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Analysis of Systems of First Order Linear Differential Equations with Non-Diagonalizable Matrices

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ANALYSIS OF SYSTEMS OF FIRST ORDER LINEAR DIFFERENTIAL EQUATIONS WITH NON-DIAGONALIZABLE MATRICES

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MATHEMATICS

ABSTRACT

Finding solutions to differential equations is notoriously computationally intensive given that the number of operations required is heavily dependent on the algorithm being implemented. This is especially true for systems described by a matrix representation because as the eigenvalues of the system approach non-uniqueness, any approximations made by the program looking for the solution can lead to undesirable precision errors when calculating the final solution. In this document, two algorithms are compared which deal with this issue of stability which can greatly affect the decision to use one method over the other. In the method of solving the system by means of matrix diagonalization can be easily found given distinct eigenvalues; however, problems come up as the eigenvalues of the system converge as this leads to undefined entries in the inverse eigenvector matrix. The Putzer algorithm, however, sidesteps these singularities, resulting in a method that produces approximations independent of the angle between the eigenvectors of the system.

DEDICATION

To the beloved Diana Etain for all her support and steadfast patience without which the completion of this work would not have been possible.

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INTRODUCTION

A long-known method of determining the independent solutions to nth-order Linear Ordinary Differential Equations (LODEs) is by representing the equation of interest as a first-order system with a matrix of constant indices. Initially, the eigenvalues of the matrix representation of said system are obtained for the calculation of both solution algorithms, and the eigenvectors that correspond to said eigenvalues are worked out for diagonalization specifically.

$$y' = Ay$$

Representation of a first-order system (Equation 1)

$$\det(A - \lambda I) = 0, (A - \lambda I)x = 0$$

Method of calculating the eigenvalues and corresponding eigenvectors of the above system (Equation 2)

Then the diagonalized representation of matrix A is constructed by writing out the eigenvector, eigenvalue, and inverse eigenvectors matrices to show an equivalent which provides a system of matrices that are easier to work with for calculating the solution of the system.

$$A = [x_1 \dots x_n] \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} [x_1 \dots x_n]^{-1} = PDP^{-1}$$

Diagonalized matrix representation of A where x_i are distinct eigenvectors of A and λ_i are eigenvalues of A where i = 1,2,…,n (Equation 3)

This matrix is then exponentiated to provide a solution to the linear system, the justification for matrix exponentiation will be provided later in the discussion. This process results in the following:

$$e^{At} = Pe^{Dt}P^{-1} = [x_1 \dots x_n] \begin{bmatrix} e^{\lambda_1 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n t} \end{bmatrix} [x_1 \dots x_n]^{-1}$$

Exponentiated solution with respect to diagonalized matrix representation of A

(Equation 4)

The alternative approach to be discussed is the Putzer Algorithm, which generates a solution identical to this e^{At} introduced in Equation 4. This method of calculation consists of the following matrix sum and eigenvalue compliment matrices, constrained by the solution to a triangular system:

$$e^{At} = \sum_{j=0}^{n-1} r_{j+1} P_j \, , P_0 = I, P_j = \prod_{k=0}^{j} (A - \lambda_k I)$$

The matrix definitions and corresponding sum of the Putzer Algorithm where I is the identity matrix and $\lambda_1, \lambda_2, \dots \lambda_n$ are the non-unique eigenvalues of A. (Equation 5)

$$r_1' = \lambda_1 r_1, r_j' = r_{j-1} + \lambda_j r_j, r_1(0) = 1, r_j(0) = 0$$

The triangular system of the Putzer Algorithm with solutions r_1, r_2, … r_n where j = 2,3,…n (Equation 6)

The purpose of comparing the two methods is to show that while both algorithms work well with explicit solutions or approximate solution with distinct or non-converging eigenvalues, that as two or more of eigenvalues of the system approach the same value, precision issues arise with the diagonalization process that do not interfere with Putzer's calculations. This is due to the instability inherent in the eigenvectors of the system. It will also be shown that in cases where the systems eigenvalues converge, that to obtain the desired precision, higher order approximations of the exponential function with be required when doing standard diagonalization whereas with the Putzer algorithm a linear approximation will suffice.

