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Pratik Talati

Nikolai Chernov

MATHEMATICS

Exploring the Variance of the Square Root of a Poisson Random Variable

Pratik Talati, Nikolai Chernov

A Poisson random variable, X, plays a fundamental role in Probability theory. It is characterized by a parameter λ > 0, and its mean value and variance are both equal to λ. As As $\lambda \to \infty$ the typical values of X get larger and larger, and their spread also gets wider. One can assume that $\sqrt{X}$ behaves similarly (i.e., it's mean value and variance grow to infinity). However, it is quite intriguing to discover that $Var(\sqrt{X})$ actually approaches a constant, ¼, which means its spread remains fixed. Dr. Nicolai Chernov found the this to be an interesting phenomenon, and I have built upon what other UAB Math Fast-Track students have done to prove this, both indirectly (through computer simulations) and directly. With Dr. Chernov's guidance, a proof has been developed that illustrates why this strange event occurs.

A Poisson distribution is a specific type of probability distribution whose mean, E(X), (denoted as λ in the following calculations) is equal to the variance, where X is a Poisson random variable. The probability P(X=k) = $\frac{\lambda^k}{k!}e^{-\lambda}$ for some k ≥ 0 and λ > 0. The variance, Var(X), can be found using the following formula: $Var(X) = E(X^2) - (E(X))^2$.
Thus, using the equation above, the $Var(\sqrt{X})$ can be expressed as $Var(\sqrt{X}) = E(X) - (E(\sqrt{X}))^2$.

The fixed value of the $Var(\sqrt{X})$ was first encountered in an application of the Poisson distribution to other fields in the sciences[1]. Even though this was an interesting phenomenon, it was unknown in probability theory[2,3]. Thus, the purpose of the project was to provide a proof that explains why the $Var(\sqrt{X})$ becomes fixed at ¼ as its parameter λ, which equals the mean and variance, approaches infinity.

Using the equation above, the mean, E(X), is already known to be λ, so once $E(\sqrt{X})$ is found, $Var(\sqrt{X})$ can be calculated. Given that

$$\sqrt{k} = \sum_{n=0}^{3} (k-\lambda)^n \frac{f^{(n)}(\lambda)}{n!} + R(k)$$

it would be difficult to calculate the summation, due to the $\sqrt{k}$ term. Therefore, an indirect approach, such as a Taylor approximation, has to be used to solve this problem, since the typical values of k lie in the range of $\lambda \pm \sqrt{\lambda}$.

The Taylor approximation for any real number $\sqrt{k}$ is:

$$(*) \quad \sqrt{k} = \sum_{n=0}^{3} (k-\lambda)^n \frac{f^{(n)}(\lambda)}{n!} + R(k),$$

where R(k) is the remainder term, $n!$ denotes the factorial of n, and $f^{(n)}(\lambda)$ is the nth derivative of a function, f(x)= $\sqrt{x}$ (for this case), evaluated at a point a = λ.

The following are the values calculated for the first 3 values of n:

For:

$$n = 0: \frac{(k-\lambda)^0\sqrt{\lambda}}{0!} = \sqrt{\lambda}$$

$$n = 1: \frac{(k-\lambda)\frac{1}{2\sqrt{\lambda}}}{1!} = \frac{(k-\lambda)}{2\sqrt{\lambda}}$$

$$n = 2: \frac{(k-\lambda)^2\frac{-1}{4\lambda^{3/2}}}{2!} = -\frac{(k-\lambda)^2}{8\lambda^{3/2}}$$

$$n = 3: \frac{(k-\lambda)^3\frac{3}{8\lambda^{5/2}}}{3!} = \frac{(k-\lambda)^3}{16\lambda^{5/2}}$$

These values can then be substituted into the summation to make the proof easier to calculate. In order to do this, we first need to create an upper bound of the remainder for all values of k.

For small values of k, the remainder can be calculated using the Lagrange form:

$$R(k) = \frac{f^{n+1}(\xi)}{(n+1)!}(x-\lambda)^{n+1} \quad \text{where} \quad \lambda < \xi < k$$

Since this remainder is evaluated at some unknown point ξ, we can create an upper bound on R(k) for small values (and eventually for all values) of k.

In this case, for R(k), the bound can be $|R(k)| \le C\frac{(k-\lambda)^4}{\lambda^{7/2}}$ for some constant C $\geq \frac{f^{n+1}(\xi)}{(n+1)!}$.

