U}(\cdot, \overline{\lambda})\alpha$ has norm 0 when $\alpha \in \mathbf{B}^\perp$ but it may not be a solution of the differential equation anymore. To cover the exceptional cases where $\mathbf{B} \neq \operatorname{ran} \mathbb{P}$ we introduce the matrix

$$\Omega(\lambda) = \mathbb{H}(\lambda)\mathcal{J}^{-1}\mathbb{P}\mathbb{F}(\overline{\lambda})^* + \mathbb{F}(\lambda)\mathbb{P}\mathcal{J}^{-1}\mathbb{H}(\overline{\lambda})^*.$$

Applying $\mathbb{P}\mathbb{F}(\lambda)^\dagger$ to $\Omega(\lambda)$ from the left and $\mathbb{F}(\overline{\lambda})^{\dagger *}\mathbb{P}$ from the right shows that $\Omega(\lambda) = 0$ implies $\mathbb{M}(\lambda) = \mathbb{M}(\overline{\lambda})^*$, a result we state as a lemma.

LEMMA 4.3. If $\lambda \in \varrho(T) \setminus \tilde{\Lambda}$ and $\Omega(\lambda) = 0$, then $\mathbb{M}(\lambda) = \mathbb{M}(\overline{\lambda})^*$.

In general Ω is a 5×5-block-matrix and we mean rows and columns of blocks when we speak simply of rows and columns in the following. Note that for $N = 0$ the first row and column are actually absent, while for $n_\pm = 0$ boundary conditions are not needed so that the fourth row and column are absent. For simplicity of notation we will, nevertheless, maintain the labeling of the remaining blocks as if those others were present.

Since the last rows of blocks in both $\mathbb{H}$ and $\mathbb{FP}$ are zero it follows that the last row of Ω is also zero. Since $\Omega(\overline{\lambda})^* = -\Omega(\lambda)$ the same is true for the last column of Ω. To investigate the remaining 4×4-block-matrix we introduce

$$X(\lambda) = \begin{pmatrix} \mathcal{B}(\lambda)\mathcal{U}^+(\lambda)\mathbb{E}_\top \\ \mathcal{Q}_-(\lambda) \\ 0 \\ \mathcal{A}_-(\lambda) \end{pmatrix} \text{ and } Y(\lambda) = \begin{pmatrix} \mathcal{B}(\overline{\lambda})^*\mathcal{U}^-(\lambda)\mathbb{E}_\perp \\ 0 \\ \mathcal{Q}_+(\lambda) \\ \mathcal{A}_+(\lambda) \end{pmatrix}.$$

From Lemma 3.1 we obtain that $(X+Y)\mathbb{P} = X + Y$. Thus, taking cancellations into account, we find that the block $\Omega_{\ell,k}$ is given by

$$\Omega_{\ell,k}(\lambda) = X_\ell(\lambda)\mathcal{J}^{-1}X_k(\overline{\lambda})^* - Y_\ell(\lambda)\mathcal{J}^{-1}Y_k(\overline{\lambda})^*$$

for $1 \le \ell, k \le 4$.

Lemma 4.4. If $\lambda \in \varrho(T) \setminus \tilde{\Lambda}$ the first row and column of $\Omega(\lambda)$ (if present) are zero.

Proof. $\Omega_{1,1}(\lambda)$ involves the term

$$\mathcal{U}^+(\lambda)\mathbb{E}_\top\mathcal{J}^{-1}\mathbb{E}_\top^*\mathcal{U}^+(\overline{\lambda})^*$$

where $\mathcal{J}^{-1}$ is an $n(N+1) \times n(N+1)$ matrix. In view of equation (2.2) and the structure of $\mathbb{E}_\top$ this product is equal to the $nN \times nN$ matrix $\mathcal{J}^{-1}$. Using the analogous argument for the term involving $\mathbb{E}_\perp$ we obtain

$$\Omega_{1,1}(\lambda) = \mathcal{B}(\lambda)\mathcal{J}^{-1}\mathcal{B}(\overline{\lambda})^* - \mathcal{B}(\overline{\lambda})^*\mathcal{J}^{-1}\mathcal{B}(\lambda),$$

a block-diagonal matrix. Since $B_+(\cdot, \overline{\lambda})^* = -B_-(\cdot, \lambda)$ and $B_\pm = J \pm A$ for a suitable matrix A we get $B_-(x_k, \lambda)J^{-1}B_+(x_k, \lambda) - B_+(x_k, \lambda)J^{-1}B_-(x_k, \lambda) = 0$ and hence $\Omega_{1,1}(\lambda) = 0$.

For $\ell = 2$ and $k = 1$ we find

$$\Omega_{2,1}(\lambda) = \mathcal{Q}_-(\lambda)\mathcal{J}^{-1}\mathbb{E}_\top^*\mathcal{U}^+(\overline{\lambda})^*\mathcal{B}(\overline{\lambda})^*.$$

This is 0 since only the first block in the row vector $\mathcal{Q}_-(\lambda)\mathcal{J}^{-1}$ is non-zero and therefore annihilated by $\mathbb{E}_\top^*$. A similar argument works to show that $\Omega_{3,1}(\lambda) = 0$ and $\Omega_{4,1}(\lambda) = 0$.

Finally, since $\Omega(\overline{\lambda})^* = -\Omega(\lambda)$ the first row of Ω is also zero. □

At present we cannot show that the remaining entries of Ω will also vanish under all circumstances even though we strongly suspect that this is the case. However, we can show that it is true when $P_\pm(\lambda) = \mathbb{1}$ for all $\lambda \in \varrho(T) \setminus \tilde{\Lambda}$ (this includes all regular problems), when $n = 1$, and when $n = 2$ with J, q and w real.

4.1.1. *The case when* $P_\pm(\lambda) = \mathbb{1}$.

Lemma 4.5. If $\lambda \in \varrho(T) \setminus \tilde{\Lambda}$ and $P_\pm(\lambda) = \mathbb{1}$, then $\Omega(\lambda) = 0$ and hence $\mathbb{M}(\lambda) = \mathbb{M}(\overline{\lambda})^*$.

Proof. Only $\Omega_{4,4}$ needs to be considered in this case. Recalling (2.2) we obtain

$$\Omega_{4,4}(\lambda) = (g^*Jg)^-(b) - (g^*Jg)^+(a)$$

which vanishes by Theorem 2.3 since we have a self-adjoint restriction of $T_{\max}$. □

4.1.2. *The case when* $n = 1$.

Lemma 4.6. If $\lambda \in \varrho(T)\setminus\tilde{\Lambda}$ and $n = 1$, then $\Omega(\lambda) = 0$ and hence $\mathbb{M}(\lambda) = \mathbb{M}(\overline{\lambda})^*$.

Proof. First assume, by way of contradiction, that $\Omega_{2,2}(\lambda) \neq 0$, i.e., $P_-(\lambda) = P_-(\overline{\lambda}) = 0$. Consider the problem where $\tilde{q} = q\chi_{(a,x_1)}$ and $\tilde{w} = w\chi_{(a,x_1)}$. We now have that b is a regular endpoint but the corresponding deficiency indices $\tilde{n}_\pm$ are still zero. In other words we have a self-adjoint situation with $\tilde{N} = 0$, $\tilde{Q}_- = 1$, $\tilde{Q}_+ = 0$, and $\tilde{\mathbb{P}} = 1$. In this situation we find $\tilde{\mathbb{M}}(\lambda) = J^{-1}/2$ in either half plane. But J^{-1} is a non-zero purely imaginary number and, according to Lemma 4.2, $\tilde{\mathbb{M}}(\lambda) = \tilde{\mathbb{M}}(\overline{\lambda})^*$, a contradiction. Thus $\Omega_{2,2}(\lambda) = 0$ and one shows similarly that $\Omega_{3,3}(\lambda) = 0$. We also have, trivially, that $\Omega_{3,2}(\lambda)$ and $\Omega_{2,3}(\lambda)$ are 0.