DIAGONALIZATION METHOD: THEORETICAL DISCUSSION

When finding the solution by means of diagonalization the value of the matrix exponential, or e^{At}, is defined by following infinite series.

$$e^{A} = \sum_{i=0}^{\infty} \left(\frac{1}{i!} * A^{i} \right)$$

Definition of the matrix exponential of the matrix A (Equation 7)

The exponential is calculated using the standard properties of scalar and matrix multiplication and is guaranteed convergence by the fact that every sequence of convergent matrices has a limit ([1], p.24). In the calculation below the derivative of this function is shown to exist and satisfy system shown in Equation 1. Note that A is not affected by taking the derivative or by the summation as it is a constant matrix.

$$\frac{d}{dt} \frac{A^{i}}{i!} t^{i} = \frac{i A^{i} t^{i-1}}{i!} = A \frac{A^{i-1} t^{i-1}}{(i-1)!}$$

Derivative of the i[th] term of the matrix exponential (Equation 8)

$$(e^{At})' = \sum_{i=1}^{\infty} A \frac{A^{i-1} t^{i-1}}{(i-1)!} = A \sum_{i=0}^{\infty} \frac{A^{i} t^{i}}{i!} = A e^{At}$$

Derivative of the matrix exponential (Equation 9)

Thus, it can be shown that the derivative of e^{At} shares the same property as that of the exponential function with respect to scalars. The eigenvector matrix as well as its inverse

do not factor into final calculation of the exponential and the only alterations necessary are that the exponential only directly affects the eigenvalue matrix as shown below.

$$e^{A} = \sum_{i=0}^{\infty} \frac{A^i}{i!} = \sum_{i=0}^{\infty} \frac{(PDP^{-1})^i}{i!} = \sum_{i=0}^{\infty} \frac{PDP^{-1}PDP^{-1}\ldots PDP^{-1}}{i!} = \sum_{i=0}^{\infty} \frac{PD^iP^{-1}}{i!} = Pe^{D}P^{-1}$$

Calculation of the exponential of the diagonalized form of A (Equation 10)

$$D^i = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}^i = \begin{bmatrix} \lambda_1^i & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n^i \end{bmatrix}, e^{D} = \begin{bmatrix} e^{\lambda_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n} \end{bmatrix}$$

Multiplicative and exponential calculations of the eigenvalue matrix D (Equation 11)

From Equations 10 and 11, a solution to the system shown in Equation 1 becomes obvious to calculate particularly for system with explicitly known matrix entries. However, the above produces issues further discussed in this paper regarding the uniqueness of system eigenvalues and eigenvectors. ([2], p. 64-67)

DIAGONALIZATION METHOD: NUMERICAL ANALYSIS

The diagonalization method is ill equipped to generate solutions regarding the case where the system does not have n unique eigenvalues and ergo n unique eigenvectors. One way of showing this discrepancy is taking for example a 2x2 linear system whose eigenvalues quickly approach the same value. This simulates the circumstance in which the angle between the two eigenvectors approaches 0 as ε goes to 0, leading to a collapse of the 2d eigenspace to that of a 1d eigenspace.

$$A_\varepsilon = \begin{bmatrix} 1 & 1 \\ 0 & 1+\varepsilon \end{bmatrix}, e^{A_\varepsilon t} = \begin{bmatrix} 1 & 1 \\ 0 & \varepsilon \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{(1+\varepsilon)t} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{\varepsilon} \\ 0 & \frac{1}{\varepsilon} \end{bmatrix}$$

A simple diagonalizable linear system which becomes non-diagonalizable for small ε (Equation 12)

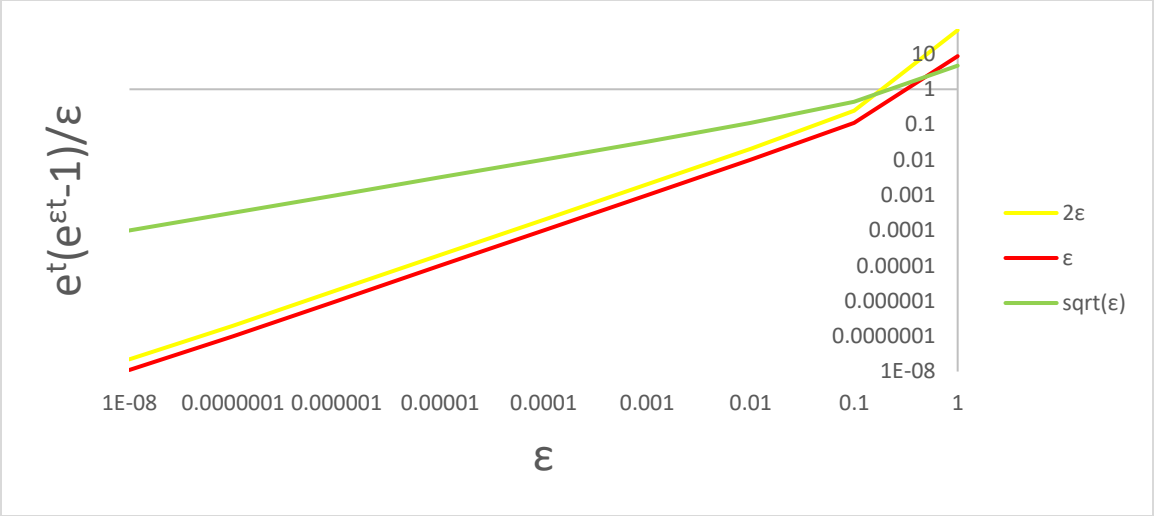
Here ε represents an interval of distinction between the two eigenvalues, but the interval is shrinking of some order with respect to t (and so is in fact a function ε(t)). The order of convergence is unknown and depends upon the inputs in A but will impact any approximate calculation of the solution e^{At}. Obviously, t is the independent variable for the solution of the system $y(t)' = Ay(t)$, but it also represents the splitting of the real line done in computation when calculating the approximate value of e^{At}. Generally, if t is smaller than ε, the problems with approximating the solution do not arise; however, if ε<<t or if ε is shrinking at a much faster rate than t, for example a quadratic relationship,

