2023

Canonical Laminations for Fixed Point Portraits

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A THESIS

Submitted to the faculty of the University of Alabama at Birmingham,
in partial fulfillment of the requirements of the degree of
Master of Science

BIRMINGHAM, ALABAMA

2023

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APPLIED MATHEMATICS

ABSTRACT

Laminations are a combinatorial and topological model for studying the Julia sets of complex polynomials. Every complex polynomial of degree d has d fixed points counted with multiplicity. From the point of view of laminations, exactly $d-1$ of these fixed points are peripheral (approachable from outside the Julia set of the polynomial). Hence, at least one of the d fixed points is "hidden" from the laminational point of view. The purpose of this thesis is to identify, classify and count the possible fixed point portraits for any lamination of degree d. We will identify the "simplest" lamination for a given fixed point portrait and will show that there are polynomials that have these simplest laminations. We extend σ_d to $\overline{\mathbb{D}}$ as a branched covering map. In future work with others, we want to apply Thurston's criterion to show there exists a complex polynomial whose lamination this is.

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1. INTRODUCTION

Thurston [**14**] introduced invariant lamination to study polynomials and their parameter space. His idea was to use *laminiations* as a tool to model the polynomial dynamics as a combinatorial or as a topological object. He discussed how to associate a lamination to a polynomial P of degree d with a locally connected Julia set.

In the paper [**12**], Goldberg and Milnor introduced the concept of fixed point portraits from the polynomial point of view. This served as motivation for our laminational study.

A natural question arises, can we reverse the process? Is it possible to realize a polynomial whose Julia set has a given lamination? To answer these questions, we need a branched covering map on the closed disk $\overline{\mathbb{D}}$. In Section 3 we extend the angle d-tupling map on the circle to a branched covering map of the unit disk. This sets the stage for applying Thurston's criterion to show the existence of a polynomial whose Julia set has this lamination.

In Section 4 we introduced the concept of fixed point portraits for laminations and count the number of fixed point portraits of a given degree d (Theorem 4.6).

In Section 5 we discuss the simplest lamination for a given fixed point portrait, which we call the canonical lamination for this fixed point portrait. To do this we discuss the critical portraits that are compatible with the given fixed point portrait (Theorems 5.4 and 5.5). We also show a canonical fixed point portrait has certain fundamental properties (Theorem 5.7). The angle d-tupling map has only $d-1$ fixed points. We also show that every complementary region of a fixed point portrait has a unique internal fixed point (Theorem 5.9).

In the final section, we discuss our ongoing and future work and provide some examples of Julia sets from the ongoing work.

I want to thank the UAB laminations seminar for helping me to understand the relationship between laminations and complex polynomials. In particular, I would like to thank Dr. John Mayer, Dr. Nikita Selinger, Dr. Lex Oversteegen, Dr. Alexander Blokh, Brittany Burdette, Adam Carty and Thomas Sirna.

2. PRELIMINARIES

Basic definitions in this section have been adapted from [**2**] and [**3**]. We measure angles on the unit circle $\mathbb{S}^1$ in revolutions rather than radians. Thus, the circle will be coordinatized by $[0, 1)$.

DEFINITION 2.1 (Angle d-tupling map). Let $d \in \mathbb{N}$. The *angle d-tupling map* $\sigma_d : \mathbb{S}^1 \to \mathbb{S}^1$ is defined by $\sigma_d(t) = dt \pmod 1$.

DEFINITION 2.2 (Lamination). A *lamination* $\mathcal{L}$ is a collection of chords of the closed unit disk $\overline{\mathbb{D}}$, which we call *leaves*, with the property that any two leaves meet, if at all, in a point of the boundary $\mathbb{S}^1$ of the disk and $\mathcal{L}^* := \mathbb{S}^1 \cup \{\cup\mathcal{L}\}$.

DEFINITION 2.3 (Rotational Set). Consider σ_d: $\mathbb{S}^1 \to \mathbb{S}^1$ for a particular $n \geq 2$.

Let $P = \{x_i | 0 < x_1 < x_2 < x_3 < ... < x_k < 1\}$ be a finite set in consecutive order in $\mathbb{S}^1$. We say that P is a *rotational set* (for σ_d) if and only if

(1) $\sigma_d(P) = P$, and
(2) For $1 \leq j \leq k$, if $\sigma_d(x_j) = x_i$, set $i(j) = i$. Then for all j, $j - i(j) (\text{mod } k)$ is the same.

If (2) holds (but possibly not (1)), we say σ_d is circular order-preserving on P. We call a rotational set which is a single periodic orbit, a *rotational orbit*.

DEFINITION 2.4 (Rotation Number). To each rotational set, we can assign a *rotation number*, a rational number $0 \leq \frac{p}{q} < 1$ in the lowest terms.

Let $\mathcal{O} = \{x_1 < x_2 < x_3 ... < x_q\}$ be a rotational periodic orbit. Suppose that $\sigma_d(x_1) = x_j$. Set $p = j - 1$. The rotation number of $\mathcal{O}$ is $\frac{p}{q}$. Our notation is $\rho(\mathcal{O}) = \rho(x_1) = \rho(x_i) = \frac{p}{q}$.

Definition 2.5 (Critical Chord). A chord under σ_d is called *critical* if, and only if, both of its endpoints (and so the whole chord) map to a single point on the circle $\mathbb{S}^1$.

Definition 2.6 (Critical Portrait). A maximal collection $\mathcal{C}$ of critical chords for σ_d, meeting at most at endpoints, is called a *critical portrait*. The closure of a component of $\overline{\mathbb{D}} \setminus \cup \mathcal{C}$ that meets $\mathbb{S}^1$ in arcs is called a *critical sector*. A polygon entirely composed of critical chords is called an *all critical polygon*.

Definition 2.7 (Sibling Invariant Lamination). A lamination $\mathcal{L}$ is said to be *sibling d-invariant* (or simply *invariant* if no confusion will result) provided that the following three statements hold:

(1) (Forward Invariant) For every $\ell \in \mathcal{L}$, $\sigma_d(\ell) \in \mathcal{L}$.
(2) (Backward Invariant) For every non-degenerate $\ell' \in \mathcal{L}$, there is a leaf $\ell \in \mathcal{L}$ such that $\sigma_d(\ell) = \ell'$.
(3) (Sibling Invariant) For every $\ell_1 \in \mathcal{L}$ with $\sigma_d(\ell_1) = \ell'$, a non-degenerate leaf, there is a full sibling collection $\{\ell_1, \ell_2, \ldots, \ell_d\} \subset \mathcal{L}$ such that $\sigma_d(\ell_i) = \ell'$.

A sibling d-invariant lamination induces an equivalence relation. Two points on $\mathbb{S}^1$ are equivalent if they are joined by a finite concatenation of leaves. We consider laminations for which this results in a closed equivalence relation. Thus, the sibling invariant laminations we will be considering have a fourth condition from [**3**] not listed in the definition:

(4) $\mathcal{L}$ has finite equivalence classes, and all leaves are boundary chords of the convex hulls of equivalence classes.

Definition 2.8 (Gap). A *gap* in a lamination $\mathcal{L}$ is the closure of a component of $\overline{\mathbb{D}} \setminus \mathcal{L}^*$. A gap is *critical* if and only if two points in its boundary map to the same point. A gap with finitely many leaves in its boundary is usually called a *polygon*. The leaves bounding a finite gap are called the *sides* of the polygon.

Definition 2.9 (Fatou gap). A *Fatou gap* in lamination is a gap whose boundary intersected with $\mathbb{S}^1$ contains a Cantor set.

Definition 2.10 (Hyperbolic Lamination). A d-invariant lamination is said to be *hyperbolic* if and only if all compatible critical chords are interior to periodic or pre-periodic Fatou gaps.

Theorem 2.11. *For σ_d, in a hyperbolic lamination, an invariant gap is either a rotational polygon or a critical Fatou gap.*

Proof. In a hyperbolic lamination, all criticality is inside Fatou gaps. We know, if A is a compact set invariant under σ_d such that σ_d is one-to-one on A, then A is finite [**10**]. Therefore, if σ_d is one-to-one, then the invariant gap is a rotational polygon. If it is not one-to-one, then the invariant gap is a critical Fatou gap. □

Theorem 2.12 (Uniqueness of Rotation Number [**9**]). *An invariant set in a given critical sector has a unique rotation number.*

The consequences of this theorem are:

- There can not be a fixed point and a rotational set with a nonzero rotation number in the same critical sector.
- Two rotational critical sectors can not be adjacent.

Definition 2.13 (Branches of the Inverse [**6**]). Let C be a critical portrait. Every critical sector S in $\overline{\mathbb{D}}$ defined by C will have a function $\tau : \mathbb{S}^1 \to S \cap \mathbb{S}^1$ that is one to one, and $\sigma_d \circ \tau$ is the identity on $\mathbb{S}^1$.

Combining the previous definitions we can now define a pullback scheme.

Definition 2.14 (Pullback Scheme [**6**]). Let C be a critical portrait and F a compatible forward invariant set. The corresponding collection $PB(F, C)$ of the branches of the inverse determined by C as in Definition 2.13 gives us a *pullback scheme* for F.

There can be multiple critical portraits compatible with F that define the pullback scheme differently. There are some critical portraits that are compatible with F, but the related pullback lamination does not satisfy condition (4) of Definition 2.7. The following theorem is well known.