We now consider the fourth row of Ω. The entry $\Omega_{4,2}(\lambda)$ contains the factor $P_-(\lambda)J^{-1}(1 - P_-(\lambda))$ which is 0 since J^{-1} is a scalar. It follows similarly $\Omega_{4,3}(\lambda)$ vanishes. Finally, since $g(\cdot, \lambda)$ vanishes near a or b when $P_-(\lambda) = 0$ or $P_+(\lambda) = 0$, respectively, we may remove the factors $P_\pm$ in the expression

$$\Omega_{4,4}(\lambda) = -\big(g^* J U_0(\cdot, \lambda) P_-(\lambda) J^{-1} P_-(\overline{\lambda}) U_0(\cdot, \overline{\lambda})^* J g^*\big)^+(a) \\ + \big(g^* J U_N(\cdot, \lambda) P_+(\lambda) J^{-1} P_+(\overline{\lambda}) U_N(\cdot, \overline{\lambda})^* J g^*\big)^-(b).$$

Thus, using (2.2) and (2.3), $\Omega_{4,4}(\lambda) = (g^* J g)^-(b) - (g^* J g)^+(a) = 0$. □

4.1.3. *The case when $n = 2$ and the coefficients are real.* The condition that J is real, skew-adjoint, and invertible implies that $J = \beta\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ for some $\beta \in \mathbb{R} \setminus \{0\}$. Without loss of generality we shall henceforth assume that $\beta = 1$.[4] Next note that u solves $Ju' + qu = \lambda w u$ if and only if $\overline{u}$ solves $J\overline{u}' + q\overline{u} = \overline{\lambda} w \overline{u}$. This implies $n_+ = n_-$ and $P_\pm(\overline{\lambda}) = \overline{P_\pm(\lambda)}$.

Lemma 4.7. If $\lambda \in \varrho(T) \setminus \tilde{\Lambda}$, $n = 2$, and J, q, and w are real, then $\Omega(\lambda) = 0$ and hence $\mathbb{M}(\lambda) = \mathbb{M}(\overline{\lambda})^*$.

Proof. As in the case $n = 1$ we will make use of the auxiliary problem where $\tilde{q} = q\chi_{(a,x_1)}$ and $\tilde{w} = w\chi_{(a,x_1)}$. Again we have that b is a regular endpoint and that $\tilde{N} = 0$. We will also denote other quantities associated with the auxiliary problem by adding the ˜ symbol.

First we show that it is impossible to have $P_\pm(\lambda) = 0$. Assuming, by way of contradiction, that $P_-(\lambda) = 0$ and hence $P_-(\overline{\lambda}) = 0$, we have $n_\pm = 0$ and obtain $\tilde{\mathbb{F}}(\lambda) = (\mathbb{1}, 0, 0)^\top$ and $\tilde{\mathbb{H}}(\lambda) = \tilde{\mathbb{F}}(\lambda)/2$ for the auxiliary problem. Consequently

[4] If $\pm\beta > 0$ we could employ the Liouville transform $v(x) = \pm\beta u(\pm x)$ to arrive at an equation of the same character but with $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

$\tilde{\mathbb{M}}(\lambda) = J^{-1}/2$ in either half-plane, which is, as before, absurd. A similar argument shows that $P_+(\lambda)$ cannot be 0.

When $P_-(\lambda) = \mathbb{1}$ then $\mathcal{Q}_-(\lambda) = 0$ which implies that the second column and the second row of $\Omega(\lambda)$ vanish.

Next assume that the rank of $P_-(\lambda)$ is 1. Any orthogonal projection P of rank 1 in $\mathbb{C}^2$ satisfies $PJ^{-1}\overline{P} = 0$ as a direct computation shows. Applying this to $P = \mathbb{1} - P_-(\lambda)$ shows that $\Omega_{2,2}(\lambda) = 0$. That $\Omega_{3,2}(\lambda) = 0$ is again trivial. Now consider $\mathcal{A}_-(\lambda) = -(g^* J U_0(\cdot,\lambda) P_-(\lambda))^+(a)$ and let η be an arbitrary element of $\mathbb{C}^2$. If $u = U_0(\cdot,\lambda)P_-(\lambda)\eta$ and $\int_{(a,x_1)} u^* w u > 0$, then our auxiliary problem is definite and we obtain $(g^* J u)^+(a) = 0$ and hence $\mathcal{A}_-(\lambda) = 0$ from Lemma 7.6. in [**10**]. If $\int_{(a,x_1)} u^* w u = 0$, i.e., if $u \in \tilde{\mathcal{L}}_0$, then u satisfies $Ju' + \tilde{q}u = \mu \tilde{w} u = 0$ for any $\mu \in \mathbb{C}$. In this case there is an $\alpha \in \mathbb{R}^2$ such that $\tilde{\mathcal{L}}_0$ is spanned by $v = U_0(\cdot, 0)\alpha$. Both u and g are then multiples of v and $(g^* J u)^+ = c\alpha^* U_0^+(\cdot,0)^* J U_0^+(\cdot,0)\alpha$ for a suitable $c \in \mathbb{C}$. Because of equation (2.2) and since $\alpha^* J \alpha = 0$, we obtain $\mathcal{A}_-(\lambda) = 0$. It follows that the second column and the second row of $\Omega(\lambda)$ vanish also when $P_-(\lambda)$ has rank 1, i.e., in any case.

We may prove similarly that the third column and the third row of $\Omega(\lambda)$ vanish whatever $P_+(\lambda)$ may be and it remains only to consider

$$\Omega_{4,4}(\lambda) = \mathcal{A}_-(\lambda)\mathcal{J}^{-1}\mathcal{A}_-(\overline{\lambda})^* - \mathcal{A}_+(\lambda)\mathcal{J}^{-1}\mathcal{A}_+(\bar{\lambda})^*.$$

If $P_-(\lambda)$ has rank 1 we have $\mathcal{A}_-(\lambda)\mathcal{J}^{-1}\mathcal{A}_-(\overline{\lambda})^* = 0 = (g^* J g)^+(a)$. If $P_-(\lambda) = \mathbb{1}$ we get $\mathcal{A}_-(\lambda)\mathcal{J}^{-1}\mathcal{A}_-(\overline{\lambda})^* = (g^* J g)^+(a)$ on account of equation (2.2). With similar considerations for $P_+(\lambda)$ we have therefore $\Omega_{4,4}(\lambda) = (g^* J g)^+(a) - (g^* J g)^+(b)$ which is 0 by (2.3). □

Later we will also need the following lemma.

Lemma 4.8. If $\mathbf{B} = \operatorname{ran} \mathbb{P}$ or if $\Omega(\lambda) = 0$, then $\operatorname{ran} \mathbb{H}(\lambda)\mathcal{J}^{-1}\mathbb{P} \subset \operatorname{ran} \mathbb{F}(\lambda)$ and $\operatorname{ran} \mathbb{F}(\lambda)^\dagger \mathbb{H}(\lambda)\mathcal{J}^{-1}\mathbb{P} \subset \operatorname{ran} \mathbb{P}$.

Proof. If $\mathbf{B} = \operatorname{ran}\mathbb{P}$ and z is any element of $\mathbb{C}^{n(N+1)}$, then there is a compactly supported $f \in L^2(w)$ such that $\mathbb{P}z = (\mathcal{F}f)(\lambda)$. Therefore (3.10) and (3.14) establish the existence of a vector $\tilde{z} = \frac{1}{2}(\tilde{u} + \tilde{v}) \in \operatorname{ran}\mathbb{P}$ such that $\mathbb{F}(\lambda)\tilde{z} = \mathbb{H}(\lambda)\mathcal{J}^{-1}\mathbb{P}z$ settling our first claim in this case. The second follows after applying $\mathbb{F}(\lambda)^\dagger$ since $\mathbb{F}(\lambda)^\dagger\mathbb{F}(\lambda) = \mathbb{1}$.