then the following analysis provides an interesting discrepancy. This fact places an important bound on how small t can be compared to ε and sets up important parameters to calculate the precision of any solutions.

The system shown normally can be solved via this method; however, as ε approaches zero, the only eigenvalue becomes one and therefore as a result the inverse matrix becomes undefined on some of its entries. Additionally, precision of the result depends heavily on the value of ε in the exponential matrix. This is despite the reality that the system converges to a well-defined behavior over time because if you take the limit of the fully multiplied out matrix, you get the following result.

$$e^{A_\varepsilon t} = \begin{bmatrix} e^t & \dfrac{e^t(e^{\varepsilon t}-1)}{\varepsilon} \\ 0 & e^{(1+\varepsilon)t} \end{bmatrix}, \lim_{\varepsilon \to 0} e^{A_\varepsilon t} = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}$$

The index a_{12} approaches te^t, an independent solution from e^t (Equation 13) te^t being the expected secondary partial solution in the case where the characteristic equation of the original ODE has a double root. The desired precision regarding this system can be determined by placing restrictions on t to prevent unwanted behavior regarding ε.



Rates of Convergence for ε-Functions (Figure 1)

In the case of the above figure, sqrt(ε) converges much more slowly when compared to the linear relationships, and therefore it can be extrapolated that values for t that growly increasingly more slowly will result in overflow errors if ε recedes too quickly. This behavior represents the numerator of index 12 in the matrix in Equation 13 converging to 0 more slowly than the in the denominator, leading to issues as the limit is taken. The figure also demonstrates that constant factor differences on t do not affect the rate of convergence to zero.

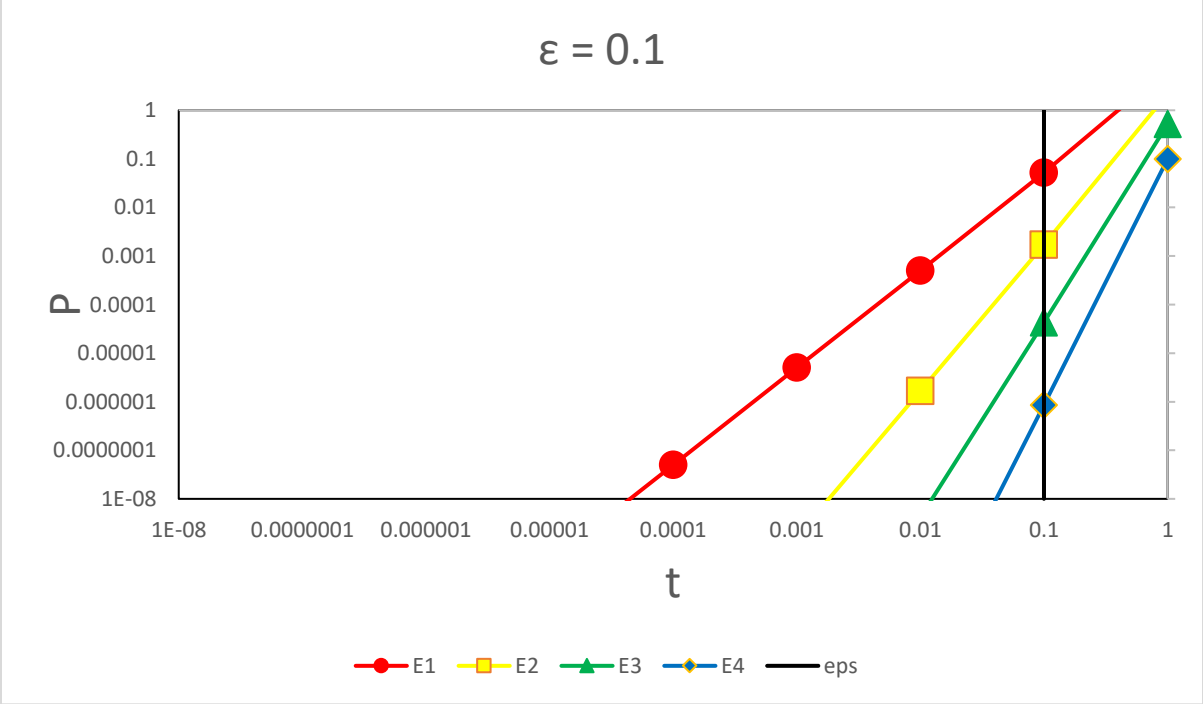
When multiplying the matrices of $e^{A_\varepsilon t}$ (Equation 1), it is apparent that the term $\frac{e^t}{\varepsilon}$ becomes problematic as ε becomes increasingly small; however, the issues regarding certain small values of ε are dependent on t. Consider the case where some ε is fixed and then e^t is approximated by the following series of partial sums $E_n(t)$ and p is represented as follows:

