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Multiplicities of Polygon Numbers

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MULTIPLICITIES OF POLYGON NUMBERS

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ABSTRACT

Imagine that we have an unlimited number of congruent equilateral triangles. We define a trapezoid number as the number of equilateral triangles used to tile a trapezoid. Similarly, we define a parallelogram number as the number of equilateral triangles needed to tile a parallelogram. We define the multiplicity of a trapezoid or parallelogram number as the number of ways we can construct the polygon for given number up to congruence.

We show a parallelogram number can always be written as $2kh$ and a trapezoid number as $h(2k+h)$, where we define the height h as the length of the non-parallel sides, and the width k as the length of the shorter of the parallel sides, all in unit triangle lengths. The relationship between multiplicities and trapezoid/parallelogram numbers can be defined through equations relating the multiplicity to the number's prime factorization.

Our findings extend to multiplicities of hexagon numbers, or the number of congruent equilateral triangles tiling an equiangular hexagon. Based on numerical evidence, Hale et. al. [3] conjecture that "the multiplicity of a hexagon number is a function of its prime factorization." They showed that if the height p of a hexagon is a prime number congruent to $1 \pmod 3$, then it produces all residues $\pmod{2p}$. Our main result is: *If the height of a hexagon is pq, where $p > 5$ is the product of prime factors congruent to* $1 \pmod 4$*, and q is the product of unique prime factors congruent to* $3 \pmod 4$*, then the height pq produces all residues* $\pmod{2pq}$.

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1 INTRODUCTION

Imagine that we have an unlimited number of congruent equilateral triangles. We can use these triangles to create various convex polygons with strict angle restrictions.

Definition 1.1 (Tiling)**.** A *tile* is an equilateral triangle with unit side length 1. The act of arranging tiles into a convex polygon is called *tiling*.

For example, we can use 4 tiles to tile a new equilateral triangle. We note that we are not tiling arbitrary polygons, but rather we are working from tiles to commensurate polygons. The following definitions are adapted from the article [3] to appear soon.

Definition 1.2 (Polygon Numbers)**.** The number of tiles used to tile a polygon is a *polygon number*. More specifically;

1. The number of tiles used to tile a triangle is a *triangle number*.

2. The number of tiles used to tile a parallelogram is a *parallelogram number*.

3. The number of tiles used to tile a trapezoid is a *trapezoid number*.

4. The number of tiles used to tile a hexagon is a *hexagon number*.

5. The number of tiles used to tile a pentagon is a *pentagon number*.

Proposition 1.3. *The polygons listed in Definition 1.2 are the only convex polygons that can be tiled with congruent equilateral triangles.*

Proof. Consider the exterior angles of the above polygons. Since we use equilateral triangles to tile the polygons, the exterior angles are set to be either $60°$ or $120°$. Additionally, we know from elementary plane geometry that the sum of the exterior angles of a convex polygon must equal $360°$. Thus, we can exhaustively find the combinations of exterior angles that create convex polygons:

1. If we have 3 exterior angles, they all must be $120°$. Thus, we can tile a triangle.

2. If we have 4 exterior angles, two must be $120°$ and two must be $60°$. If the angles alternate, we tile a parallelogram. If the angles don't alternate, we tile a trapezoid.

3. If we have 5 exterior angles, we must have one $120°$ angle and four $60°$ angles. Thus, we tile a pentagon.

4. If we have 6 exterior angles, all the angles must be $60°$, tiling a hexagon.

If we have more than 6 exterior angles, one angle will not be $120°$ or $60°$. Thus, the list polygons are the only polygons that can be tiled with equilateral triangles. □

The initial motivation for this research was the question, *how many geometric non-congruent ways can we tile one type of polygon with the same number of tiles?* To explore this, we first must establish an algebraic parameterization of each type of polygon.