If $\Omega(\lambda) = 0$ we have that $\mathbb{H}(\lambda)\mathcal{J}^{-1}\mathbb{P}\mathbb{F}(\overline{\lambda})^* = -\mathbb{F}(\lambda)\mathbb{P}\mathcal{J}^{-1}\mathbb{H}(\overline{\lambda})^*$. Here we use $\mathbb{F}(\overline{\lambda})^*\mathbb{F}(\overline{\lambda})^{\dagger *} = \mathbb{1}$ to get $\mathbb{H}(\lambda)\mathcal{J}^{-1}\mathbb{P} = -\mathbb{F}(\lambda)\mathbb{P}\mathcal{J}^{-1}\mathbb{H}(\overline{\lambda})^*\mathbb{P}\mathbb{F}(\overline{\lambda})^{\dagger *}$. □

Remark 4.1. *If* $\mathbf{B} = \operatorname{ran}\mathbb{P}$ *we have now that* $\operatorname{ran}\mathbb{F}(\lambda)^\dagger\Omega(\lambda) \subset \operatorname{ran}\mathbb{P}$*. Since* $\mathbb{F}(\lambda)\mathbb{F}(\lambda)^\dagger$ *is the orthogonal projection onto the range of* $\mathbb{F}(\lambda)$*, this shows that* $\mathbb{F}(\lambda)(\mathbb{M}(\lambda) - \mathbb{M}(\overline{\lambda})^*)\mathbb{F}(\overline{\lambda})^* = \Omega(\lambda) = 0$*. In other words, the requirement* $\Omega(\lambda) = 0$ *is satisfied when* $\mathbf{B} = \operatorname{ran}\mathbb{P}$*.*

4.2. The imaginary part of $\mathbb{M}$. In this section we assume that λ is in the upper half-plane but not in $\tilde{\Lambda}$ and that $\Omega(\lambda) = 0$ (which holds when $\mathbf{B} = \operatorname{ran}\mathbb{P}$). Let $\mathcal{S}(x)$ be the diagonal block matrix whose entries are, in this order, the $n \times n$ blocks $\operatorname{sgn}(x - \xi_j)\mathbb{1}$, $j = 0, ..., N$. Recall from the beginning of Section 3 that $U_j(\xi_j, \lambda) = \mathbb{1}$.

Define θ by

$$\theta(x, \lambda) = \big(\mathbb{M}(\lambda) + \frac{1}{2}\mathcal{S}(x)\mathcal{J}^{-1}\mathbb{P}\big)z$$

with $z \in \mathbb{C}^{n(N+1)}$. Note that $\mathcal{U}(\cdot, \lambda)\theta(\cdot, \lambda)$ is a solution of $Ju' + (q - \lambda w)u = 0$ in each of the intervals (x_k, ξ_k) and (ξ_k, x_{k+1}), $k = 0, ..., N$. We want to show that it actually satisfies the differential equation on (ξ_{k-1}, ξ_k), $k = 1, ..., N$, as well as the boundary conditions including the requirement that $\mathcal{U}(\cdot, \lambda)\theta(\cdot, \lambda)$ is in $\mathcal{L}^2(w)$. For the first claim we need to show that

$$\mathbb{F}_k(\lambda)\theta(x_k, \lambda) = 0 \tag{4.2}$$

where $\mathbb{F}_k$ denotes the k-th row of $n \times n$ blocks of $\mathbb{F}$ and hence of $\mathbb{B}$. Using $\mathbb{F}_k\mathbb{P} = \mathbb{F}_k$, see Lemma 3.1, and $\mathbb{F}^\dagger\mathbb{F} = \mathbb{1}$, the left-hand side of (4.2) becomes

$$\mathbb{F}_k(\lambda)\mathbb{F}(\lambda)^\dagger\big(\mathbb{H}(\lambda) + \frac{1}{2}\mathbb{F}(\lambda)S(x_k)\big)\mathcal{J}^{-1}\mathbb{P}z.$$

Since $\mathbb{F}(\lambda)\mathbb{F}(\lambda)^\dagger$ is the orthogonal projection onto $\operatorname{ran}\mathbb{F}(\lambda)$ and since, by Lemma 4.8, $\mathbb{H}(\lambda)\mathcal{J}^{-1}\mathbb{P}z$ is in $\operatorname{ran}\mathbb{F}(\lambda)$ we get next

$$\frac{1}{2}\big(\tilde{\mathbb{B}}_k(\lambda) + \mathbb{B}_k(\lambda)S(x_k)\big)\mathcal{J}^{-1}\mathbb{P}z.$$

This does indeed vanish due to the special structure of $\mathbb{B}$ and $\tilde{\mathbb{B}}$ and since $\mathcal{S}(x_k)$ is a diagonal block matrix whose first k blocks are $\mathbb{1}_n$ while the remaining $N+1-k$ blocks are $-\mathbb{1}_n$.

To show that $\mathcal{U}(\cdot,\lambda)\theta(\cdot,\lambda)$ is in $L^2(w)$ we are following a very similar strategy. On the left-hand side of (4.2) we have to choose $k = N+1$ so that $\mathbb{F}_k = \mathcal{Q}_-$ or $k = N+2$ so that $\mathbb{F}_k = \mathcal{Q}_+$. We also have to choose x in either (a, ξ_0) or else in (ξ_N, b) so that $S(x)$ is either $-\mathbb{1}_{n(N+1)}$ or else $\mathbb{1}_{n(N+1)}$. Since $\mathcal{Q}_\pm\mathbb{P} = \mathcal{Q}_\pm$ we get $\mathbb{F}_k(\lambda)\mathbb{M}(\lambda) = \mathbb{H}_k(\lambda)\mathcal{J}^{-1}\mathbb{P} = \pm\frac{1}{2}\mathbb{F}_k(\lambda)\mathcal{J}^{-1}\mathbb{P}$. This proves (4.2) for $k = N+1$ and $k = N+2$.

Finally, the boundary condition translates to $\mathcal{A}_+(\lambda)\theta_+(\lambda) + \mathcal{A}_-(\lambda)\theta_-(\lambda) = 0$ where $\theta_\pm(\lambda) = \theta(x,\lambda)$ with $x > \xi_N$ for the upper sign and $x < \xi_0$ for the lower sign. Note that, imitating previous arguments,

$$(\mathcal{A}_+(\lambda) + \mathcal{A}_-(\lambda))\theta_+(\lambda) = (\mathbb{H}_{N+3}(\lambda) + \frac{1}{2}\mathbb{F}_{N+3}(\lambda))\mathcal{J}^{-1}\mathbb{P}z = \mathcal{A}_-(\lambda)\mathcal{J}^{-1}\mathbb{P}z$$

gives $\mathcal{A}_+(\lambda)\theta_+(\lambda) = \mathcal{A}_-(\lambda)(\frac{1}{2}\mathcal{J}^{-1}\mathbb{P}z - \mathbb{M}(\lambda)z) = -\mathcal{A}_-(\lambda)\theta_-(\lambda)$, our desired result.

We now abbreviate $\mathcal{U}\theta$ by s. Fix $\lambda, \mu \in \varrho(T) \setminus \tilde{\Lambda}$ and define $h = s(\cdot,\lambda) - s(\cdot,\mu)$. Thus h satisfies the differential equation

$$Jh' + (q - \mu w)h = (\lambda - \mu)ws(\cdot,\lambda)$$

in the intervals (a,ξ_0), (ξ_{k-1},ξ_k), $k=1,...,N$, and in (ξ_N,b). Since the jump of $s(\cdot,\lambda)$ at ξ_k is equal to the k-th row of blocks $\mathcal{J}^{-1}\mathbb{P}z$ and hence independent of λ it follows that h is continuous at those points which entails that it satisfies the above equation in all of (a,b). Since h is also in $\mathcal{L}^2(w)$ and satisfies the boundary conditions (if any) it is thus the case that h is in the class of $(\lambda-\mu)R_\mu s(\cdot,\lambda)$. Evaluating h at ξ_{k-1} for $k=1,...,N+1$ gives

$$(h(\xi_0),...,h(\xi_n))^\diamond = (\mathbb{M}(\lambda)-\mathbb{M}(\mu))z \tag{4.3}$$

showing that $(h(\xi_0),...,h(\xi_n))^\diamond \in N_0^\perp$ so that $h=(\lambda-\mu)\mathcal{R}_\mu s(\cdot,\lambda)$.