THEOREM 2.15 (Pullback [**6**]). *Let F be a periodic forward invariant set under σ_d, C a compatible critical portrait, and the pullback scheme $PB(F, C) = \{\tau_1, \tau_2, \ldots, \tau_d\}$. Let $F_0 = F$, and $F_1 = F_0 \cup \tau_1(F_0) \cup \tau_2(F_0) \cup \cdots \cup \tau_d(F_0)$. In general, for any given stage n of the pull back $F_n = F_{n-1} \cup \tau_1(F_{n-1}) \cup \cdots \cup \tau_d(F_{n-1})$. Let $F_\infty = \bigcup_{n=0}^{\infty} F_n$, and let $\mathcal{L} = \overline{F_\infty}$. Then, $\mathcal{L}$ is a sibling d-invariant lamination.*

DEFINITION 2.16 (Covering Map [**13**]). A *covering map* is a surjective open map $f : X \to Y$ that is locally a homeomorphism, meaning that each point in X has a neighborhood that is the same after mapping f in Y.

DEFINITION 2.17 (Branched Covering Map [**15**]). In topology, a map is a *branched covering* if it is a covering map everywhere except for a nowhere-dense set known as the branch set.

3. EXTENDING σ_d TO THE DISC $\overline{\mathbb{D}}$ AS A BRANCHED COVERING MAP

THEOREM 3.1. *Let $\mathcal{L}$ be a hyperbolic lamination. Then there is a branched covering map $\sigma_d^{\#}$, a continuous extension of σ_d, from $\overline{\mathbb{D}}$ to $\overline{\mathbb{D}}$ that maps $\mathcal{L}$ to $\mathcal{L}$.*

We will construct the branched covering map by extending the map σ_d in several steps.

Step 1: In this step, we will extend σ_d from $\mathbb{S}^1$ to $\mathcal{L}^*$ as defined in 2.2.

We define action of σ_d on $\mathcal{L}$ (see 2.2) by

$$\sigma_d(\overline{ab}) = \overline{\sigma_d(a)\sigma_d(b)}, \forall \overline{ab} \in \mathcal{L}$$

We parametrize each leaf $\overline{ab}$ by [0, 1], i.e., each point on $\overline{ab}$ lies on the line segment $(1-t)*a+t*b$ for $t \in [0,1]$. By $(1-t)*a+t*b$, we mean the point on the leaf $\overline{ab}$, t part of the distance from a to b in the Euclidean plane.

We define the extension of σ_d on $\mathcal{L}^*$ by

$$\sigma_d^*((1-t)*a+t*b) = (1-t)*\sigma_d(a)+t*\sigma_d(b)$$

Using the limit definition of continuity, we will show that σ_d^* is continuous on $\mathcal{L}^*$.

(i) It is continuous from one leaf to its image.

PROOF. Let $(p_i)_{i\in\mathbb{N}}$ be a sequence of points on a leaf $\overline{ab} \in \mathcal{L}$ converges to a point p on $\overline{ab}$. We need to show $\sigma_d^*(p_i)$ converges to $\sigma_d^*(p)$ on $\overline{\sigma_d(a)\sigma_d(b)}$. Let $p_i = (1-t_i)*a+t_i*b$ and $p = (1-t)*a+t*b$, where, $t_i, t \in [0,1]$. Since $(p_i)_{i\in\mathbb{N}}$ converges to p, the t_i associated with p_i converge to t associated with p on $\overline{ab}$. Therefore, t_i associated with $\sigma_d^*(p_i)$ converge to t associated with $\sigma_d^*(p)$ on $\overline{\sigma_d(a)\sigma_d(b)}$. Thus, $\sigma_d^*(p_i)$ converge to $\sigma_d^*(p)$ on $\overline{\sigma_d(a)\sigma_d(b)}$. Hence, σ_d^* is continuous from one leaf to its image. □

(ii) It is continuous on $\mathcal{L}^*$.

PROOF. Let $(p_i)_{i\in\mathbb{N}}$ be a sequence of points on $\mathcal{L}^*$ converges to a point p on $\mathcal{L}^*$. We need to show $\sigma_d^*(p_i)$ converges to $\sigma_d^*(p)$.

Let each point p_i of the sequence $(p_i)_{i\in\mathbb{N}}$ lies on some leaf $\overline{a_ib_i}$ on $\mathcal{L}^*$.

Case I: Suppose there are infinitely many points lie on the same leaf that contains p. We consider these points $(p_i)_{i\in\mathbb{N}}$ as a co-final sub-sequence $(p_{n_i})_{n_i\in\mathbb{N}}$. Thus $(p_{n_i})_{n_i\in\mathbb{N}}$ is an infinite subsequence of $(p_i)_{i\in\mathbb{N}}$. Since $(p_i)_{i\in\mathbb{N}}$ converges to p, $(p_{n_i})_{n_i\in\mathbb{N}}$ also converges to p. Since σ_d^* is continuous from one leaf to its image, $\sigma_d^*(p_{n_i})$ converges to $\sigma_d^*(p)$. Therefore, $\sigma_d^*(p_i)$ converges to $\sigma_d^*(p)$. Hence, σ_d^* is continuous on $\mathcal{L}^*$.

Case II: The leaf on $\mathcal{L}^*$ that contains p, contains only finitely many points of $(p_i)_{i\in\mathbb{N}}$. First, we construct a co-final sub-sequence $(p_{n_i})_{n_i\in\mathbb{N}}$ out of the points of $(p_i)_{i\in\mathbb{N}}$ that lie on different leaves of $\mathcal{L}^*$. To do this, we consider p_1 as p_{n_1}. Now, if p_2 lies on the same leaf as p_1, we do not include p_2 in $(p_{n_i})_{n_i\in\mathbb{N}}$, instead we move to the point p_3. If p_3 lies on the different leaf as p_1, then we consider p_3 as p_{n_2}, and so on. This way we take exactly one point from each leaf. Now, since $(p_{n_i})_{n_i\in\mathbb{N}}$ is co-final, the sequence of the endpoints (a_{n_i}) and (b_{n_i}) of the leaf that contains (p_{n_i}) are also co-final. Therefore, the sequences (a_{n_i}) and (b_{n_i}) converges to the endpoints a and b respectively of the leaf that contains p. Thus, for every $\epsilon > 0$, there exists a neighborhood $N_\epsilon(\overline{ab})$ that contains all but finitely many points (a_{n_i}) and (b_{n_i}). Since the leaves never cross each other and are straight lines, all but finitely many leaves lie inside $N_\epsilon(\overline{ab})$. As a result, $p_{n_i} = (1 - t_{n_i}) * a_{n_i} + t_{n_i} * b_{n_i}$ converges to $p = (1 - t) * a + t * b$. Again, since p_{n_i} converges to p, there is a $N_\delta(p) \subset N_\epsilon(\overline{ab})$ that contains all but finitely many points of (p_{n_i}). Now, as δ gets smaller (p_{n_i}) get closer to p. We have the point $p = (1 - t) * a + t * b$ lies on t part of the distance from a to b in the Euclidean plane, and the point $p_{n_i} = (1 - t_{n_i}) * a_{n_i} + t_{n_i} * b_{n_i}$ lies on the t_{n_i} part of the distance from a_{n_i} to b_{n_i} in the Euclidean plane. As a result, whenever p_{n_i} gets closer to p, we have t_{n_i} gets closer to t. Therefore, every neighborhood of t contains finitely many t_{n_i}. Thus, t_{n_i} associated with (p_{n_i}) on $\overline{a_{n_i}b_{n_i}}$ converges to t associated with p on $\overline{ab}$. Now, since σ_d^* is continuous from one leaf to its image, (t_{n_i}) associated with $\sigma_d^*(p_{n_i})$ on $\overline{\sigma_d^*(a_{n_i})\sigma_d^*(b_{n_i})}$ converges to t associated with $\sigma_d^*(p)$ on $\overline{\sigma_d^*(a)\sigma_d^*(b)}$. So, $\sigma_d^*(p_{n_i})$ converges to $\sigma_d^*(p)$. Therefore, σ_d^* is continuous on $\mathcal{L}^*$ in this case.

Case III: The point lies on the circle $\mathbb{S}^1$.

Sub-case I: Infinitely many of the leaves $\overline{a_{n_i}b_{n_i}}$ that contain points of $(p_{n_i})_{n_i\in\mathbb{N}}$ go over p. In this case, as the leaves get nearer to p, their lengths shorten, and both a_{n_i} and b_{n_i} converge to p. Since σ_d is continuous on $\mathbb{S}^1$, $\sigma_d(a_{n_i})$ and $\sigma_d(b_{n_i})$ also converge to

$\sigma_d(p)$ on $\mathbb{S}^1$. Therefore, σ_d^* is continuous.

Sub-case II: Infinitely many of the leaves $\overline{a_{n_i}b_{n_i}}$ that contain points of $(p_{n_i})_{n_i\in\mathbb{N}}$ go over p, and only the (a_{n_i}) or (b_{n_i}) converges to p. In this case, (t_{n_i}) converges to 0 or 1 respectively. But the parametrization is a homomorphism from [0,1] to the segment of the leaves. Therefore, convergence in the domain and convergence in the range is equivalent. Hence, σ_d^* is continuous.

Sub-case III: Infinitely many of the leaves $\overline{a_{n_i}b_{n_i}}$ that contain $(p_{n_i})_{n_i\in\mathbb{N}}$ lie on one side of p. In this case, as the leaves get nearer to p, their lengths shorten. Therefore, the image of parametrization [0,1] goes to 0. Hence, both (a_{n_i}) and (b_{n_i}) converge to p. Now, since σ_d is continuous on $\mathbb{S}^1$, $\sigma_d(a_{n_i})$ and $\sigma_d(b_{n_i})$ also converge to $\sigma_d(p)$ on $\mathbb{S}^1$. Therefore, σ_d^* is continuous in this case.

□

Hence, σ_d^* is continuous on $\mathcal{L}^*$.

Step 2: In this step we want to extend σ_d^* from $\mathcal{L}^*$ to $\overline{\mathbb{D}}$. So, we need to extend the map to the gaps.