$$E_n(t) = \sum_{i=0}^{n} \frac{t^i}{i!}$$

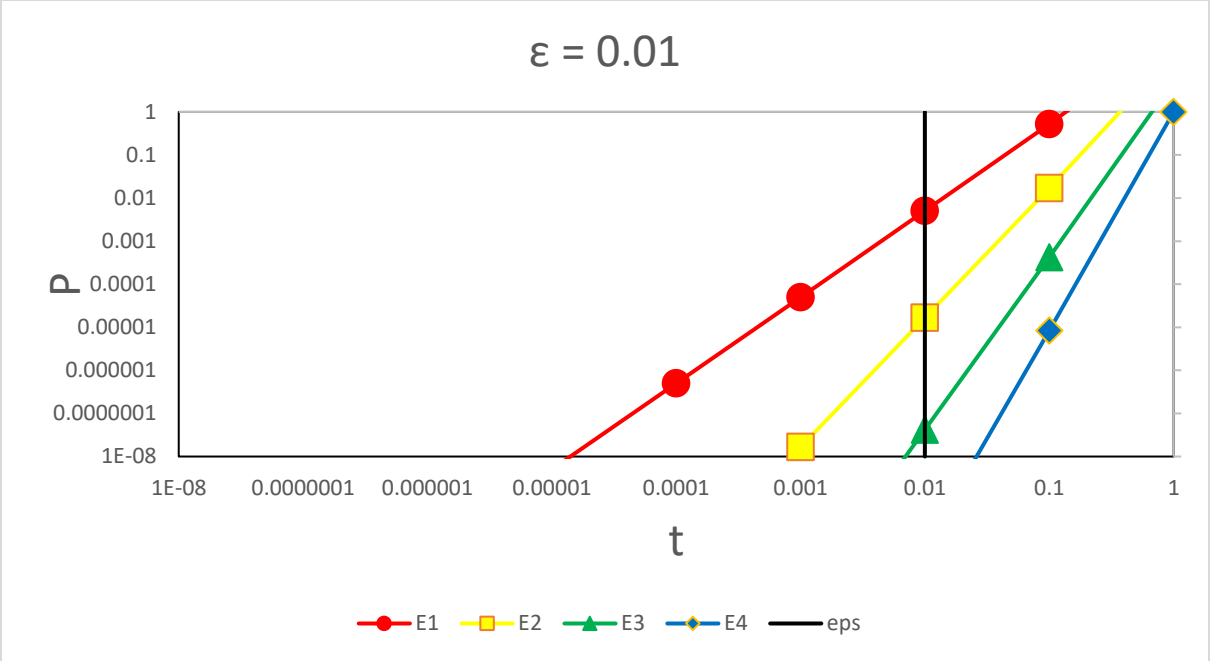
The series of partial sums of the Taylor approximation of e^t (Equation 14)