Multiply (4.3) by z^* on the left to get

$$\frac{z^*\mathbb{M}(\lambda)z - z^*\mathbb{M}(\mu)z}{\lambda-\mu} = \sum_{k=1}^{N+1} z_k^*(\mathcal{R}_\mu s(\cdot,\lambda))(\xi_{k-1}). \tag{4.4}$$

Using (3.17) and $\mathcal{U}(\xi_{k-1},\mu)=e_k^*$, where e_k^* is a row of $n\times n$ blocks all zero except for the k-th which is $\mathbb{1}_n$, we get

$$(\mathcal{R}_\mu s(\cdot,\lambda))(\xi_{k-1}) = e_k^*\int \big(\mathbb{M}(\mu)+\frac{1}{2}\mathcal{J}^{-1}\operatorname{sgn}(\xi_{k-1}-\cdot)\big)\mathcal{U}(\cdot,\overline{\mu})^* w s(\cdot,\lambda).$$

Now let $\mu=\overline{\lambda}$ and recall that $\mathbb{M}(\overline{\lambda})=\mathbb{M}(\lambda)^*$. Then

$$z^*\frac{\mathbb{M}(\lambda)-\mathbb{M}(\lambda)^*}{\lambda-\overline{\lambda}}z = \int\sum_{k=1}^{N+1}\Big[\mathcal{U}(\cdot,\lambda)\big(\mathbb{M}(\lambda)-\frac{1}{2}\mathcal{J}^{-1}\operatorname{sgn}(\xi_{k-1}-\cdot)\big)e_k z_k\Big]^* w s(\cdot,\lambda).$$

Using the identities $\sum_{k=1}^{N+1} e_k z_k = z$ and $\sum_{k=1}^{N+1}\operatorname{sgn}(\xi_{k-1}-x)e_k z_k = -\mathcal{S}(x)z$ and the fact that the matrices $\mathcal{J}^{-1}$ and $\mathcal{S}(x)$ commute, we get now

$$z^*\frac{\mathbb{M}(\lambda)-\mathbb{M}(\overline{\lambda})}{\lambda-\overline{\lambda}}z = \int s(\cdot,\lambda)^* w s(\cdot,\lambda)$$

which is non-negative.

We have proved the following lemma.

Lemma 4.9. *If $\operatorname{Im}\lambda>0$, $\lambda\notin\tilde{\Lambda}$ and either $\mathbf{B}=\operatorname{ran}\mathbb{P}$ or $\Omega(\lambda)=0$, then $\operatorname{Im}M(\lambda)\geq 0$.*

4.3. Analyticity. Suppose λ is a non-real complex number for which there is a neighborhood which does not intersect $\tilde{\Lambda}$. The resolvent identity for R_μ and the boundedness of the operator selecting a representative of $([u],[f]) \in T_{\max}$ (see Lemma 5.2 in [**16**]) show that the map $\mu \mapsto \mathcal{R}_\mu f$ is continuous at λ. Using now equation (4.4) shows that $z^*(\mathbb{M}(\mu) - \mathbb{M}(\lambda))z/(\mu - \lambda)$ has a limit as μ tends to λ. Thus $z^*\mathbb{M}z$ is analytic near λ.

4.4. $\mathbb{M}$ is Nevanlinna. In the course of our investigations we have added several hypotheses to the basic Hypothesis 2.1. We will add one more and collect them in the following Hypothesis 4.10.

HYPOTHESIS 4.10. (a,b) is a real interval. J is a constant, invertible, skew-hermitian $n \times n$-matrix. q and w are $n \times n$-matrices whose entries are distributions of order 0 on (a,b); q is hermitian and w is non-negative. $\Lambda_x \cap \mathbb{R}$ is empty unless $x \in \{x_1, ..., x_N\} \subset (a,b)$ and $\tilde{\Lambda} = \bigcup_{x \notin \{x_1,...,x_N\}} \Lambda_x$ is a closed set of isolated points. Finally, we require $\Omega = 0$ (which is satisfied when $\mathbf{B} = \operatorname{ran} \mathbb{P}$).

Under this hypothesis we can prove the following theorem.

THEOREM 4.11. Assume the validity of Hypothesis 4.10. Then the function $\mathbb{M}$ may be extended to all of $\mathbb{C} \setminus \mathbb{R}$ as a matrix-valued Nevanlinna function.

PROOF. Let $z \in \mathbb{C}^{n(N+1)}$ and $m = z^*\mathbb{M}z$. The singularities of m are the points in $\tilde{\Lambda}$, a closed set of isolated points by hypothesis. Suppose now that μ is one of these points and that B is a ball centered at μ not intersecting $\mathbb{R}$ or $\tilde{\Lambda} \setminus \{\mu\}$. Note that $\operatorname{Im} m(\lambda)/\operatorname{Im}\lambda \geq 0$ in $B \setminus \{\mu\}$ and hence μ is a removable singularity of m. Since this is so for any $z \in \mathbb{C}^{n(N+1)}$ we may extend $\mathbb{M}$ to all $\mathbb{C} \setminus \mathbb{R}$ as an analytic function. The properties of symmetry and of the sign of $\operatorname{Im} \mathbb{M}$ are also retained. □

5. The Fourier expansion

We begin with a few words about the spectral theorem for self-adjoint relations T. The closure $\mathcal{H}_0$ of the domain of T is the orthogonal complement of $\mathcal{H}_\infty =$

$\{f \in L^2(w) : (0,f) \in T\}$ in $L^2(w)$. It follows that $T = T_0 \oplus (\{0\} \times \mathcal{H}_\infty)$ where $T_0 = T \cap (\mathcal{H}_0 \times \mathcal{H}_0)$ is a self-adjoint operator densely defined in $\mathcal{H}_0$. The spectral theorem for self-adjoint operators guarantees the existence of a resolution of the identity π such that

$$\langle f, T_0 g\rangle = \int t \, d\langle f, \pi((-\infty,t))g\rangle.$$

One may now extend the domain of definition of the spectral projections $\pi(B)$ from $\mathcal{H}_0$ to $\mathcal{H}$ by setting $\pi(B)f = 0$ whenever $f \in \mathcal{H}_\infty$. Thus $\pi(\mathbb{R})$ becomes the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_0$. For more information we refer the reader to Appendix B of [**10**] or to Section 2.2 of [**3**].

In the following we require Hypothesis 4.10 to hold. Then, as we just proved $\mathbb{M}$ is a Nevanlinna function and therefore defines a spectral measure τ, see equation (4.1). For compactly supported functions f of $L^2(w)$ and if $\lambda \in \mathbb{C} \setminus \tilde{\Lambda}$ we defined earlier the Fourier transform $\mathcal{F}$ by

$$(\mathcal{F}f)(\lambda) = \int \mathcal{U}(\cdot, \overline{\lambda})^* w f.$$

Note that $\mathbb{C} \setminus \tilde{\Lambda}$ is an open set containing the real line, so $\mathcal{F}f$ is defined everywhere in a neighborhood of $\mathbb{R}$. In fact, we are mostly concerned with the restriction of $\mathcal{F}f$ to $\mathbb{R}$ but will not use different notation.

The remainder of this section is devoted to the proof of the following theorem, the main theorem of this paper. The outline of the proof follows, with one exception, the proof of Theorem 15.5 in Bennewitz [**2**]. The spirit of that proof was also used in [**3**] and in [**10**] which may be consulted for some of the details we skip here in the interest of brevity. The exception concerns the proof of the fact that $\ker \mathcal{F}^*$ is trivial, see Lemma 5.6 below where $\mathcal{F}^*$ is called $\mathcal{G}$.

Theorem 5.1. Suppose T is a self-adjoint restriction of a relation $T_{\max}$ whose coefficients q and w satisfy Hypothesis 4.10. Let τ be the measure generated by the associated $\mathbb{M}$-function. Then the following statements hold.

(1) There is a continuous map $\mathcal{F} : L^2(w) \to L^2(\tau)$ which assigns to a compactly supported element $f \in L^2(w)$ the function defined by $(\mathcal{F}f)(t) = \int \mathcal{U}(\cdot, t)^* wf$. The kernel of $\mathcal{F}$ is the space $\mathcal{H}_\infty = \{f \in L^2(w) : (0, f) \in T\}$.