LEMMA 3.2. *The area of gaps goes to 0 for any infinite sequence of disjoint gaps.*

PROOF. We know the total area of gaps is bounded by the area of the unit circle which is finite. By the way of contradiction, let every gap has an area some $\epsilon > 0$. Then the total area of gaps for any infinite sequence of disjoint gaps is $\infty.\epsilon = \infty$, which is a contradiction. Therefore, The area of gaps goes to 0 for any infinite sequence of disjoint gaps. □

LEMMA 3.3. *Given any infinite sequence of disjoint gaps, either: (a) Their diameters converge to 0. In this case, the gaps converge to points on the circle. Because the vertices of the gaps lie on the circle, all of them meet at a point on $\mathbb{S}^1$.*

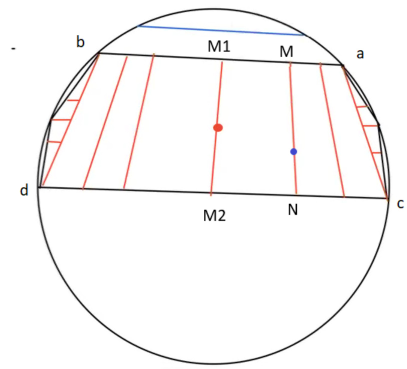

FIGURE 3.1. Coordination of center and other points inside a gap with two longest sides

Or, (b) All but finitely many gaps have two longest sides and other sides converging to length 0. In this case, the gaps converge to leaves of $\mathcal{L}$.

DEFINITION 3.4 (Center of a gap). Let $\overline{ab}$ and $\overline{cd}$ be two longest sides of a gap G. As we have parametrized each leaf of $\mathcal{L}$ by $[0,1]$, suppose the midpoint of $\overline{ab}$ is $M_1 = \frac{1}{2}(a+b)$ and the midpoint of $\overline{cd}$ is $M_2 = \frac{1}{2}(c+d)$. We define the midpoint c of M_1M_2 as the *center of the gap* G (see Figure 3.1). If G does not have two longest sides or sibling to a critical gap, let c denote its barycenter (see Figure 3.2).

Sub-step 1: First, we want to extend σ_d^* to the centers of gaps.

Let $C = \{c \in \overline{D} : c \text{ is the center of a gap}\}$. Let c_1 and c_2 be the centers of gaps G_1 and G_2 respectively, where $\sigma_d^*(G_1) = G_2$.
We define the extension of σ_d^* on C by $\sigma_d^{\#}(c_1) = c_2$.

We will show that $\sigma_d^{\#}$ is continuous in this case, i.e., $\sigma_d^{\#} : \mathcal{L}^* \cup C \to \mathcal{L}^* \cup C$ is continuous.
Let $(c_i)_{i\in\mathbb{N}}$ be a sequence of centers of the gaps that converges to some point $p \in \mathcal{L}^* \cup C$.

Case I: p is on $\mathbb{S}^1$.

PROOF. Since p is on $\mathbb{S}^1$ and $(c_i)_{i\in\mathbb{N}}$ converges to p, the diameters of the gaps G_i containing c_i converge to 0. Consequently, the gaps entirely converge to p. Now since p lies on $\mathbb{S}^1$, $\sigma_d^{\#}(p)$ also lies on $\mathbb{S}^1$; and the gaps $\sigma_d^{\#}(G_i)$ entirely converge to $\sigma_d^{\#}(p)$. Consequently, $\sigma_d^{\#}(c_i)$ converges to $\sigma_d^{\#}(p)$. Therefore, $\sigma_d^{\#}$ is continuous in this case. □

Case II: p is on some leaf $\overline{ab}$.

PROOF. Since the longest two sides of each gap converge to the leaf $\overline{ab}$ and t are preserved on the leaves, every point on the line segment joining the midpoints converges to the midpoint of $\overline{ab}$. Thus, p is the midpoint of $\overline{ab}$.
Let M_{1_i} and M_{2_i} be the midpoints of the two longest sides of the gaps G_i. Now, since $(c_i)_{i\in\mathbb{N}}$ converges to p, $(M_{1_i})_{i\in\mathbb{N}}$ and $(M_{2_i})_{i\in\mathbb{N}}$ converge to p. We know σ_d^* is continuous on $\mathcal{L}^*$. Thus, $\sigma_d^*(M_{1_i})$ and $\sigma_d^*(M_{2_i})$ converge to $\sigma_d^*(p)$. Consequently, every point on the line segment $\overline{\sigma_d^*(M_{1_i})\sigma_d^*(M_{2_i})}$ converge to $\sigma_d^*(p)$. Therefore, the images of the centers $\sigma_d^{\#}(c_i)$ converge to $\sigma_d^*(p)$ i.e., to $\sigma_d^{\#}(p)$. Hence, $\sigma_d^{\#}$ is continuous in this case. □

Sub-step II: Second, we want to extend σ_d^* inside a gap, and to prove that $\sigma_d^{\#}$ is continuous from one gap to its image.

DEFINITION 3.5 (Coordinate of a point inside a gap). Let p be a point inside a gap G, and let c be the center of G.

- *Fill-up construction:* If the gap has two longest sides and is not a sibling of a critical gap, then we do the fill-up construction. Let $\overline{ab}$ and $\overline{cd}$ be the two longest sides of this gap G. As we have parametrized each leaf of $\mathcal{L}$ by $[0,1]$, suppose the midpoint of $\overline{ab}$ is $M_1 = \frac{1}{2}(a+b)$ and the midpoint of $\overline{cd}$ is $M_2 = \frac{1}{2}(c+d)$. We connect M_1 and M_2 by the line segment M_1M_2. Let p be any point inside G. We draw a unique line segment M_tN_t through p where M_t and N_t have the same t parameter on lines $\overline{ab}$ and $\overline{cd}$ respectively (see Figure 3.1). We denote the coordinate of p as $(1-s)*M_t + s*N_t$.

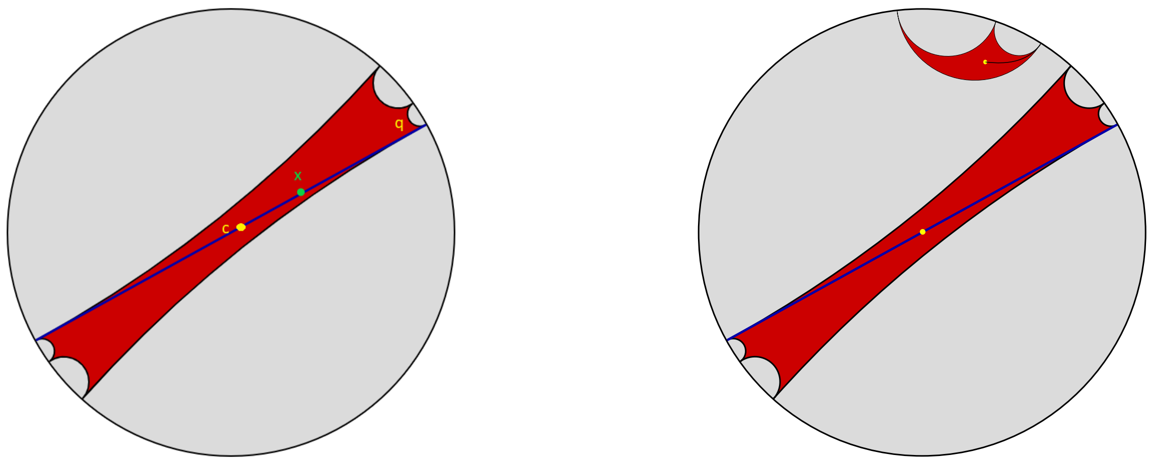

FIGURE 3.2. Coordination of center and other points inside a critical gap

We define the extension of σ_d^* in G as

$$\sigma_d^{\#}((1-s)*M_t + s*N_t) = (1-s)*\sigma_d^*(M_t) + s*\sigma_d^*(N_t).$$

This covers the part of the gap bounded by M_0N_0 and M_1N_1. For the remaining part, we could do a "horizontal" foliation as shown in Figure 3.1. Since the regions that are bounded by M_0N_0 and the part of the arc of ∂G between a and c will be shown to converge to a point on a leaf, we omit the details of this foliation.

- *Coning construction:* On the remaining gaps we do the barycentric construction. Let the barycenter c be the coning point for this kind of gap G. Again let p be any point in G. We connect c and p by a line segment and extend it to the boundary of G. Let the line segment touches the boundary of G at q (see Figure 3.2). We denote the coordinate of p as $(1-s)*c+s*q$. This coordination is unique since the line segment joining c and q is unique. We define the extension of σ^* in G as

$$\sigma_d^{\#}((1-s)*c+s*q) = (1-s)*\sigma_d^{\#}(c) + s*\sigma_d^*(q).$$

Let $(p_i)_{i\in\mathbb{N}}$ be sequence of points inside a gap G converges to a point p inside G.

Case I: The gap G has fill-up construction.

PROOF. We draw the line segments $M_{t_i}N_{t_i}$ through the points p_i connecting the points on the longest two sides of G with the same t_i values. Similarly, we draw the line segment M_tN_t through p. Since p_i converges to p, $M_{t_i}N_{t_i}$ converges to M_tN_t. Again since $\sigma_d^{\#}$ is continuous on $\mathcal{L}^*$, $\sigma_d^{\#}(M_{t_i}N_{t_i})$ converges to $\sigma_d^{\#}(M_tN_t)$. Let $p_i = (1-s_i)*M_{t_i} + s_i * N_{t_i}$ and $p = (1-s)*M_t + s*N_t$, where, $s_i, s \in [0,1]$. Since $(p_i)_{i\in\mathbb{N}}$ converges to p, the s_i associated with p_i converges to s associated with p on M_tN_t. Therefore, s_i associated with $\sigma_d^{\#}(p_i)$ converge to s associated with $\sigma_d^{\#}(p)$ on $\sigma_d^{\#}(M_t)\sigma_d^{\#}(N_t)$. Thus, $\sigma_d^{\#}(p_i)$ converge to $\sigma_d^{\#}(p)$ on $\sigma_d^{\#}(M_t)\sigma_d^{\#}(N_t)$. Hence, $\sigma_d^{\#}$ is continuous.