$$p = \frac{(e^t - E_n(t))}{\varepsilon}$$

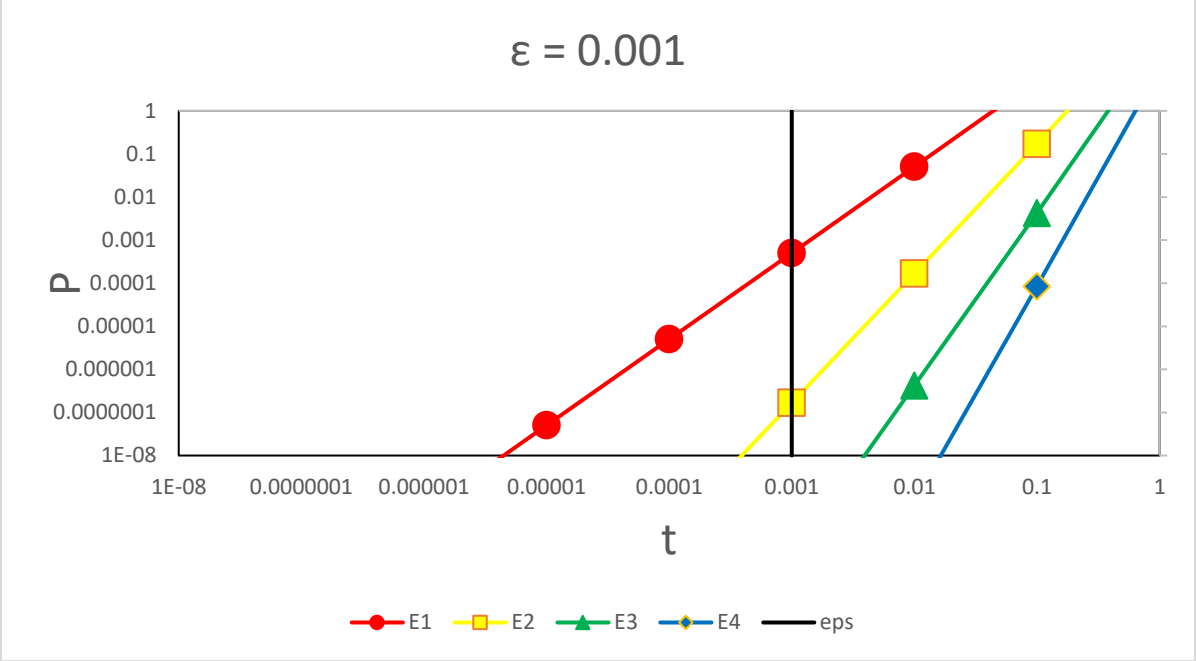
The precision or error of the Taylor approximation dependent on ε (Equation 15)



Log-log plot of p vs t when ε = 0.1 (Figure 2)



Log-log plot of p vs t when ε = 0.01 (Figure 3)



Log-log plot of p vs t when ε = 0.001 (Figure 4)

Consider some desired precision p; the interval of splitting t required to obtain this precision is governed strictly by ε. The ε-boundary (provided by the vertical bound in Figures 2-4) designates the point where $t<\varepsilon$ and thus this analysis does not apply as the issues being discussed should not appear. If $t \approx \varepsilon$ (or of order ε) then the linear approximation of e^t is all that is required provided the splitting is right of the boundary line and therefore valid via the analysis. However, if the interval is of order $\sqrt{\varepsilon}$ then either a quadratic approximation of e^t is required or a much higher error must be tolerated. This relationship continues to hold for the cases where t is of order $\sqrt[3]{\varepsilon}$, $\sqrt[4]{\varepsilon}$, … and so on such that for the same initial precision the required approximation's number of terms is equal to the order comparison between t and ε. Notice, additionally as ε approaches zero the approximations on the above figure shift to the right with the slope of each line increasing, indicating that the difference in precision for each approximation given some t grows in magnitude as ε becomes extremely small.

PUTZER ALGORITHM: THEORETICAL DISCUSSION

The solution to the system, represented by e^{At} can also be calculated by the process put forward by Putzer introduced in previous sections. The matrix sum proposed in Equation 5 can be set equal to some matrix that will be shown to be equivalent to e^{At}.

$$S(t) = \sum_{j=0}^{n-1} r_{j+1} P_j$$

Matrix Summation from Putzer (Equation 16)

Next the derivative of the matrix S is taken by differentiating every term in the sum with respect to t, where P_j is a product of constant matrices with respect to t and therefore the following becomes clear:

$$S'(t) = \sum_{j=0}^{n-1} r'_{j+1} P_j$$

Derivative of the Matrix Summation (Equation 17)