(2) There is a continuous map $\mathcal{G} : L^2(\tau) \to L^2(w)$ which assigns to a compactly supported element $\hat{f} \in L^2(\tau)$ the function defined by $(\mathcal{G}\hat{f})(x) = \int \mathcal{U}(x, \cdot)\tau \hat{f}$. The range of $\mathcal{G}$ is the space $\mathcal{H}_0 = \overline{\operatorname{dom} T}$.

(3) $\mathcal{F} \circ \mathcal{G}$ is the identity operator on $L^2(\tau)$, $\mathcal{G} \circ \mathcal{F}$ is the orthogonal projection from $L^2(w)$ onto $\mathcal{H}_0$, and the restriction of $\mathcal{F}$ to $\mathcal{H}_0$ is unitary.

(4) If $(u, f) \in T$ then $(\mathcal{F}f)(t) = t(\mathcal{F}u)(t)$. Conversely, if $t \mapsto \hat{u}(t)$ and $t \mapsto \hat{f}(t) = t\hat{u}(t)$ are both in $L^2(\tau)$, then $(\mathcal{G}\hat{u}, \mathcal{G}\hat{f}) \in T$.

REMARK 5.1. *When q and w are finite measures on (a, b), i.e., when the endpoints a and b are regular, the conditions that $\tilde{\Lambda}$ is a closed set of isolated points and that $\Omega = 0$ in Hypothesis 4.10 are automatically satisfied.*

Since the functions $\mathcal{U}(x, \cdot)$ are analytic on $\mathbb{C} \setminus \tilde{\Lambda}$ it follows immediately that

$$\langle g, R_\lambda f \rangle - (\mathcal{F}g)(\overline{\lambda})^* \mathbb{M}(\lambda)(\mathcal{F}f)(\lambda)$$

extends to an analytic function on $\mathbb{C} \setminus \tilde{\Lambda}$ so that we obtain the identity

$$\oint_\Gamma \langle f, R_\lambda f \rangle \, d\lambda = \oint_\Gamma (\mathcal{F}f)(\overline{\lambda})^* \mathbb{M}(\lambda)(\mathcal{F}f)(\lambda) \, d\lambda \tag{5.1}$$

when Γ is the contour described by the rectangle with vertices $c \pm i\varepsilon$ and $d \pm i\varepsilon$ for $c < d$ and sufficiently small but positive ε, provided the integrals exist.

Let π be the resolution of the identity for our self-adjoint relation T and define

$$\Pi_{f,g}(t) = \langle f, \pi((-\infty, t))g \rangle_w.$$

With the aid of the spectral theorem, Fubini's theorem, and Cauchy's integral formula one shows now that

$$\oint_\Gamma \langle f, R_\lambda f\rangle d\lambda = -2\pi i \int_{[c,d)} d\Pi_{f,f}. \tag{5.2}$$

when c and d are points of differentiability of $\Pi_{f,f}$. Similarly, using the Nevanlinna representation of $\mathbb{M}$, Fubini's theorem, and Cauchy's integral formula gives

$$\oint_\Gamma (\mathcal{F}f)(\overline{\lambda})^*\mathbb{M}(\lambda)(\mathcal{F}f)(\lambda)d\lambda = -2\pi i \int_{[c,d)} (\mathcal{F}f)^*\tau(\mathcal{F}f) \tag{5.3}$$

when c and d are points of differentiability of an antiderivative $\mathfrak{T}$ of τ. Thus we obtain

$$\int_{[c,d)} (\mathcal{F}f)^*\tau(\mathcal{F}f) = \int_{[c,d)} d\Pi_{f,f}. \tag{5.4}$$

Since $\Pi_{f,f}$ is left-continuous and $\mathfrak{T}$ may be chosen to be, equation (5.4) actually holds for all c, d with $c < d$.

With these preparations we may prove statement (1) of Theorem 5.1.

Lemma 5.2. When $f \in L^2(w)$ is compactly supported, then $\mathcal{F}f$ is in $L^2(\tau)$. The map $\mathcal{F}$ extends, by continuity, to all of $L^2(w)$. Moreover,

$$\Pi_{f,g}(t) = \langle f, \pi((-\infty,t))g\rangle_w = \int_{(-\infty,t)} (\mathcal{F}f)^*\tau(\mathcal{F}g) \tag{5.5}$$

whenever $f, g \in L^2(w)$. In particular, $\langle f, \pi(\mathbb{R})g\rangle_w = \langle \mathcal{F}f, \mathcal{F}g\rangle_\tau$ and $\ker \mathcal{F} = \mathcal{H}_\infty$.

Proof. Suppose $f \in L^2(w)$ is compactly supported. Choosing the interval $[c,d)$ sufficiently large in equation (5.4) proves that $\mathcal{F}f \in L^2(\tau)$. If f is an arbitrary element of $L^2(w)$ and $n \mapsto [a_n, b_n]$ a sequence of intervals in (a,b) converging to (a,b), set $f_n = f\chi_{[a_n,b_n]}$. Then

$$\|\mathcal{F}f_n - \mathcal{F}f_m\|_\tau = \|\pi(\mathbb{R})(f_n - f_m)\|_w \le \|f_n - f_m\|_w$$

showing that $n \mapsto \mathcal{F}f_n$ is a Cauchy sequence in $L^2(\tau)$ and thus convergent. By interweaving sequences it follows that the limit of this Cauchy sequence does not depend on how f is approximated. We denote the limit by $\mathcal{F}f$ thereby extending our definition of the Fourier transform to all of $L^2(w)$. Equation (5.5) holds when $g = f$ and otherwise by polarization. $\square$

Now we define a transform $\mathcal{G} : L^2(\tau) \to L^2(w)$. We begin by setting

$$(\mathcal{G}\hat{f})(x) = \int_{\mathbb{R}} \mathcal{U}(x, \cdot)\tau\hat{f}$$

whenever $\hat{f}$ is compactly supported. Note that $\mathcal{G}\hat{f}$ is locally of bounded variation. Then we have the following result which proves statement (2) of Theorem 5.1.

Lemma 5.3. When $\hat{f} \in L^2(\tau)$ is compactly supported, then $\mathcal{G}\hat{f}$ is in $L^2(w)$. The map $\mathcal{G}$ extends, by continuity, to all of $L^2(\tau)$. We have that

$$\langle g, \mathcal{G}\hat{f}\rangle_w = \langle \mathcal{F}g, \hat{f}\rangle_\tau \tag{5.6}$$

for all $g \in L^2(w)$ and all $\hat{f} \in L^2(\tau)$. Moreover, $\ker \mathcal{G} = (\operatorname{ran} \mathcal{F})^\perp$, $\operatorname{ran} \mathcal{G} = H_0$, and $\mathcal{G} \circ \mathcal{F} = \pi(\mathbb{R})$.

Proof. Suppose that $\hat{f} \in L^2(\tau)$ is compactly supported, denote $\mathcal{G}\hat{f}$ by f, and let $f_n = f\chi_{[a_n,b_n]}$. Upon changing the order of integration we get

$$\|f_n\|_w^2 = \langle f_n, f\rangle_w = \langle \mathcal{F}f_n, \hat{f}\rangle_\tau \le \|\mathcal{F}f_n\|_\tau \|\hat{f}\|_\tau.$$

Lemma 5.2 implies $\|\mathcal{F}f_n\|_\tau = \|\pi(\mathbb{R})f_n\|_w \le \|f_n\|_w$ and hence $\|f_n\|_w \le \|\hat{f}\|_\tau$. This is the case for every interval $[a_n, b_n] \subset (a, b)$ so it follows that $\mathcal{G}\hat{f} \in L^2(w)$.

As before we extend the domain of definition of $\mathcal{G}$ from the compactly supported functions in $L^2(\tau)$ to all of $L^2(\tau)$. Specifically, for a general element $\hat{f}$ in $L^2(\tau)$ set $\hat{f}_n = \hat{f}\chi_{[-n,n]}$. Then, according to what we just proved, $\|\mathcal{G}\hat{f}_n - \mathcal{G}\hat{f}_m\|_w \le \|\hat{f}_n - \hat{f}_m\|_\tau$ which implies that $n \mapsto \mathcal{G}\hat{f}_n$ is a Cauchy sequence in $L^2(w)$ and thus convergent. We denote the limit by $\mathcal{G}\hat{f}$.