Also, let infinitely many points of p_i and p lie on the regions of "horizontal" foliation of G. Then since M_0N_0 and M_1N_1 map to $\sigma_d^{\#}(M_0)\sigma_d^{\#}(N_0)$ and $\sigma_d^{\#}(M_1)\sigma_d^{\#}(N_1)$ respectively, the points $\sigma_d^{\#}(p_i)$ and $\sigma_d^{\#}(p)$ lie on the "horizontal" foliation of $\sigma_d^{\#}(G)$. Therefore, the continuity of $\sigma_d^{\#}$ follows from the facts that $\sigma_d^{\#}(p_i)$ is continuous on $\mathcal{L}^*$ and on "vertical" foliation from one gap to its image.

Hence, $\sigma_d^{\#}$ is continuous from G to its image. $\square$

Case II: The gap G has coning construction.

PROOF. Let c be the center of G. We draw line segments from c to p_i and extend them till they intersect sides of G. Let the intersection points be q_i. Also, we draw a line segment from c to p and extend it till it intersects a side of G. Let the intersection point be q. Since p_i converges to p, q_i converges to q. Again since $\sigma_d^{\#}$ is continuous on $\mathcal{L}^*$, $\sigma_d^{\#}(q_i)$ converges to $\sigma_d^{\#}(q)$. Therefore, the line segments $\overline{\sigma_d^{\#}(c)\sigma_d^{\#}(q_i)}$ converge to the line segment $\overline{\sigma_d^{\#}(c)\sigma_d^{\#}(q)}$. Thus, $\sigma_d^{\#}(p_i)$ on $\overline{\sigma_d^{\#}(c)\sigma_d^{\#}(q_i)}$ converge to $\sigma_d^{\#}(p)$ on $\overline{\sigma_d^{\#}(c)\sigma_d^{\#}(q)}$. Hence, $\sigma_d^{\#}$ is continuous. $\square$

Sub-step III: Third, we want to prove that $\sigma_d^{\#}$ is continuous on $\overline{\mathbb{D}}$.

Let $(p_i)_{i\in\mathbb{N}}$ be sequence of points on $\overline{\mathbb{D}}$ converges to a point p on $\overline{\mathbb{D}}$. Also, let $(p_i)_{i\in\mathbb{N}}$ on different gaps $(G_i)_{i\in\mathbb{N}}$.

Case I: p is on $\mathbb{S}^1$ and not on a leaf.

PROOF. Since p is on $\mathbb{S}^1$ and $(p_i)_{i\in\mathbb{N}}$ converges to p, the diameters of the gaps G_i containing p_i converge to 0. Consequently, the gaps entirely converge to p. Now, since p lies on $\mathbb{S}^1$, $\sigma_d^\#(p)$ also lies on $\mathbb{S}^1$; and the gaps $\sigma_d^\#(G_i)$ entirely converge to $\sigma_d^\#(p)$. Consequently, $\sigma_d^\#(p_i)$ converges to $\sigma_d^\#(p)$. Therefore, $\sigma_d^\#$ is continuous. □

Case II: p is on some leaf $\overline{ab}$.

Sub-case I: The gaps $(G_i)_{i\in\mathbb{N}}$ have fill-up construction.

PROOF. We draw the line segments $M_{t_i}N_{t_i}$ through the points p_i connecting the points on the longest two sides of G_i with the same t_i values. Since p_i converges to p, t_i associated with $M_{t_i}N_{t_i}$ converges to t associated with p on $\overline{ab}$. Thus, every point on $M_{t_i}N_{t_i}$ converge to p. Again since $\sigma_d^\#$ is continuous on $\mathcal{L}^*$, t_i associated with $\sigma_d^\#(M_{t_i}N_{t_i})$ converges to t associated with $\sigma_d^\#(p)$ on $\overline{\sigma_d^\#(a)\sigma_d^\#(b)}$. Therefore, every point on $\sigma_d^\#(M_{t_i}N_{t_i})$ converge to $\sigma_d^\#(p)$. Consequently, $\sigma_d^\#(p_i)$ converge to $\sigma_d^\#(p)$. Hence, $\sigma_d^\#$ is continuous.

Also, if infinitely many points of p_i lie on the regions of "horizontal" foliation of G_i, then they converge to one of the endpoints of $\overline{ab}$ for $M_{0_i}N_{0_i}$ converge to a and $M_{1_i}N_{1_i}$ converge to b. Since $\sigma_d^\#$ is continuous on $\mathbb{S}^1$, the images $\sigma_d^\#(p_i)$ of p_i converge to the corresponding endpoint of $\overline{\sigma_d^\#(a)\sigma_d^\#(b)}$. Hence, $\sigma_d^\#$ is continuous. □

Sub-case II: The gaps $(G_i)_{i\in\mathbb{N}}$ have coning construction.

PROOF. Let c_i be the centers of the gaps G_i. We draw line segments from c_i to p_i on a gap G_i and extend them till they intersect sides of G_i. Let the intersection points be q_i. Since p_i converges to p, the longest two sides of G_i converge to the leaf. Therefore, the midpoints of the longest two sides of G_i converge to the midpoint $\frac{1}{2}(a+b)$ of $\overline{ab}$. Consequently, the centers c_i of G_i converge to $\frac{1}{2}(a+b)$. Again since

$\sigma_d^{\#}$ is continuous on $\mathcal{L}^*$, $\sigma_d^{\#}(q_i)$ converges to $\sigma_d^{\#}(q)$. Therefore, the line segments $\overline{\sigma_d^{\#}(c)\sigma_d^{\#}(q_i)}$ converge to the line segment $\overline{\sigma_d^{\#}(c)\sigma_d^{\#}(q)}$. Thus, $\sigma_d^{\#}(p_i)$ on $\overline{\sigma_d^{\#}(c)\sigma_d^{\#}(q_i)}$ converge to $\sigma_d^{\#}(p)$ on $\overline{\sigma_d^{\#}(c)\sigma_d^{\#}(q)}$. Hence, $\sigma_d^{\#}$ is continuous. □

Sub-case III: Infinitely many points of p_i lie in gaps that have fill-up construction and infinitely many points of p_i lie in gaps that have coning construction.

Proof. The continuity of $\sigma_d^{\#}$ follows from applying the arguments of sub-case I and sub-case II on these gaps. □

Step 3: In this step we want to show that $\sigma_d^{\#}$ is a branched covering map. We will show that in four sub-steps.

Sub-step I: There are finitely many branch points.

Proof. Centers of critical gaps are branch points. Since there are finitely many critical gaps, there are finitely many branch points. □

Sub-step II: On non-branch value points the map $\sigma_d^{\#}$ is d to 1.

Proof. Let $(x_n)_{n\in\mathbb{N}} \to x \in \overline{\mathbb{D}}$, and let us consider that x is not an image of a branch point.

Case-I: x lies on the circle $\mathbb{S}^1$. By the virtue of σ_d, we know that every point on $\mathbb{S}^1$ has exactly d pre-images. Thus, $\sigma_d^{\#}$ is d to 1.

Case-II: $x \in \mathcal{L}/\mathbb{S}^1$.

Lemma 3.6. *Let $\mathcal{L}$ a q-lamination with all criticality inside Fatou gaps (Hyperbolic lamination). Let $\ell \in \mathcal{L}$ be a non-degenerate leaf. Also, let $\sigma_d^{\#}(\ell_1) = \ell$. Then there exists exactly one full sibling collection $\{\ell_1, \ell_2, \dots, \ell_d\}$ such that $\sigma_d^{\#}(\ell_i) = \ell\ \forall i \in \{1, 2, \dots, d\}$.*

Proof. Let $\overline{ab} = \ell \in \mathcal{L}$ be a leaf on the lamination $\mathcal{L}$. Then we have d pre-images for both the endpoints a and b of $\overline{ab}$. Let $a_1, a_2, \dots, a_d$ be the pre-images of a and

$b_1, b_2, \ldots, b_d$ be the pre-images of b. Also, let $\overline{a_i b_i}, i = 1, 2, \ldots, d$ be the pre-image leaves of $\overline{ab}$.

By the way of contradiction, suppose there is another leaf $\overline{a_1 b_2}$. Then the equivalence class of a_1 is the same as the equivalence class of b_2. This means a_1, a_2, b_1, b_2 are all in the same equivalence class. Thus, a_1 and a_2 map to the same point, which means the class is critical. This is a contradiction because according to the hypothesis, all criticality is inside Fatou gaps. □

As a consequence of Lemma 3.6, we have exactly d disjoint pre-images of the leaf $\overline{ab}$. Hence, x has exactly d pre-images, one at every leaf.

Case III: x is in a gap.

Sub-case I: x is in a regular gap.

In this case, every gap has exactly d pre-image gaps, because each leaf of the gap has exactly d disjoint pre-images. Therefore, x has d pre-images, one at every gap.

Sub-case II: x is in an image of a critical gap.

Let x is in the gap G which is an image of a critical gap G_d, and let G_d maps to G by k to 1. Also, Let c and c_d be the centers of the gaps G and G_d respectively. We draw the arc from c through x till it intersects with a side of G. Then we have k similar arcs from c_d to the boundary of G_d. Each of the k arcs contains a pre-image of x. Thus, we have k pre-images of x for k to 1. Ultimately, we have d to 1 map. □

Sub-step III: $\sigma_d^{\#}$ is an open map.