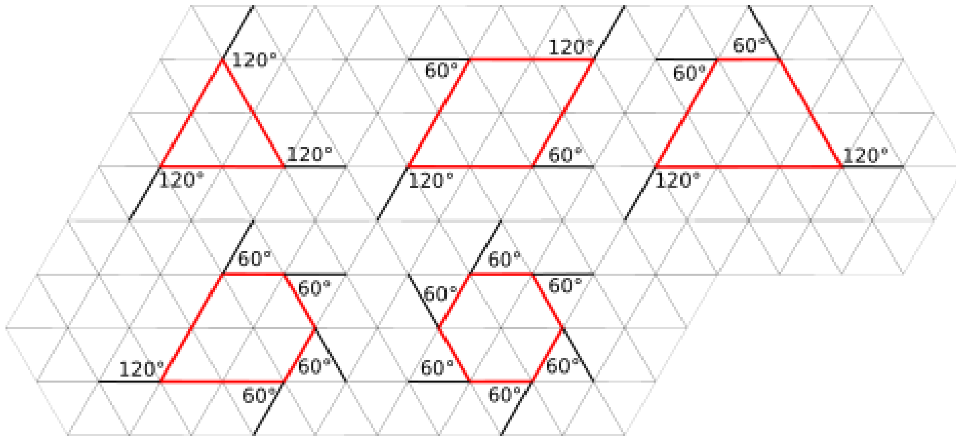

Figure 1: Exterior angles of tiled polygons.

1.1 Height and Width

This introduction to triangle numbers is adapted from the poster [6].

Definition 1.4 (Side Length)**.** The number of triangles along one side of a tiled polygon is called the *side length*. Note, we only count the triangles that have an entire side lined up with the side of the polygon. The triangles that have points along the side are not counted. See Figures 2 and 3 for examples.

Definition 1.5 (Height and Width)**.** The *height* and the *width* of a polygon varies depending on the specific polygon:

1. The height and width of a tiled triangle is its side length in tiles. [5]
2. The height of a tiled parallelogram is either side length in tiles. The width of a tiled parallelogram is the other side length in tiles.. [6]
3. The height of a tiled trapezoid is the side length of a non-parallel side in tiles. The width of a tiled trapezoid is the side length of the shorter of the parallel sides in tiles. [6]
4. The height of a tiled pentagon is either one of the side lengths of the sides that meet at a 60° angle. The width of a tiled pentagon is the side length of the other side at the 60° angle.
5. The height of a tiled hexagon is the sum of two adjacent side lengths in tiles. The width of a tiled hexagon is the side length of either parallel sides not used to define the height. [3]

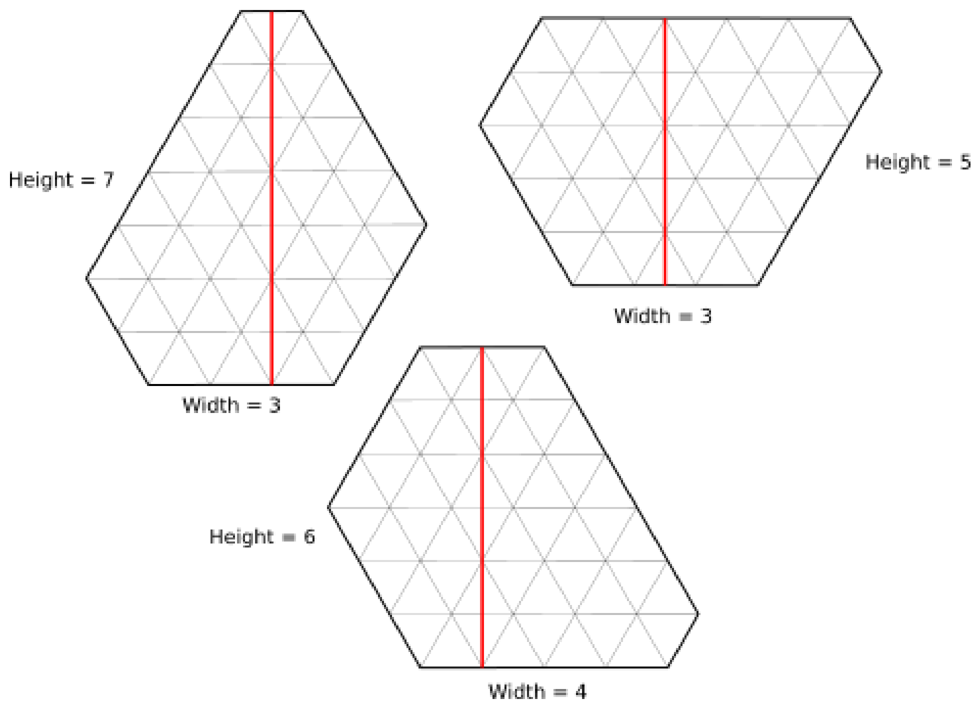

Figure 2: An example of a tiled hexagon with hexagon number 50. The same hexagon is shown rotated 60°. This shows how a hexagon can have 3 different pairs of height and width, and still be the same hexagon geometrically.