Now suppose $g \in L^2(w)$, $\hat{f} \in L^2(\tau)$, $[a_k, b_k] \subset (a,b)$ and $[-n,n] \subset \mathbb{R}$. Upon changing the order of integration we get (as before)

$$\langle g\chi_{[a_k,b_k]}, \mathcal{G}(\hat{f}\chi_{[-n,n]})\rangle_w = \langle \mathcal{F}(g\chi_{[a_k,b_k]}), \hat{f}\chi_{[-n,n]}\rangle_\tau.$$

Now let $[a_k, b_k] \times [-n,n]$ approach $(a,b) \times \mathbb{R}$ to obtain equation (5.6).

Equation (5.6) implies $\ker \mathcal{G} = (\operatorname{ran} \mathcal{F})^\perp$ and $\operatorname{ran} \mathcal{G} \subset \mathcal{H}_\infty^\perp = \mathcal{H}_0$. Choosing $\hat{f} = \mathcal{F}f$ in (5.6) implies, using Lemma 5.2, that $\langle g, (\mathcal{G} \circ \mathcal{F})f\rangle = \langle g, \pi(\mathbb{R})f\rangle$. Since this is so for all $g \in L^2(w)$ we get $\mathcal{G} \circ \mathcal{F} = \pi(\mathbb{R})$ and, in particular $\mathcal{H}_0 \subset \operatorname{ran} \mathcal{G}$. □

Our next goal is to show that $\ker \mathcal{G}$ is trivial so that $\operatorname{ran} \mathcal{F}$ is dense in $L^2(\tau)$. This will show that statement (3) of Theorem 5.1 holds as can be seen as follows. Using Lemma 5.2, Lemma 5.3, and the self-adjointness of $\pi(\mathbb{R})$ we obtain

$$\langle \mathcal{F}g, (\mathcal{F} \circ \mathcal{G} - \mathbb{1})\hat{f}\rangle = \langle \mathcal{F}g, \mathcal{F}(\mathcal{G}\hat{f})\rangle - \langle \mathcal{F}g, \hat{f}\rangle = 0.$$

This implies $\mathcal{F} \circ \mathcal{G} = \mathbb{1}$ and that $\mathcal{G}$ is an isometry. We have already that $\mathcal{G} \circ \mathcal{F} = \pi(\mathbb{R})$ and that $\mathcal{F}|_{\mathcal{H}_0}$ is an isometry. However, the proof of $\ker \mathcal{G} = \{0\}$, see Lemma 5.6, requires more preparation.

Lemma 5.4. If $\operatorname{Im}(\lambda) \neq 0$, then $(\mathcal{F}(R_\lambda g))(t) = (\mathcal{F}g)(t)/(t - \lambda)$.

Proof. First note that $t \mapsto \hat{g}(t)/(t-\lambda)$ is in $L^2(\tau)$ if $\hat{g}$ is. The spectral theorem and Lemma 5.2 give

$$\langle f, R_\lambda g\rangle_w = \int \frac{1}{t-\lambda}\, d\Pi_{f,g}(t) = \int \frac{(\mathcal{F}f)(t)^*\tau(t)(\mathcal{F}g)(t)}{t-\lambda} = \langle \mathcal{F}f, (\mathcal{F}g)/(\cdot - \lambda)\rangle_\tau.$$

In particular, $\|R_\lambda g\|_w^2 = \langle \mathcal{F}(R_\lambda g), (\mathcal{F}g)/(\cdot - \lambda)\rangle_\tau$. On the other hand

$$\|R_\lambda g\|_w^2 = \langle g, R_{\overline{\lambda}} R_\lambda g\rangle_w = \frac{1}{\lambda - \overline{\lambda}}\langle g, (R_\lambda - R_{\overline{\lambda}})g\rangle_w = \|\mathcal{F}g/(\cdot - \lambda)\|_\tau^2.$$

Lemma 5.2 also implies that $\|R_\lambda g\|_w^2 = \|\mathcal{F}(R_\lambda g)\|_\tau^2$. Thus the four terms appearing in the expansion of $\|\mathcal{F}(R_\lambda g) - \mathcal{F}g/(\cdot - \lambda)\|^2$ cancel each other leaving 0. □

Let $\mathbb{T} = (\tau/\operatorname{tr}\tau)$ be the Radon-Nikodym derivative of τ with respect to $\operatorname{tr}\tau$. Note that $\mathbb{T} \in L^\infty(\operatorname{tr}\tau)$. By the Lebesgue-Radon-Nikodym theorem we have $\operatorname{tr}\tau = h\,\mathrm{m} + \sigma$ where m denotes Lebesgue measure, h is a non-negative function, σ is a non-negative measure, and σ and m are mutually singular. Define

$$\omega(s,\varepsilon) = \int_{\mathbb{R}} \frac{\varepsilon}{(s-t)^2+\varepsilon^2} \operatorname{tr}\tau(t).$$

By Fatou's theorem (see, e.g., Theorem 5.5 in Rosenblum and Rovnyak [**17**]) $\omega(s,\varepsilon)$ converges to $\pi h(s)$ for m-almost every $s \in \mathbb{R}$ as $\varepsilon \downarrow 0$. In fact $h(s) > 0$ m-almost everywhere. The measure σ is concentrated on the set

$$S = \{s \in \mathbb{R} : \lim_{r\downarrow 0} \operatorname{tr}\tau((s-r,s+r))/(2r) = \infty\}$$

and, consequently, when $s \in S$ then $\omega(s,\varepsilon)$ tends to ∞ as $\varepsilon \downarrow 0$. It follows that $1/\omega(s,\varepsilon)$ is bounded above when s is in a set of full $\operatorname{tr}\tau$-measure.

Lemma 5.5. $\mathbb{B}\mathbb{T} = 0$ and $(1-\mathbb{P})\mathbb{T} = 0$ almost everywhere with respect to $\operatorname{tr}\tau$.

Proof. Since $\mathbb{F}(\lambda)\mathbb{F}(\lambda)^\dagger$ is the orthogonal projection onto the range of $\mathbb{F}(\lambda)$ Lemma 4.8 shows that

$$\mathbb{H}(\lambda)\mathcal{J}^{-1}\mathbb{P} = \mathbb{F}(\lambda)\mathbb{F}(\lambda)^\dagger\mathbb{H}(\lambda)\mathcal{J}^{-1}\mathbb{P}.$$

Using that $\mathbb{B} = \mathbb{B}\mathbb{P}$ (see Lemma 3.1) the first nN rows of this identity are

$$\mathbb{B}(\lambda)\mathbb{M}(\lambda) = \frac{1}{2}\tilde{\mathbb{B}}(\lambda)\mathcal{J}^{-1}\mathbb{P}.$$

Subtract from this identity the one where λ is replaced by $\overline{\lambda}$ to get

$$\mathbb{B}(\lambda)\mathbb{M}(\lambda) - \mathbb{B}(\overline{\lambda})\mathbb{M}(\overline{\lambda}) = \frac{1}{2}(\tilde{\mathbb{B}}(\lambda) - \tilde{\mathbb{B}}(\overline{\lambda}))\mathcal{J}^{-1}\mathbb{P}. \tag{5.7}$$

We will now take the limit as $\varepsilon = \operatorname{Im}\lambda \downarrow 0$. First note that the right-hand side of (5.7) will tend to 0. Next, the integral occurring in $\mathrm{i}\,\varepsilon\mathbb{M}(s \pm \mathrm{i}\,\varepsilon)$ is

$$\int_{\mathbb{R}} \frac{\mathrm{i}\,\varepsilon(1 + ts \pm \mathrm{i}\,t\varepsilon)}{t - s \mp \mathrm{i}\,\varepsilon} \frac{\tau(t)}{t^2 + 1}.$$

For $\varepsilon \in [0, 1]$ and $t \in \mathbb{R}$ the first fraction may be bounded by $5(s^2 + 1)$. Since the measure $\tau(t)/(t^2 + 1)$ is finite, the dominated convergence theorem shows that

$$\lim_{\varepsilon\downarrow 0} \mathrm{i}\,\varepsilon\mathbb{M}(s \pm \mathrm{i}\,\varepsilon) = \mp\tau(\{s\}).$$