LEMMA 3.7. *If x is not a critical point, then there exists a neighbourhood of x that maps locally one-to-one on a sufficiently small neighbourhood of $\sigma_d^{\#}(x)$.*

PROOF. Let $\sigma_d^{\#}(x)$ not be an image of a critical value point.

Case-I: x is in the interior of a gap.

In this case, every gap has d pre-images which map one-to-one if all the pre-image gaps are regular. Thus, we can always find a small neighbourhood of x inside every pre-image gap that maps one-to-one to a small neighborhood of $\sigma_d^{\#}(x)$ inside the image gap.

Case-II: x is on a leaf. On one side we have a gap and on the other side, we have a limit of leaves.

In this case, if the gap is a regular gap, then we have a one-to-one mapping between the gap and its image.
If the gap is critical, we consider the neighborhood which has a boundary arc from the critical point to the boundary and hits the boundary before x, and another side of the neighborhood hits the boundary of the gap after x but before the next sibling of the first arc. Then this neighborhood maps one-to-one forward. On the other side of the gap, there is a sequence of leaves converging to the leaf containing x. We know, σ_d is one-to-one on laves. Also, the gaps between the leaves map one-to-one because their boundaries do so. Therefore, every neighborhood around x maps forward one-to-one on this side.

Hence, in every case, there exists a neighbourhood of x that maps locally one-to-one. $\square$

THEOREM 3.8. *Let $f : X \to Y$ be a subjective function. Then f is open if and only if $\forall \{y_i\}_{i\in\mathbb{N}} \to y \in Y$, $\forall x \in f^{-1}(y)$, there exists a sequence $\{x_i\}_{i\in\mathbb{N}}$ in X such that $\{x_i\}_{i\in\mathbb{N}} \to x \in X$ and $f(x_i) = y_i$.*

PROOF. Well known. $\square$

We will use Lemma 3.7 and Theorem 3.8 to prove $\sigma_d^{\#}$ is an open map.

Case I: $y_n, \forall n \in \mathbb{N}$ belong to the adjacent gap of the leaf that contains y.

Proof. Since y is not an image of a critical point, from the Lemma 3.7 we know that $\forall x \in f^{-1}(y)$ there exists a neighbourhood of x that maps locally one-to-one on a sufficiently small neighbourhood of y. Thus, if $\forall \{y_i\}_{i\in\mathbb{N}} \to y$ in image gaps, $\forall x \in (\sigma_d^{\#})^{-1}(y)$, there exists a sequence $\{x_i\}_{i\in\mathbb{N}}$ in pre-image gaps such that $\{x_i\}_{i\in\mathbb{N}} \to x$ in pre-image gaps and $\sigma_d^{\#}(x_i) = y_i$. Hence, $\sigma_d^{\#}$ is open (Theorem 3.8). □

Case II: $y_n, \forall n \in \mathbb{N}$ belong to different gaps.

Proof. Let y lies on the leaf $\overline{ab}$ and y_n lies on the gap that has two longest sides $\overline{a_{n-1}b_{n-1}}$ and $\overline{a_nb_n}$. Since $y_n \to y$, we have $a_n \to a$ and $b_n \to b$. Also, let $\{x_1, x_2, \ldots, x_d\}$ be the pre-images of y under $\sigma_d^{\#}$. Suppose one of the pre-images x_d of y lies on the leaf $\overline{a_db_d}$, and pre-images of $\overline{a_nb_n}$ are $\overline{a_n^{'}b_n^{'}}$. Since $\sigma_d^{\#}$ is continuous and $\overline{a_nb_n}$ converge to $\overline{ab}$, we have $\overline{a_n^{'}b_n^{'}}$ converges to $\overline{a_db_d}$. If possible let $\overline{a_n^{'}b_n^{'}}$ converges to $\overline{a_db_d}$ from one side, i.e., $a_n^{'} \to a_d$ but $b_n^{'} \to b_\infty^{'}$. In this case, we have the leaf $\overline{a_db_\infty^{'}}$. This means we have a finite gap with an even number of sides whose two points map to the same point. This a contradiction, because all criticality is inside Fatou gap. Therefore, we must have $a_n^{'} \to a_d$ and $b_n^{'} \to b_d$. Now since $\sigma_d^{\#}$ is continuous and y_n converges to y, then the pre-images of y_n, x_n converges to x_d. Thus, if $\forall \{y_i\}_{i\in\mathbb{N}} \to y$ in image gaps, $\forall x \in (\sigma_d^{\#})^{-1}(y)$, there exists a sequence $\{x_i\}_{i\in\mathbb{N}}$ in pre-image gaps such that $\{x_i\}_{i\in\mathbb{N}} \to x$ in pre-image gaps and $\sigma_d^{\#}(x_i) = y_i$. Hence, $\sigma_d^{\#}$ is open (Theorem 3.8). □

Sub-step IV: $\sigma_d^{\#}$ is evenly covering mapping.

Proof. Let $y \in \overline{\mathbb{D}}$, and y is not a critical value point. Then by Lemma 3.6, y has d pre-images. Let the pre-images be $\{x_1, x_2, \ldots, x_d\}$. Since $\sigma_d^{\#}$ is open for every open U_i around $x_i, \forall i \in \{1, 2, \ldots, d\}$, there exists an open set $\sigma_d^{\#}(U_i)$ around y. From the Lemma 3.7, we have $\sigma_d^{\#}\Big|_{U_i}$ is one-to-one. Also, let $U_y = \bigcap_{i=1}^{d} \sigma_d^{\#}(U_i)$. This is an open set containing y. Again let $B_y \subset U_y$ be an open ball around y inside U_y. Now, we pull back B_y inside each U_i. We call $V_i = (\sigma_d^{\#})^{-1}(B_y)$. Then V_i's are disjoint and maps one-to-one to B_y. Therefore, $(\sigma_d^{\#})^{-1}(B_y) = \bigcup_{i=1}^{d} V_i$, which means V_y is evenly covered. Hence, $\sigma_d^{\#}$ is evenly covering mapping. □

Therefore, $\sigma_d^{\#}$ is an open map which is locally homeomorphism except at finite branch points. Hence, it is a branched covering map on $\overline{\mathbb{D}}$.

4. FIXED POINT PORTRAITS

DEFINITION 4.1 (Fixed Point). We say a point x is *fixed* by a map f, if and only if, $f(x) = x$.

PROPOSITION 4.2. *For the map σ_d on the circle, the fixed points are $\frac{i}{d-1}$, where $0 \le i < d-1$, for $i \in \mathbb{Z}^+$.*

4.1. Counting the Number of Fixed Point Portraits. Recall that we defined the map σ_d in Definition 2.1. Also, recall that Proposition 4.2 states that there are $d-1$ fixed points on the boundary of $\mathbb{S}^1$ for σ_d.

DEFINITION 4.3 (Fixed Point Portrait(FPP)). We call a partition of $\mathbb{S}^1$, obtained by connecting any number of fixed points with leaves in a manner that two leaves only meet at a fixed point, a *fixed point portrait*.

EXAMPLE 4.4. Each of the following is a fixed point portrait for σ_6. (see Figure 4.1)

DEFINITION 4.5. We call a sector of $\mathbb{D}$ bounded by fixed leaves and arcs of $\mathbb{S}^1$ a *fixed sector* and a polygon inside $\mathbb{D}$ bounded by fixed leaves a *fixed region.*

THEOREM 4.6 (Counting Theorem). *The number of fixed point portraits for σ_d are the Catalan numbers $\frac{(2n)!}{(n+1)!(n)!}$, where $n = d-1$, the number of fixed points of σ_d.*

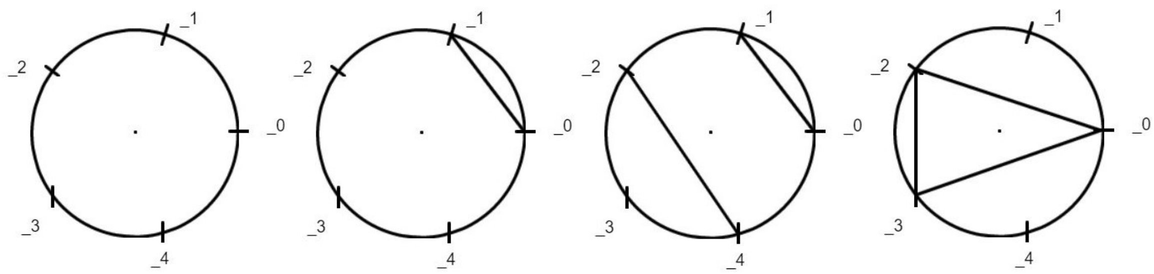

FIGURE 4.1. Examples of fixed point portraits for σ_6

PROOF. Let $d \geq 2$. By Proposition 4.2, the map σ_d has $d-1$ fixed points $\{\overline{0}, \overline{1}, \ldots, \overline{n-1}\}$. We will show the number of fixed point portraits satisfies the recurrence relation of Catalan numbers.

The Catalan numbers are defined by the recurrence

$$C_0 = 1$$

and

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$$

for $n \geq 0$.

By way of induction, let $f(n)$ be the number of fixed point portraits for σ_d using the fixed points $\{\overline{0}, \overline{1}, \ldots, \overline{n-1}\}$. We need to show $f(n)$ satisfies the above recurrence relation.

We define, $f(0) = 1$.