Given the fact that r'_{j+1} already has a defined value given in equation 5, the following series of calculations can be made:

$$S'(t) = \sum_{j=0}^{n-1} \left(\lambda_{j+1} r_{j+1} + r_j\right) P_j$$

$$S'(t) - \lambda_n S(t) = \sum_{j=0}^{n-1} \left(\lambda_{j+1} r_{j+1} + r_j\right) P_j - \lambda_n S(t)$$

$$S'(t) - \lambda_n S(t) = \sum_{j=0}^{n-1} \left(\lambda_{j+1} r_{j+1} + r_j\right) P_j - \lambda_n \sum_{j=0}^{n-1} r_{j+1} P_j$$

$$S'(t) - \lambda_n S(t) = \sum_{j=0}^{n-1} \left(\lambda_{j+1} - \lambda_n\right) r_{j+1} P_j + \sum_{j=0}^{n-1} r_j P_j$$

for convenience $r_0(t)$ is defined to be equal to 0, and by equation 5, $P_{j+1} = \left(A - \lambda_{j+1} I\right) P_j$:

$$S'(t) - \lambda_n S(t) = \sum_{j=0}^{n-2} \left(\lambda_{j+1} - \lambda_n\right) r_{j+1} P_j + \sum_{j=0}^{n-2} r_{j+1} P_{j+1}$$

$$S'(t) - \lambda_n S(t) = \sum_{j=0}^{n-2} \left(\lambda_{j+1} - \lambda_n\right) r_{j+1} P_j + \sum_{j=0}^{n-2} r_{j+1} \left(A - \lambda_{j+1} I\right) P_j$$

$$S'(t) - \lambda_n S(t) = \sum_{j=0}^{n-2} (A - \lambda_n I) r_{j+1} P_j = (A - \lambda_n I) \sum_{j=0}^{n-2} r_{j+1} P_j$$

$$= (A - \lambda_n I) \left[\left\{ \sum_{j=0}^{n-1} r_{j+1} P_j \right\} - r_n P_{n-1} \right] = (A - \lambda_n I)[S(t) - r_n P_{n-1}]$$

$$= (A - \lambda_n I) S(t) - r_n P_{n-1} (A - \lambda_n I) = \ (A - \lambda_n I) S(t) - r_n P_n$$

Derivation of $S'(t) - \lambda_n S(t)$ put in characteristic form (Equation 18)

In order to finish the full derivative, the characteristic equation of A must be considered for application of the Cayley-Hamilton Theorem ([3], p. 317-318):

$$p(A) = (A - \lambda_1 I)(A - \lambda_2 I) \ldots (A - \lambda_n I) = P_n$$

Characteristic Equation of A where $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigenvalues of A (Equation 19)

$$p(A) = 0$$

Cayley-Hamilton Theorem (Equation 20)

Equations 19 and 20 together imply that P_n is the zero matrix which means that the final form of $S'(t) - \lambda_n S(t)$ can be continued from Equation 18:

$$S'(t) - \lambda_n S(t) = (A - \lambda_n I)S(t) = A * S(t) - \lambda_n S(t)$$

$$S'(t) = A * S(t)$$

S(t) as a solution to the first order system of Equation 1 (Equation 21)

The initial condition introduced in equation 5 is also satisfied by S(t) as shown below:

$$S(0) = \sum_{j=0}^{n-1} r_{j+1}(0)P_j = r_1(0)P_0 + \sum_{j=1}^{n-1} 0 = I$$

Initial Condition of S(t) (Equation 22)

Therefore S(t) satisfies Equation 1 and is a equivalent calculation when compared to the standard diagonalization method for calculating e^{At}([1], p. 49-51).

PUTZER ALGORITHM: NUMERICAL ANALYSIS

The example from the previous analysis section will be used but this time calculations will be performed by the Putzer algorithm.

$$\lambda_1 = 1, \lambda_2 = 1 + \varepsilon$$

$$P_0 = I, P_1 = \prod_{k=1}^{1} A - \lambda_k I = A - \lambda_1 I = \begin{bmatrix} 1 & 1 \\ 0 & 1+\varepsilon \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & \varepsilon \end{bmatrix}$$

$$r_1' = r_1, \qquad \frac{dr_1}{dt} = r_1, \qquad \int \frac{1}{r_1} dr_1 = \int dt\,, \qquad \ln r_1 = t + C, \qquad r_1 = c_1 e^t$$

$$r_2' = r_1 + (1+\varepsilon) r_2, \qquad r_2' - (1+\varepsilon) r_2 = c_1 e^t, \qquad I(t) = e^{-\int (1+\varepsilon) dt} = e^{-(1+\varepsilon)t}$$

$$e^{-(1+\varepsilon)t} r_2' - (1+\varepsilon) e^{-(1+\varepsilon)t} r_2 = e^{-(1+\varepsilon)t} * c_1 e^t, \qquad \left(e^{-(1+\varepsilon)t} r_2\right)' = c_1 e^{-\varepsilon t}$$

$$e^{-(1+\varepsilon)t} r_2 = \int c_1 e^{-\varepsilon t}, \qquad e^{-(1+\varepsilon)t} r_2 = -\frac{c_1}{\varepsilon} e^{-\varepsilon t} + c_2, \qquad r_2 = -\frac{c_1}{\varepsilon} e^t + c_2 e^{(1+\varepsilon)t}$$

$$r_1 = e^t, \qquad r_2 = \frac{e^t (e^{\varepsilon t} - 1)}{\varepsilon}$$

$$e^{A_\varepsilon t} = e^t I + \frac{e^t (e^{\varepsilon t} - 1)}{\varepsilon} \begin{bmatrix} 0 & 1 \\ 0 & \varepsilon \end{bmatrix}$$