Also $\lim_{\varepsilon\downarrow 0}(\mathbb{B}(s \pm \mathrm{i}\,\varepsilon) - \mathbb{B}(s))/(\mathrm{i}\,\varepsilon) = \pm\dot{\mathbb{B}}(s)$. These facts may be combined to give

$$\lim_{\varepsilon\downarrow 0} \mathbb{B}(s)(\mathbb{M}(\lambda) - \mathbb{M}(\overline{\lambda})) = 0.$$

The Nevanlinna representation of M gives

$$\mathbb{M}(s + \mathrm{i}\,\varepsilon) - \mathbb{M}(s - \mathrm{i}\,\varepsilon) = 2\,\mathrm{i}\,\varepsilon B + 2\,\mathrm{i} \int_{\mathbb{R}} \frac{\varepsilon}{(t - s)^2 + \varepsilon^2} \mathbb{T}(t) \operatorname{tr}\tau(t).$$

Using the fact that the function $1/\omega(s, \varepsilon)$ is bounded above for $\operatorname{tr}\tau$-almost every s, and Theorem A.2 we get now

$$\mathbb{B}(s)\mathbb{T}(s) = \lim_{\varepsilon\downarrow 0} \mathbb{B}(s) \int_{\mathbb{R}} \frac{\varepsilon}{(t - s)^2 + \varepsilon^2} \mathbb{T}(t) \operatorname{tr}\tau(t)/\omega(s, \varepsilon) = 0.$$

The proof of the identity $(1 - \mathbb{P})\mathbb{T} = 0$ is very similar after remembering that $\mathbb{P}$ is the left-most factor of $\mathbb{M}$ and hence

$$0 = (1 - \mathbb{P})(\mathbb{M}(\lambda) - \mathbb{M}(\overline{\lambda})) = 2\,\mathrm{i}(1 - \mathbb{P}) \int_{\mathbb{R}} \frac{\varepsilon}{(t - s)^2 + \varepsilon^2} \mathbb{T}(t) \operatorname{tr}\tau(t).$$

This completes the proof. □

LEMMA 5.6. $\ker\mathcal{G}$ is trivial.

PROOF. Suppose $\hat{u} \in \ker\mathcal{G}$. Then, using Lemmas 5.3 and 5.4, we find

$$0 = \langle R_{\overline{\lambda}}g, \mathcal{G}\hat{u}\rangle_w = \langle \mathcal{F}R_{\overline{\lambda}}g, \hat{u}\rangle_\tau = \int \frac{(\mathcal{F}g)(t)^*}{t - \lambda} \tau(t)\hat{u}(t)$$

for any $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and any $g \in L^2(w)$. The Stieltjes inversion formula shows that $(\mathcal{F}g)^* \tau \hat{u}$ is the zero measure. If K is a compact subset of $\mathbb{R}$ and $\psi_K(t) = t\chi_K(t)$, then

$$\langle g, \mathcal{G}\chi_K \hat{u}\rangle_w = \langle \mathcal{F}g, \chi_K \hat{u}\rangle_\tau = 0 \text{ and } \langle g, \mathcal{G}\psi_K \hat{u}\rangle_w = \langle \mathcal{F}g, \psi_K \hat{u}\rangle_\tau = 0$$

show that $\chi_K \hat{u}$ and $\psi_K \hat{u}$ are also in $\ker \mathcal{G}$.

Define $u(x) = \int_K \mathcal{U}(x,t)\mathbb{T}(t)\hat{u}(t) \operatorname{tr}\tau(t)$ and $f(x) = \int_K \mathcal{U}(x,t)\mathbb{T}(t)t\hat{u}(t) \operatorname{tr}\tau(t)$ which are representatives of $\mathcal{G}\chi_K \hat{u} = [0]$ and $\mathcal{G}\psi_K \hat{u} = [0]$, respectively.

We proved in Lemma 5.5 that $\mathbb{B}\mathbb{T}\hat{u} = 0$ almost everywhere with respect to $\operatorname{tr}\tau$. Therefore, by equation (3.3) for $f = 0$, $\mathcal{U}(\cdot,t)\mathbb{T}(t)\hat{u}(t)$ is a solution of $Jv' + qv = twv$ on (a,b) for almost every $t \in K$. Using Fubini's theorem this implies that $Ju' + qu = wf$. Since u and f are in $[0]$, it follows that $u \in \mathcal{L}_0$. Therefore $\tilde{u}_0 = (u(\xi_0), ..., u(\xi_N))^\diamond = \int_K \mathbb{T}\hat{u} \operatorname{tr}\tau$ is in $\ker \mathbb{P}$. By Lemma 5.5 we also have that $(\mathbb{1} - \mathbb{P})\mathbb{T} = 0$ and hence $\tilde{u}_0 = 0$. This is only possible when u is identically equal to 0.

Choosing $K = [0,s]$ or $K = [s,0]$ shows that the cumulative distribution function of the measure $\mathcal{U}(x,\cdot)\mathbb{T}\hat{u} \operatorname{tr}\tau$ is zero, i.e., this measure is the zero measure. In particular, if $x = \xi_k \in (x_k, x_{k+1})$ this is the $(k+1)$-st block of $\mathbb{T}\hat{u} \operatorname{tr}\tau = \tau\hat{u}$. But this means that $\hat{u}$ is zero almost everywhere with respect to τ. □

Our last lemma provides the proof of statement (4) of Theorem 5.1.

LEMMA 5.7. If $(u, f) \in T$ then $(\mathcal{F}f)(t) = t(\mathcal{F}u)(t)$. Conversely, if $t \mapsto \hat{u}(t)$ and $t \mapsto \hat{f}(t) = t\hat{u}(t)$ are both in $L^2(\tau)$, then $(\mathcal{G}\hat{u}, \mathcal{G}\hat{f}) \in T$.

PROOF. Suppose $(u, f) \in T$ and hence that $(f - \lambda u, u) \in R_\lambda$. Then Lemma 5.4 gives $(\mathcal{F}u)(t) = (\mathcal{F}(f - \lambda u))(t)/(t - \lambda)$ which simplifies to $t(\mathcal{F}u)(t) = (\mathcal{F}f)(t)$.

Now suppose that $\hat{u}, \hat{f} \in L^2(\tau)$ where $\hat{f}(t) = t\hat{u}(t)$. Define $f = \mathcal{G}\hat{f}$ and $u = \mathcal{G}\hat{u}$ and pick a λ in $\varrho(T) \setminus \tilde{\Lambda}$. Then, using Lemma 5.4,

$$\mathcal{F}u = \hat{u} = \frac{\hat{f} - \lambda\hat{u}}{t - \lambda} = \frac{\mathcal{F}(f - \lambda u)}{t - \lambda} = \mathcal{F}(R_\lambda(f - \lambda u)).$$

Applying $\mathcal{G}$ gives $u = R_\lambda(f - \lambda u)$ which is equivalent to $(u, f) \in T$. □

A. An extension of Fatou's theorem

In 1906 Fatou [**7**] investigated the limiting behavior of holomorphic functions defined on the unit disk. Analogous considerations for holomorphic functions defined on the upper half-plane lead to the following theorem, commonly called Fatou's theorem, see, e.g., Theorem 5.5 in Rosenblum and Rovnyak [**17**].

Theorem A.1. Let

$$V(s+\mathrm{i}\,r)=\int_{\mathbb{R}}\frac{r\mu(t)}{(t-s)^2+r^2}$$

where μ is a non-negative Borel measure satisfying $\int_{\mathbb{R}}\mu(t)/(t^2+1)<\infty$, $r>0$, and $s\in\mathbb{R}$. If the Lebesgue-Radon-Nikodym decomposition of μ is $h\,\mathrm{m}+\sigma$ (where m denotes Lebesgue measure), then

$$\lim_{r\downarrow 0}V(s+\mathrm{i}\,r)=\pi h(s)$$

almost everywhere with respect to Lebesgue measure.