Now, consider a portrait for σ_{d+1} using fixed points $\{\overline{0}, \overline{1}, \ldots, \overline{n}\}$ that contains $\overline{n}$. Also, let $\overline{k}$ be the least fixed point in this portrait, i.e., $\overline{k}$ and $\overline{n}$ are connected and no fixed points of $\{\overline{0}, \overline{1}, \ldots, \overline{k-1}\}$ is connected with any fixed points of $\{\overline{k}, \ldots, \overline{n}\}$. Therefore, $\{\overline{0}, \overline{1}, \ldots, \overline{k-1}\}$ can form portraits on their own and from the induction hypothesis, there are $f(k)$ such portraits. Also, there are $(n-k+1)$ fixed points from $\overline{k}$ to $\overline{n}$. All of these fixed points lie on the arc of the circle that is bounded by the chord $\overline{k}\overline{n}$. Therefore, any portraits among $\{\overline{k}, \ldots, \overline{n}\}$ will not cross the the chord $\overline{k}\overline{n}$. Furthermore, since $\overline{k}$ and $\overline{n}$ are already connected there are $(n-k)$ fixed points to form portraits from $\overline{k}$ to $\overline{n}$. See Figure 4.2. Hence, by the induction hypothesis, there are $f(n-k)$ portraits.

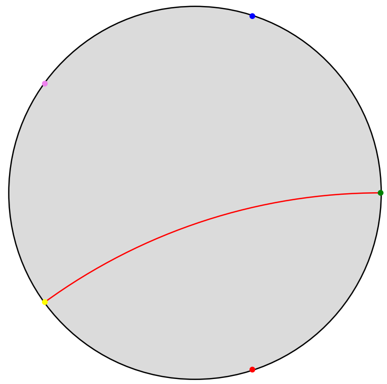

FIGURE 4.2. A fixed point portrait for σ_6 that partitions $\overline{\mathbb{D}}$ into two lower degree fixed point portraits.

Now, the set $\{\overline{0}, \ldots, \overline{n}\}$ has $(n+1)$ fixed points and $\overline{k}$ ranges ranges from $\overline{0}$ to $\overline{n}$. Thus we have,

$$f(n+1) = \sum_{k=0}^{n} f(k)f(n-k)$$

Therefore, by induction, we can conclude that the number of fixed point portraits of σ_d satisfies the recurrence relation of the Catalan numbers. □

5. CANONICAL LAMINATIONS FOR FIXED POINT PORTRAITS

5.1. Critical Portraits for Fixed Point Portraits.

DEFINITION 5.1 (Degree of a fixed region). The *degree of a fixed region* is the cardinality of the maximal number of disjoint critical chords that is compatible with the region.

THEOREM 5.2. *The degree of a fixed region is the number of arcs of the circle $\mathbb{S}^1$ between the adjacent fixed points in the boundary of that region.*

PROOF. Each arc between two adjacent fixed points can contain only one critical chord. □

Definition 5.3 (Canonical Critical Portrait for Fixed Point Portrait). Let $\mathcal{P}$ be a fixed point portrait. For each fixed sector in $\mathcal{P}$ a maximal all-critical polygon that touches a fixed point in the boundary of that sector is called a *canonical critical portrait.*

In the next theorem, we will show that it does not matter which fixed point in a given fixed sector the all-critical polygon touches.

Theorem 5.4. *There are infinitely many critical portraits compatible with the initial data of leaves connecting fixed points but there are finitely many critical portraits that are canonical, i.e., touch the fixed points.*

Proof. For a d-tupling map there are $d-1$ fixed points on $\mathbb{S}^1$, and the fixed points are $\frac{i}{d-1}$ where, $0 \leq i \leq d-1$, and the length of each critical chord is $\frac{i}{d}$ where, $1 \leq i \leq \frac{d}{2}$. First, we consider a critical sector bounded by a fixed leaf connecting two consecutive fixed points. Here, the distance between two successive fixed points is $\frac{1}{d-1}$. On the other hand, the length of the critical chord in this sector is $\frac{1}{d}$. Thus, the maximum distance between a fixed point and the endpoint of the critical chord is $\frac{1}{d-1} - \frac{1}{d} = \frac{1}{d(d-1)}$, so that the critical chord doesn't leave the sector. But since $[0,1)$ is complete, there are infinitely many points in this distance (see Figure 5.1). Therefore, we have infinitely many choices for the endpoints of the critical chord, and different endpoints correspond to different critical chords.

Now, we consider a critical sector bounded by a fixed leaf connecting two fixed points leaving a fixed point in the middle. Then the distance between the end-fixed points is $\frac{2}{d-1}$. This sector can contain a critical triangle whose longest side is $\frac{2}{d}$. Thus, the maximum distance between an end-fixed point and the next vertex point of the critical triangle is $\frac{2}{d-1} - \frac{2}{d} = \frac{2}{d(d-1)}$, so that the all-critical triangle doesn't leave the sector. Similarly, if a critical sector can contain an n-sided all-critical polygon then the maximum distance between an end-fixed point and the next vertex point of the all-critical polygon is $\frac{n-1}{d(d-1)}$. Therefore, according to the previous argument we have

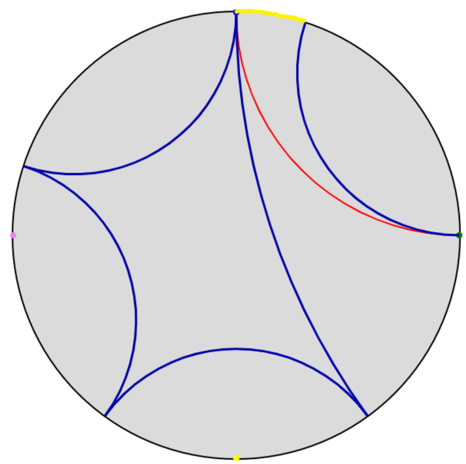

FIGURE 5.1. Finitely many canonical portraits

infinitely many choices for the vertices of the all-critical polygon. Hence, there are infinitely many critical portraits compatible with a given fixed point portrait.

However, if we restrict an all-critical polygon must touch any of the fixed points in its sector then we have a finite number of choices for the vertices of the all-critical polygon because for a finite d we have a finite number of fixed points. Therefore, every critical sector of a fixed point portrait has a finite number of the all-critical polygon. Hence, we will get finitely many canonical portraits for a given fixed point portrait. □

THEOREM 5.5. *Canonical critical portraits compatible with a given fixed point portrait result in the same sibling portrait in the first pullback.*

PROOF. Suppose for a fixed d, we are given a fixed point portrait. We shall show that every canonical critical portrait compatible with this given fixed point portrait results in the same sibling portrait in the first pullback. By Proposition 4.2, the map σ_d has $d-1$ fixed points on $\mathbb{S}^1$ and their pre-images as points are invariant. Also, each of the $d-1$ fixed points pullbacks to d pre-images separated evenly on $\mathbb{S}^1$, one of them the fixed point itself. Since fixed points on $\mathbb{S}^1$ are separated evenly and pre-images of individual fixed points are also separated evenly on $\mathbb{S}^1$, between any two consecutive pre-images of a fixed point there is exactly one pre-image of all other fixed points.

First, we consider a simple case where the fixed point portrait is just a leaf connecting any two consecutive fixed points. Then the canonical critical portrait compatible with this fixed point portrait will consist of a critical chord and an all-critical $(d-1)$-gon. The chord lies in the smaller critical sector of $\mathbb{S}^1$ bounded by the fixed leaf, and one of its ends touches one of the two fixed points (see Figure 5.2). Therefore, the critical chord will connect the fixed endpoint and the adjacent pre-image of that fixed point in this sector in the first pullback. The $(d-1)$-gon lies on the other critical sector of $\mathbb{S}^1$ and one of its vertices touches one of the $(d-1)$ fixed points in this sector. Therefore, the sides of the $(d-1)$-gon will connect the adjacent pre-images of the vertex fixed point starting from the vertex fixed point itself. We call these segments of $\mathbb{S}^1$ bounded by the critical chord and by the sides of the $(d-1)$-gon as critical intervals. As we have seen earlier every critical interval contains pre-images of each fixed point exactly once except for the touching fixed point, which has two pre-images in every interval. Now, in the critical interval bounded by the critical