Calculation of explicit solution of $e^{A_\varepsilon t}$ using Putzer Algorithm with $\varepsilon \neq 0$ (Equation 24)

$$\lambda_1, \lambda_2 = 1$$

$$P_0 = I, P_1 = \prod_{k=1}^{1} A - \lambda_k I = A - \lambda_1 I = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$r_1' = r_1, \qquad \frac{dr_1}{dt} = r_1, \qquad \int \frac{1}{r_1} dr_1 = \int dt\,, \qquad \ln r_1 = t + C, \qquad r_1 = c_1 e^t$$

$$r_2' = r_1 + r_2, \qquad r_2' - r_2 = c_1 e^t, \qquad I(t) = e^{-\int dt} = e^{-t}$$

$$e^{-t} r_2' - e^{-t} r_2 = e^{-t} * c_1 e^t, \qquad (e^{-t} r_2)' = c_1$$

$$e^{-t} r_2 = \int c_1, \qquad e^{-t} r_2 = c_1 t + c_2, \qquad r_2 = c_1 t e^t + c_2 e^t$$

$$r_1 = e^t, \qquad r_2 = t e^t$$

$$e^{A_\varepsilon t} = e^t I + t e^t \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Calculation of explicit solution of $e^{A_\varepsilon t}$ with using Putzer Algorithm ε = 0 (Equation 25)

This algorithm generates the same explicit solution for the linear system as the diagonalization method. However, considering the limit of the solution as ε becomes increasingly small is not required as the eigenvalues for A are not required to be distinct and thus there is no need to numerically substitute this limit in the analysis.

Additionally, this method allows for a linear approximation of the exponential function to be considered without ε being a major consideration in the calculation of the error p:

$$r_1(t) - r_1(0) = \lambda_1 r_1(0) t, r_j(t) - r_j(0) = t\left(r_{j-1}(t) + \lambda_j r_j(0)\right)$$

Difference equation method for approximation of e^{At} (Equation 26)