Applying this result to the measure $f\mu$ when $f\in L^\infty(\mu)$ gives

$$\lim_{r\downarrow 0}\left[\int_{\mathbb{R}}\frac{rf(t)\mu(t)}{(s-t)^2+r^2}\Big/\int_{\mathbb{R}}\frac{r\mu(t)}{(s-t)^2+r^2}\right]=f(s) \tag{A.1}$$

almost everywhere with respect to Lebesgue measure.

We are interested in the behavior of the quotient on the left-hand side of (A.1) on a set of full μ-measure, that is, also at points where σ is concentrated. This was achieved in the following theorem whose proof is due to Björn and Christer Bennewitz [**1**].

Theorem A.2. Suppose μ is a non-negative measure on $\mathbb{R}$ such that $\int_{\mathbb{R}}\mu(t)/(t^2+1)$ is finite. If $f\in L^\infty(\mu)$, then

$$\lim_{r\downarrow 0}\left[\int_{\mathbb{R}}\frac{rf(t)\mu(t)}{(s-t)^2+r^2}\Big/\int_{\mathbb{R}}\frac{r\mu(t)}{(s-t)^2+r^2}\right]=f(s)$$

for μ-almost every $s \in \mathbb{R}$.

Proof. Suppose that s is a μ-Lebesgue point of f, i.e., for every positive ε there is a positive δ such that

$$\int_I |f(t) - f(s)|\mu(t) < \varepsilon\mu(I) \tag{A.2}$$

for any interval $I \subset [s-\delta, s+\delta]$ containing s. Without loss of generality one may assume that $\delta \leq 1$ and that $s-\delta$ and $s+\delta$ are points of continuity for μ. The differentiation theorem (see, e.g., Theorem B.8.8 of [**3**]) implies that μ-almost every point is a μ-Lebesgue point. Moreover assume that

$$\inf\{\mu((s-r, s+r))/(2r) : r > 0\} > 0. \tag{A.3}$$

Again, this is true for μ-almost all s by Hardy's maximal theorem (see, e.g., Theorem B.8.9 of [**3**]) applied to the function $s \mapsto \sup\{2r/\mu((s-r, s+r)) : r > 0\}$.

Let F be the left-continuous antiderivative of $|f(\cdot) - f(s)|\mu$ satisfying $F(s) = 0$ and, similarly, M the left-continuous antiderivative of μ satisfying $M(s) = 0$. Also abbreviate the expression $r/((s-t)^2 + r^2)$ by $p(t)$. Then the claim may be written as

$$\int_{\mathbb{R}} p\, dF \Big/ \int_{\mathbb{R}} p\, dM \to 0$$

as $r \downarrow 0$.

Equation (A.2) may be rephrased in terms of F and M as follows. If $t > s$

$$F(t) \leq \varepsilon\mu([s,t)) = \varepsilon M(t) \tag{A.4}$$

and, if $t < s$,

$$-F(t) \leq \varepsilon\mu([t,s]) = \varepsilon(\mu(\{s\}) - M(t)). \tag{A.5}$$

Split the integral $\int_{\mathbb{R}} p\,dF$ into two parts, namely $I_1 = \int_{(s-\delta,s+\delta)} p\,dF$ and $I_2 = \int_{(s-\delta,s+\delta)^c} p\,dF$. Since p is continuous integration by parts yields

$$I_1 = (pF)(s+\delta) - (pF)(s-\delta) - \int_{(s-\delta,s+\delta)} F\,dp.$$

Now use the estimates (A.4) and (A.5) to obtain

$$I_1 \leq \varepsilon\Big((pM)(s+\delta) - (pM)(s-\delta) + \mu(\{s\})p(s) - \int_{(s-\delta,s+\delta)} M\,dp\Big).$$

Integrating by parts once more one gets

$$I_1 \leq \varepsilon\mu(\{s\})p(s) + \varepsilon \int_{(s-\delta,s+\delta)} p\,dM \leq 2\varepsilon \int_{\mathbb{R}} p\,dM.$$

To estimate I_2 note that, if $|s-t| \geq \delta$ and $\delta \in (0,1]$, then

$$\frac{r(t^2+1)}{(s-t)^2+r^2} \leq 8\frac{s^2+1}{\delta^2}r.$$

To obtain this estimate use $t^2+1 \leq 4s^2 + 2(s^2+1) + 2 \leq 8(s^2+1)$ when $|t| \leq 2|s|+1$ and $t^2+1 \leq 2t^2$ and $(s-t)^2 \leq t^2/4$ when $|t| \geq 2|s|+1$. Therefore

$$I_2 \leq 16\|f\|_\infty \frac{s^2+1}{\delta^2} r \int \frac{\mu(t)}{t^2+1}$$

which tends to 0 as $r \downarrow 0$. Also

$$\int_{\mathbb{R}} p\,dM \geq \int_{(s-r,s+r)} p\,dM \geq \frac{\mu((s-r,s+r))}{2r}$$

which is bounded away from 0 according to equation (A.3). Hence $I_2 / \int_{\mathbb{R}} p\,dM < \varepsilon$ for sufficiently small r.

Combining this estimate with the one for I_1 proves the claim, since ε may be arbitrarily small. □

REFERENCES

[1] Björn Bennewitz and Christer Bennewitz. Private communication, 2022.

[2] Christer Bennewitz. Spectral theory in Hilbert space. Lecture Notes, 2008.

[3] Christer Bennewitz, Malcolm Brown, and Rudi Weikard. *Spectral and scattering theory for ordinary differential equations. Vol. I.* Universitext. Springer, Cham, 2020. Sturm–Liouville equations.

[4] Kevin Campbell, Minh Nguyen, and Rudi Weikard. On the spectral theory for first-order systems without the unique continuation property. *Linear Multilinear Algebra*, 69(12):2315–2323, 2021. Published online: 04 Oct 2019.

[5] Jonathan Eckhardt, Fritz Gesztesy, Roger Nichols, and Gerald Teschl. Weyl-Titchmarsh theory for Sturm-Liouville operators with distributional potentials. *Opuscula Math.*, 33(3):467–563, 2013.

[6] Jonathan Eckhardt and Gerald Teschl. Sturm-Liouville operators with measure-valued coefficients. *J. Anal. Math.*, 120:151–224, 2013.

[7] P. Fatou. Séries trigonométriques et séries de Taylor. *Acta Math.*, 30(1):335–400, 1906.

[8] William Feller. On second order differential operators. *Ann. of Math. (2)*, 61:90–105, 1955.

[9] F. Gesztesy and H. Holden. A new class of solvable models in quantum mechanics describing point interactions on the line. *J. Phys. A*, 20(15):5157–5177, 1987.

[10] Ahmed Ghatasheh and Rudi Weikard. Spectral theory for systems of ordinary differential equations with distributional coefficients. *J. Differential Equations*, 268(6):2752–2801, 2020.

[11] I. S. Kac. On the existence of spectral functions of certain second-order singular differential systems. *Dokl. Akad. Nauk SSSR (N.S.)*, 106:15–18, 1956.

[12] M. G. Kreĭn. On a generalization of investigations of Stieltjes. *Doklady Akad. Nauk SSSR (N.S.)*, 87:881–884, 1952.

[13] P. Kurasov. Distribution theory for discontinuous test functions and differential operators with generalized coefficients. *J. Math. Anal. Appl.*, 201(1):297–323, 1996.

[14] P. Kurasov and J. Boman. Finite rank singular perturbations and distributions with discontinuous test functions. *Proc. Amer. Math. Soc.*, 126(6):1673–1683, 1998.

[15] Angelo B. Mingarelli. *Volterra-Stieltjes integral equations and generalized ordinary differential expressions*, volume 989 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1983.

[16] Steven Redolfi and Rudi Weikard. Green's functions for first-order systems of ordinary differential equations without the unique continuation property. *Integral Equations Operator Theory*, 94(2):Paper No. 23, 19, 2022.

[17] Marvin Rosenblum and James Rovnyak. *Topics in Hardy classes and univalent functions*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Verlag, Basel, 1994.

[18] A. M. Savchuk and A. A. Shkalikov. Sturm-Liouville operators with singular potentials. *Mathematical Notes*, 66(6):741–753, 1999. Translated from Mat. Zametki, Vol. 66, pp. 897–912 (1999).