The remainder of large values of k can be calculated by the following:

Given that $\sqrt{k} = \sum_{n=0}^{3}(k-\lambda)^n \frac{f^n(\lambda)}{n!} + R(k)$,

$$R(k) = \sqrt{k} - \sqrt{\lambda} - \frac{k-\lambda}{2\sqrt{\lambda}} + \frac{(k-\lambda)^2}{8\lambda^{3/2}} - \frac{(k-\lambda)^3}{16\lambda^{5/2}}$$

In this term, the k^3 term is dominant, so the equation of R(k) will act like a cubic for large values of k. In order to create an upper bound for this function, we will need to choose a quartic equation because it grows faster than a cubic function for large values of k. Therefore, we can select a large constant, C_2, for the quartic function such that it grows faster than the cubic function at a point we desire. In this situation, since we already have an upper bound for small values of k, if we choose C_2 such that for all values of k excluding the small values of k, the quartic function grows faster than the cubic function, then we can establish an upper bound for the remainder term that can be used to evaluate the error term for the Taylor polynomial.

Therefore, it can be stated that a bound for R(k) exists such that $|R(k)| \leq C_2\frac{(k-\lambda)^4}{\lambda^{7/2}}$ for a large value of C_2 that satisfies the above condition. Thus, substituting the value of the Taylor approximation of $\sqrt{k}$ into $E(\sqrt{X}) = \sum_{k=1}^{\infty}\sqrt{k}\frac{\lambda^k}{k!}e^{-\lambda}$, we have

$$E(\sqrt{X}) = \sum_{k=1}^{\infty}\left(\sqrt{\lambda} + \frac{k-\lambda}{2\sqrt{\lambda}} - \frac{(k-\lambda)^2}{8\lambda^{3/2}} + \frac{(k-\lambda)^3}{16\lambda^{5/2}} + R(k)\right)\frac{\lambda^k}{k!}e^{-\lambda}$$

where $|R(k)| \leq C_3\frac{(k-\lambda)^4}{\lambda^{7/2}}$ for some C_3 larger than C and C_2.

Multiplying and separating out the terms, we have the following:

$$E(\sqrt{X}) = \frac{5\sqrt{\lambda}}{16}\sum_{k=1}^{\infty}\left(\frac{\lambda^k}{k!}e^{-\lambda}\right) + \frac{15}{16\sqrt{\lambda}}\sum_{k=1}^{\infty}\left(k\frac{\lambda^k}{k!}e^{-\lambda}\right) - \frac{5}{16\lambda^{3/2}}\sum_{k=1}^{\infty}\left(k^2\frac{\lambda^k}{k!}e^{-\lambda}\right) + \frac{1}{16\lambda^{5/2}}\sum_{k=1}^{\infty}\left(k^3\frac{\lambda^k}{k!}e^{-\lambda}\right) + \left|\sum_{k=1}^{\infty}\left(R(k)\frac{\lambda^k}{k!}e^{-\lambda}\right)\right|$$

$$E(\sqrt{X}) = \frac{5\sqrt{\lambda}}{16}\sum_{k=1}^{\infty}\left(\frac{\lambda^k}{k!}e^{-\lambda}\right) + \frac{15}{16\sqrt{\lambda}}\sum_{k=1}^{\infty}\left(k\frac{\lambda^k}{k!}e^{-\lambda}\right) - \frac{5}{16\lambda^{3/2}}\sum_{k=1}^{\infty}\left(k^2\frac{\lambda^k}{k!}e^{-\lambda}\right) + \frac{1}{16\lambda^{5/2}}\sum_{k=1}^{\infty}\left(k^3\frac{\lambda^k}{k!}e^{-\lambda}\right) + \sum_{k=1}^{\infty}\left(R(k)\frac{\lambda^k}{k!}e^{-\lambda}\right) \quad (**)$$

Many of these summations are very difficult to solve. With the help of Moment Generating Functions, however, the complex summations can be evaluated.

A Moment Generating Function, in probability, generates the moments of a probability distribution. The moments can be generated using M(t) = E(e^{Xt}), the expectation value of e^{Xt}.