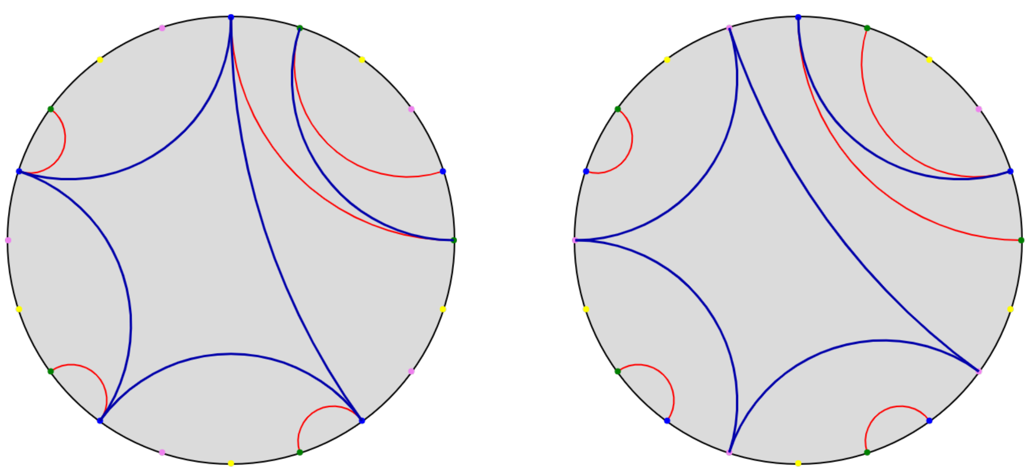

FIGURE 5.2. First pullback lamination results in the same sibling portrait

chord, the pullback of the fixed leaf will connect the pre-image of the fixed point where one end of the critical chord touches, which is basically the other end of the critical chord, with the pre-image of the non-touching fixed point in this interval. We can observe that this is the only option to pullback the fixed leaf in this sector of $\mathbb{S}^1$. Thus, if we change the touching fixed point of the critical chord, it wouldn't affect the pullback leaf and the leaf will remain the same. In the critical intervals bounded by the sides of the $(d-1)$-gon, since we have exactly one pre-image of each fixed point except for the touching fixed point, which has two pre-images, we have exactly one option to pullback the fixed leaf in every interval if neither of the fixed points of the pullback leaf is the touching fixed point of the critical d-gon. If one of the fixed points is the touching fixed point then we have two pre-images of that fixed point to pullback the fixed leaf. In this case, the critical interval where the $(d-1)$-gon touches the fixed point connecting the fixed leaf, there is only one option to pullback the fixed leaf because the pullbacks are disjoint. One fixed point cannot be used for two pullbacks. Consequently, every other interval will have exactly one option to pullback the fixed leaf. Now in a critical interval, the distance between two endpoints is $\frac{1}{d}$. And since our fixed leaf connects two adjacent fixed points the length of the pullback leaf will be $\frac{1}{d(d-1)}$. Also, the maximum distance between two pre-images of two different fixed points in a critical sector is $\frac{1}{d} - \frac{1}{d(d-1)}$. We have, $\frac{1}{d} = \frac{1}{d} - \frac{1}{d(d-1)} + \frac{1}{d(d-1)}$. Thus, if we change the touching fixed point of the critical $(d-1)$-gon the fixed points of the pullback leaves will remain on the same critical interval. Moreover, since the pullback leaves are unique in every interval, the change will not affect the pullback leaves and they will remain the same. Therefore, we can conclude that we will have the same sibling irrespective of the choices of the touching fixed point.

Now, since we have the same sibling portrait in the first pullback and the branches of inverse are also fixed, we will have the same sibling portrait in every pullback. Therefore, the lamination is canonical. □

Algorithm for counting the number of canonical critical portraits compatible with some fixed point portrait for a given d.

To count the number of canonical critical portraits compatible with some fixed point portrait we need to find the number of fixed points in every critical sector described by the fixed point portrait. If a given sector has n fixed points, then the component of the canonical critical portrait will be an n-gon in that sector. This n-gon can touch any of the n fixed points in that sector, and the touching of different fixed points corresponds to components of different canonical portraits. Also, components of critical portraits in each sector are independent of components of critical portraits in other sectors. Now, every canonical critical portrait has exactly one component in each critical sector, so the total number of critical portraits is the combination of components from each critical sector. Since each sector is independent, the total number of critical portraits is the product of the number of components from each sector.

5.2. Canonical Laminations for Fixed Point Portraits.

Definition 5.6 (Canonical Laminations for Fixed Point Portrait). Let $\mathcal{P}$ be a fixed point portrait. Choose for each fixed sector in $\mathcal{P}$ a maximal all-critical polygon touching a fixed point in the boundary of that sector. Then the pullback lamination $\mathcal{L}$ with respect to the chosen critical portrait is the *canonical lamination* for $\mathcal{P}$ (see Figure 5.3).

It follows from Theorems 2.15 and 5.5 that Definition 5.6 is well-defined. That is it does not matter which canonical critical portrait is used in the pullback scheme. See Figure 5.2.

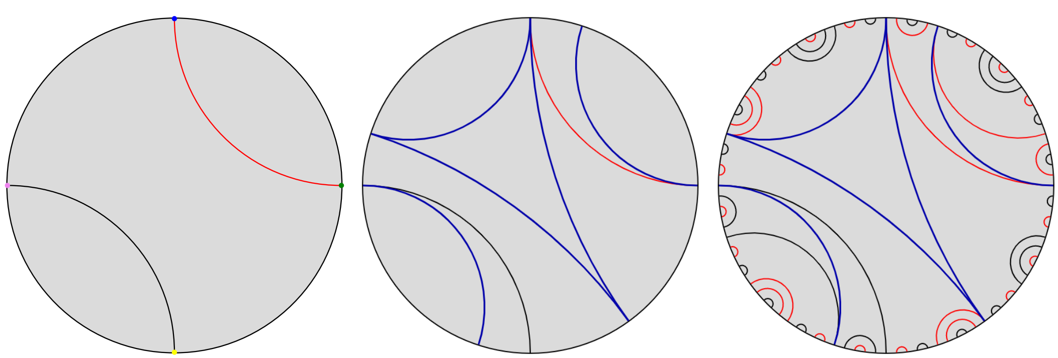

Figure 5.3. Canonical lamination for σ_5 with two fixed leaves.

Theorem 5.7 (Properties of Canonical Fixed Point Lamination). *Let $\mathcal{P}$ be a fixed point portrait and $\mathcal{L}$ be its canonical pullback lamination. Then $\mathcal{L}$ has the following properties:*

(1) *Every pullback leaf is a preimage of a leaf of $\mathcal{P}$.*

(2) *Pullback leaves converge to points of $\mathbb{S}^1$.*

(3) *There are no limit leaves.*

(4) *Every fixed sector contains a unique invariant critical Fatou gap whose center is a fixed point.*

(5) *The boundary leaves of the Fatou gap are preimages of boundary leaves of that fixed sector.*

Proof. (1) Every point on $\mathbb{S}^1$ for the pullback lamination $\mathcal{L}$ is a pre-image of the endpoints of the leaves of $\mathcal{P}$. So in $\mathcal{L}$, if we connect two pre-images of two endpoints of a leaf of $\mathcal{P}$ by a leaf, then this leaf in $\mathcal{L}$ maps forward to the corresponding leaf of $\mathcal{P}$. Therefore, every pullback leaf is a pre-image of a leaf of $\mathcal{P}$.

(2) Let $\mathcal{L}'$ be a canonical lamination of the fixed point portrait $\mathcal{P}$, i.e., $\mathcal{L}'$ touches some fixed points of $\mathcal{P}$. We know, for a d-tupling map there are $d-1$ fixed points with lengths $\frac{i}{d-1}$ where, $0 \leq i \leq d-1$, and the length of each critical chord is $\frac{i}{d}$ where, $1 \leq i \leq \frac{d}{2}$. Each of the critical sectors maps one-to-one onto $\mathbb{S}^1$. Thus, the first pullback lamination $\mathcal{L}$ contains pre-images of every fixed leaf in every critical sector. Since the pre-image leaves map forward under σ_d to a fixed leaf of length $\frac{1}{d}$, the length of the pullback leaves become $\frac{1}{d(d-1)}$. Similarly, the second pullback leaves map forward to the first pullback leaves. So the length of the second pullback leaves becomes $\frac{1}{d^2(d-1)}$. Consequently, after n pullbacks, the length becomes $\frac{1}{d^n(d-1)}$. Since $d > 1$, the length converges to 0. Hence, pullback leaves converge to points of $\mathbb{S}^1$.

(3) Since from (2) we have pullback leaves converge to points, there are no limit leaves.

(4) Existence: Let F be a fixed sector for $\mathcal{P}$. Let the degree of the sector F be n. Then, there is an all-critical n-gon, touching one of the fixed points on the boundary of F, which is compatible with F. Now, the chords of the all-critical polygon describe critical sectors on F. So, when we pull back each of the critical sectors has a pre-image of every fixed point inside it, because the critical sectors map one-to-one onto $\mathbb{S}^1$ except at the endpoints. If we connect pre-images of the fixed points by leaves in accordance with the initial fixed leaves of $\mathcal{P}$, without crossing the lamination, we will get a pre-image of every fixed leaf inside F in the first pullback. Since the pullback of the fixed points preserves counterclockwise circular order, the other leaves that are not boundary leaves of F will be subtended by the boundary leaves of F. The arcs of $\mathbb{S}^1$ subtended by these leaves map outside of F. In the second pullback, we will get pre-images of the first pullback leaves on the remaining part of F. Again the arcs of $\mathbb{S}^1$ subtended by these leaves maps outside of F. If we go on infinitely, we will have an infinite gap in F that is invariant which is by 2.11 a Fatou gap.

Uniqueness: Suppose there are two invariant Fatou gaps G and H. Since the gaps are invariant, they have to have fixed leaves or fixed points in their boundary. If the two gaps share a common fixed leaf, then it contradicts the fact that they are both in the same fixed sector F. If they share a common fixed point, the point is either a vertex of a fixed region between the two gaps, which again contradicts the fact that they are in the same fixed sector F. And if the fixed point is just a point then G and H are the same invariant Fatou gap. Thus, we have only one invariant Fatou gap.

(5) From (4) we have seen that in the first pullback inside F, the pullback leaves that are not the boundary leaves of F are subtended by the boundary leaves of F. Again, since pre-images of the fixed points preserve circular order in every pullback and we connect the pre-images at each stage of pullback to draw leaves the same

thing happens at every pullback. Therefore, the boundary leaves of the Fatou gap are pre-images of the boundary leaves of that fixed sector. □

5.3. Fixed Points in a Lamination. Every lamination of degree d is supposed to correspond to a polynomial of degree d. Every polynomial of degree d has d fixed points. But on the boundary of $\mathbb{S}^1$ we have exactly $d-1$ fixed points. If in the lamination two fixed points are joined by a leaf, then in the corresponding Julia set this will be a multiple fixed point. In Figure 6.1 four peripheral fixed points in the lamination have become two multiple fixed points in the Julia set. But there is an attractive fixed point in each invariant Fatou gap as in Theorem 5.7. In this subsection, we will show that there exists a unique fixed point in the interior of every fixed sector using the Brouwer fixed point theorem.