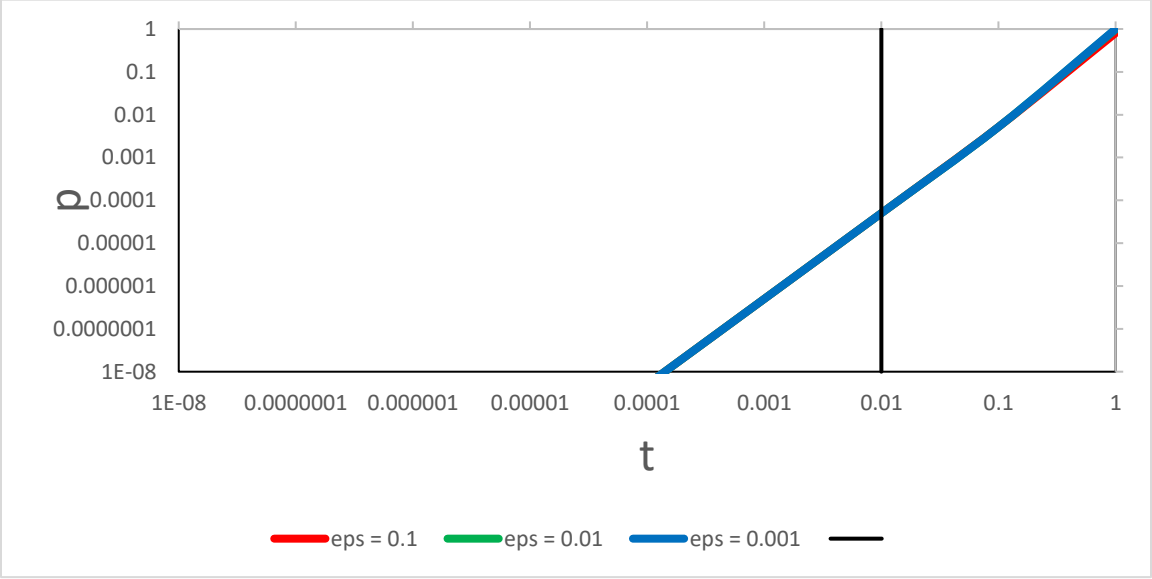
$$r_1 - 1 = t, \qquad r_1 = 1 + t$$

$$r_2 - 0 = t\big((1 + t) + 0\big), \qquad r_2 = t(1 + t)$$

$$e^{A_\varepsilon t} \approx (1 + t)I + (t + t^2)\begin{bmatrix} 0 & 1 \\ 0 & \varepsilon \end{bmatrix} = \begin{bmatrix} 1 + t & t(1 + t) \\ 0 & 1 + (1 + \varepsilon)t + \varepsilon t^2 \end{bmatrix}$$

Linear Approximation of $e^{A_\varepsilon t}$ by the difference equation method (Equation 27)

Note that when the algorithm is used in this approximation via the difference equations, the linear approximations of the two independent solutions, $r_1 = e^t,\ r_2 = te^t$, are generated automatically. In the final matrix provided by Equation 27 there is no division by ε which means that the calculation needed to substitute the limit in the analysis of the diagonalization method are not needed here. In fact, as ε goes to 0, the error term εt^2 goes to 0 and has less of an effect over time in the case of this Putzer approximation.



Plot of p vs t when ε = 0.1, 0.01, 0.001 for Putzer approximation (Figure 5)

As shown by Figure 5, the system is no longer heavily dependent on ε, so generally the only variable restricting the desired precision is t. This means that the precision described in Equation 15 is no longer of the order εp, but just that of p so the precision does not need to consider ε in advance when the Putzer calculations are run.

CONCLUSION

It is clear from the analysis that when the Putzer algorithm is used there is no significant deviation in the relationship between p and t as ε becomes increasingly small. This implies that higher order approximations of e^t are not required to obtain the needed precision. This occurs since there are no division by extremely small numbers by the preferred method and as such the error of the approximation is p as opposed to εp. Therefore, when calculating an approximation for $e^{A_\varepsilon t}$ the values of the exponentials needed will only require calculations with two terms per exponential which will be as good as the diagonalization method when ε is of order t but will be significantly better when $\varepsilon \approx t^2, t^3, \ldots$

In general, while it is trivial to show in the case of the 2x2 matrix used throughout the example above, the same emergent properties generalize nicely to arbitrary matrices. If the eigenvalues of a given matrix are known and are distinct up to some known bound, then the diagonalization method of solving the first order series by constructing the eigenvector and eigenvalue matrices will have no fundamental issues computationally. It should be noted that the general first order systems have variable coefficients and are not particular values of ε per say. If all inputs to the system are known explicitly there should be no significant difference between either algorithm. However, in circumstances where the eigenvalues are converging as in the case of limits or recursive calculations, that introducing the step parameter t is essential in order to make a meaningful distinction

between the computational power of both procedures. t's relationship with ε is what establishes distinguishable differences between solution by diagonalization or solution by Putzer.

Additionally, as shown in the example when the eigenvalues are distinct, both diagonalization and Putzer will calculate the same solution. However, if two or more eigenvalues are the same in a given matrix or have the same limit in a series of matrices, then the traditional method will produce either a singular or near-singular eigenvector matrix which will therefore be non-invertible and therefore non-existent or extremely difficult to invert computationally as some of the inputs will increase without bound. Therefore, in the singular case, the traditional method cannot product an explicit answer because this inverse matrix does not exist; however, the limit of matrix series when put through the traditional method will go to the correct solution but this cannot be computed as the determinate of the inverse matrix with go to infinity as ε approaches 0. The Putzer method has no such limitation because it does not have an issue with any of the eigenvalues being the same, it will simply calculate the triangular system differently dependent on said eigenvalues.

In the near-singular case or when taking approximations of the series as the eigenvalues converge, issues arise in the final precision of the traditional method that do not occur with Putzer. If the eigenvalues converge linearly then a linear approximation in both methods with suffice and therefore it would be expected to reach the desired precision within roughly the same number of floating-point operations. However, if the eigenvalues converge even slightly more quickly than linearly, a quadratic approximation of the Taylor series must be used in the diagonal matrix of the traditional method else the

desired precision will never be reached. The Putzer method will continue to only require a linear approximation regardless of how the eigenvalues converge as the precision of this method does not depend on ε and therefore the approximation will continue to work even when several terms are required with diagonalization. This shows why the Putzer method is preferrable especially in cases with extremely large matrices with unknown initial eigenvalues as it can be near guaranteed that it will require less calculations to calculate the solution up to the precision that is needed.

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