Thus, using M(t) = $e^{\lambda(e^t-1)}$, the different moments can be calculated by taking derivatives of the function. This function can be evaluated at time t = 0 to generate the following values:

$$M(0) = \sum_{k=1}^{\infty}\frac{\lambda^k}{k!}e^{-\lambda} = e^{-\lambda}e^{\lambda e}\Big|_{t=0} = 1$$

$$M'(0) = \sum_{k=1}^{\infty}k\frac{\lambda^k}{k!}e^{-\lambda} = \lambda e^t e^{\lambda(e^t-1)}\Big|_{t=0} = \lambda$$

Similarly, if further derivatives are taken, the following moments are generated:

$$M''(0) = \sum_{k=1}^{\infty}k^2\frac{\lambda^k}{k!}e^{-\lambda} = (\lambda e^t)^2 e^{\lambda(e^t-1)} + \lambda e^t e^{\lambda(e^t-1)}\Big|_{t=0} = \lambda^2 + \lambda$$

$$M'''(0) = \sum_{k=1}^{\infty}k^3\frac{\lambda^k}{k!}e^{-\lambda} = \lambda e^{\lambda(e^t-1)}\left(3\lambda e^{2t} + \lambda^2 e^{3t} + e^t\right)\Big|_{t=0} = \lambda^3 + 3\lambda^2 + \lambda$$

$M^{(4)}(0) = \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda$ (Figure 1)

$$M^{(4)}(0) = \sum_{k=1}^{\infty}k^4\frac{\lambda^k}{k!}e^{-\lambda} = \lambda e^{\lambda(e^t-1)}\left(6\lambda e^{2t} + 3\lambda^2 e^{3t} + e^t\right) + \lambda^2 e^t e^{\lambda(e^t-1)}\left(3\lambda e^{2t} + \lambda^2 e^{3t} + e^t\right)\Big|_{t=0} = \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda$$

Figure 1.

Thus, with substitution into (**), the following is obtained:

$$E\left(\sqrt{X}\right)=\frac{5\sqrt{\lambda}}{16}+\frac{15\lambda}{16\sqrt{\lambda}}-\frac{5\lambda^2-5\lambda}{16\lambda^{3/2}}+\frac{\lambda^3+3\lambda^2+\lambda}{16\lambda^{5/2}}+\sum_{k=1}^{\infty}R(k)\frac{\lambda^k}{k!}e^{-\lambda}$$

$$\begin{aligned}\left|\sum_{k=1}^{\infty}R(k)\frac{\lambda^k}{k!}e^{-\lambda}\right| &\le C_3\sum_{k=1}^{\infty}\frac{k^4-4k^3\lambda+6k^2\lambda^2-4k\lambda^3+\lambda^4}{\lambda^{7/2}}\frac{\lambda^k}{k!}e^{-\lambda}\\ &\le C_3\frac{(\lambda^4+6\lambda^3+7\lambda^2+\lambda)-4(\lambda^3+3\lambda^2+\lambda)\lambda+6(\lambda^2+\lambda)\lambda^2-4(\lambda)\lambda^3+\lambda^4}{\lambda^{7/2}}\\ &\le C_3\frac{3\lambda^2+\lambda}{\lambda^{7/2}}\\ &\le \frac{C_4}{\lambda^{3/2}}\end{aligned}$$

for some C_4 larger than C_3

$$E\left(\sqrt{X}\right)=\sqrt{\lambda}-\frac{1}{8\sqrt{\lambda}}+\frac{1}{16\lambda\sqrt{\lambda}}+\frac{O}{\lambda^{3/2}}$$

$$\left(E\left(\sqrt{X}\right)\right)^2=-\frac{1}{4}+\lambda+\frac{9}{64\lambda}-\frac{1}{64\lambda^2}+\frac{O}{\lambda}=-\frac{1}{4}+\lambda+\frac{O}{\lambda}$$

Thus,

$$\begin{aligned}Var\left(\sqrt{X}\right)&=E(X)-\left(E\left(\sqrt{X}\right)\right)^2=\lambda+\frac{1}{4}-\lambda+\frac{O}{\lambda}\\ &=\frac{1}{4}+\frac{O}{\lambda}\end{aligned}$$

Finally, $\lim_{\lambda\to\infty}Var(\sqrt{X})=\frac{1}{4}$ because $\frac{O}{\lambda}\to 0$ as $\lambda\to\infty$

Remark: By using more terms in the Taylor expression (*), we can obtain more accurate asymptotical formulas for $Var\left(\sqrt{X}\right)$. In particular, we can show that

$Var\left(\sqrt{X}\right)=\frac{1}{4}+\frac{b_1}{\lambda}+\frac{b_2}{\lambda^2}+$ K and compute the values of b1, b2, etc.