PROPOSITION 5.8. *Let G be an invariant Fatou gap in a hyperbolic lamination. Then G is critical.*

PROOF. Since in a hyperbolic lamination, a Fatou gap is infinite, This follows immediately from 2.11. □

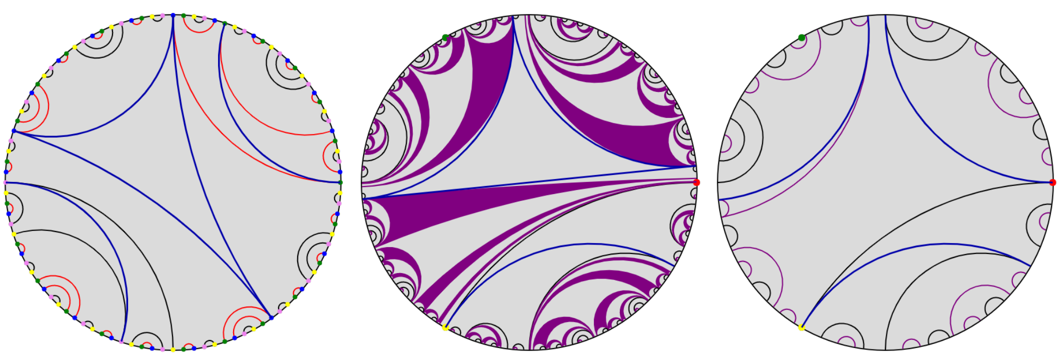

FIGURE 5.4. Left to right cases I, II and III of Theorem 5.9

THEOREM 5.9. *Let $\mathcal{L}$ be a hyperbolic lamination containing a particular FPP $\mathcal{P}$. In each fixed sector of $\mathcal{P}$ there is a fixed object of one of the following types:*

(1) *The fixed point is the center of an invariant Fatou gap.*

(2) *The fixed object is a rotational polygon.*

Moreover, the canonical fixed point lamination for $\mathcal{P}$ is a sub-lamination of $\mathcal{L}$.

Proof. Let F be a fixed sector $\mathcal{P}$.
Case I: No fixed point or fixed leaf of F is subtended by the leaves of the lamination (see leftmost picture of Figure 5.4).

Existence: In this case, every fixed object is on the boundary of a fixed critical gap. As we have defined $\sigma_d^{\#}$, the center of this invariant critical Fatou gap is fixed.

Uniqueness: Since every fixed object is on the boundary of an invariant Fatou gap, the lamination in F is canonical as in 5.6. Therefore, sector F contains a unique fixed point.
Therefore, we have a fixed point in the fixed object F, which is the center of an invariant Fatou gap.

Case II: Every fixed object of F is subtended by leaves of the lamination (see middle picture of Figure 5.4).

Existence: For each fixed object, we pick a sufficiently close leaf l of the lamination $\mathcal{L}$. So that l is repelled from the fixed object, but the image of l remains in the fixed sector F. Let $\widetilde{\mathbb{D}} \subset F$ be the disk with our chosen leaves and the portions of $\mathbb{S}^1$ that are not subtended by these leaves as the boundary of it. Now, we slide every point of $\mathbb{D}/\widetilde{\mathbb{D}}$ to the nearest point of the boundary of $\widetilde{\mathbb{D}}$ by a retraction mapping r. Therefore, $\widetilde{\mathbb{D}}$ is a retract of $\mathbb{D}$, and we have, $r(\sigma_d^{\#}(\widetilde{\mathbb{D}})) \subset \widetilde{\mathbb{D}}$. We can observe that $\widetilde{\mathbb{D}}$ does not contain any fixed point under $\sigma_d^{\#}$. But since $\widetilde{\mathbb{D}} \subset \mathbb{D}$, $\sigma_d^{\#}$ must be continuous on from $\widetilde{\mathbb{D}}$ to $\widetilde{\mathbb{D}}$. Therefore, Brouwer's fixed point theorem shows at least one fixed point exists on $\widetilde{\mathbb{D}}$. However, these fixed points can not occur on $\widetilde{\mathbb{D}} \cap \mathbb{S}^1$, so they must occur inside

F as $\widetilde{\mathbb{D}} \subset F$.

Uniqueness: In a lamination, each invariant object, not a point on $\mathbb{S}^1$, is either a leaf or a gap. Also, if it is a gap, then by Theorem 2.11 either it is a polygon or a critical Fatou gap. In this case, all of the fixed objects are subtended by the leaves of the lamination. So, there is no invariant Fatou gap. Thus, the fixed objects are either in a leaf or in a polygon. Now, if we have two or more rotational objects, a combination of leaf and/or polygon, they must be separated by some critical sector of $\mathcal{L}$ with rotation number zero by Theorem 2.12. Therefore, they do not lie on a single fixed sector F. Hence, the fixed object in F is unique.

Case III: Some fixed objects of F are subtended by leaves of the lamination, and some are not (see rightmost picture of Figure 5.4). As in case I, the fixed objects not subtended by the leaves of the lamination are on the boundary of an invariant critical gap. As we have defined $\sigma_d^{\#}$, the center of this invariant critical Fatou gap is fixed.

Uniqueness: Suppose there are two invariant Fatou gaps G and H. Since the gaps are invariant, they have to have fixed leaves or fixed points in their boundary. If the two gaps share a common fixed leaf then it contradicts the fact that they are both in the same fixed sector F. If they share a common fixed point, the point is either a vertex of a fixed region between the two gaps, which again contradicts the fact that they are in the same fixed sector F, or the fixed point is just a point then G and H are the same invariant Fatou gap. Thus, we have only one invariant Fatou gap G.

If there were another fixed object, it is either a rotational polygon or a leaf. But it can not be on the boundary of G, because G is invariant. Then it has to be in a different critical sector other than G. So, there is another critical gap say G' between G and the rotational object that shares boundaries with the rotational object and

with the invariant gap G. Since it shares a boundary with the rotational object, it is not invariant. Again since it shares a boundary with G, it contradicts the invariance of G. Therefore, there can not be another fixed object other than G.
From Theorem 5.7, we have the fixed leaves of the fixed point portraits and their pullbacks are in the lamination, and they converge only to points of $\mathbb{S}^1$. Therefore, fixed leaves, their pullbacks and $\mathbb{S}^1$ are a closed sub-lamination of $\mathcal{L}$. □

6. CONTINUING AND FUTURE WORK

We can extend our work to realize canonical fixed point portrait laminations as polynomial Julia sets. The step of extending σ_d to the disk $\overline{\mathbb{D}}$ is a step in that direction of getting a topological polynomial to which we can apply Thurston's criterion for the existence of polynomial with our lamination, and therefore the existence of Julia sets corresponding to that polynomial. For example, the followings are some of the fixed point portraits for σ_5 and their corresponding Julia sets.

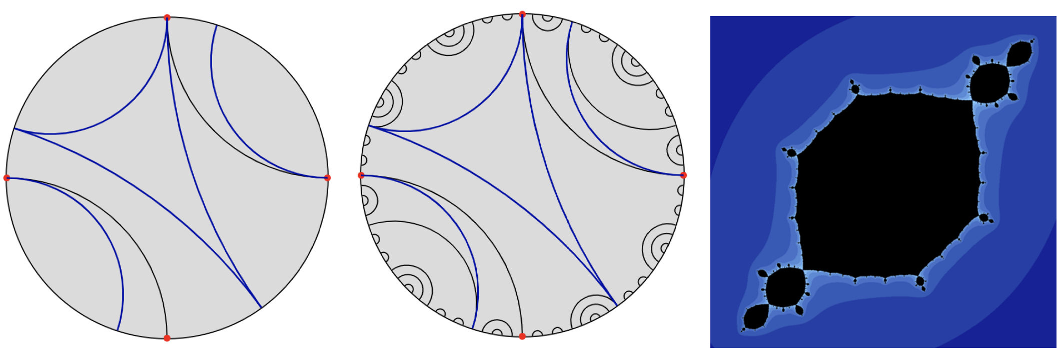

FIGURE 6.1. This is the fixed point portrait with two fixed leaves, one from _0 to _1 and another from _2 to _3 and the corresponding Julia set [**1**].

In the paper [**6**], the authors studied the correspondence between unicritical and maximally critical rotational sets. Combining that with the fixed point portrait we are working on [**1**] to study the relationship between locally unicritical and maximally locally critical rotational sets.

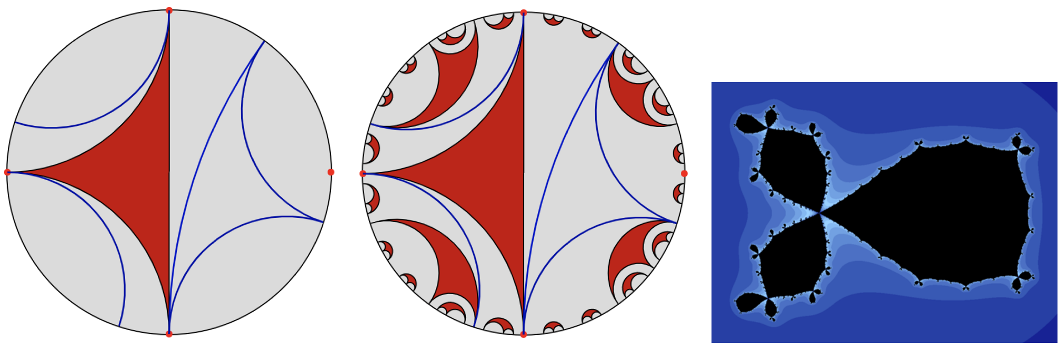

FIGURE 6.2. This is the fixed point portrait with a fixed triangle connecting _1, _2, and _3 and the corresponding Julia set [**1**].

If we want to explore further, we can study how fixed point portraits help us to understand the structure of the parameter space for degree d polynomials.

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