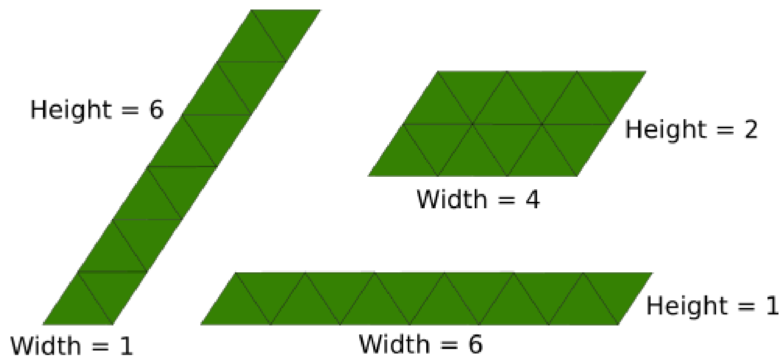

Figure 3: Examples of parallelogram number 12. The geometric multiplicity here is 2, as there are only 2 geometrically different parallelograms that can be created with 12 tiles. We can also see how the parameters of the parallelogram vary, creating a rotated copy of one parallelogram. In this case, the algebraic multiplicity is 4 (one rotation not shown).

Remark 1.6. As we can see, some polygons can have more than one height, depending on which height we choose. Figure 2 has a visual depiction.

Theorem 1.7 (Triangle Number Parameterization [6]). *For any tiled equilateral triangle, its triangle number T is its side length a squared:*

$$T = a^2$$

In the following sections, we parameterize tiled polygons from tiled equilateral triangles.

1.2 Multiplicity

Definition 1.8 (Multiplicity). The number of ways we can construct a polygon for a given polygon number up to geometric congruence is called the *multiplicity* of said polygon number.

For example, a trapezoid number with height 15 can have height one and width seven, or it can have height three and width one, yielding a multiplicity of 2. Note, when we refer to multiplicity, we are specifically referring to *geometric multiplicity*. That is, we want to only count geometrically non-congruent polygons, discounting any rotations or reflections of these polygons. If we were to count rotations and reflections, we would be finding the *algebraic multiplicity*. See Figure 3 for an example.

Initial research in tiling polygons was done by Hertel answering the question, *are there only finitely many numbers that are not hexagon numbers?* affirmatively [5], [1]. This leads to a more in depth motivating question, *are there finitely many hexagon numbers that have only multiplicity 1? What about hexagon numbers that have only multiplicity 2, and so on?*

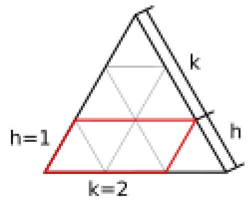

Figure 4: An example of parallelogram number 4 derived from triangle number 9.

In this thesis, we find answers to our motivating questions, and we discover other questions. In each section, we discuss what makes any natural number a specific polygon number, as well as how to find the multiplicity of said polygon number. We discover that multiplicity of a polygon number depends on its prime factorization. In Section 2, we prove there exist closed form formulas to find the multiplicity of both parallelogram and trapezoid numbers. Section 3 discusses hexagon number multiplicity, in which we review much of the content of [3]. We investigate the multiplicity of tiled hexagons with respect to the height, and we talk about the differences between prime and composite heights. Finally, in section 4, we discuss an algorithm to determine pentagon numbers and their multiplicity.

2 TRAPEZOID AND PARALLELOGRAM NUMBERS

2.1 Parallelogram Numbers

To parameterize a parallelogram number, we can view a parallelogram as being tiled from an equilateral triangle.

Theorem 2.1. *A parallelogram number P can be written as $P = 2kh$, for width k and height h.*

Proof. As we can see in Figure 4, we can tile a parallelogram within a tiled triangle. Then, we can see that the area of the triangle is $(k+h)^2$ from 1.7. If we subtract the area of the equilateral triangles on top and to the right of our desired parallelogram, we then get the area of the parallelogram to be $P = (k+h)^2 - h^2 - k^2$. This simplifies to $P = 2kh$. □

We can also see this to be true geometrically. By our definition of side length (1.4), if we multiply our base length k by our height h, we are essentially missing half the triangles hidden within the parallelogram. Thus, we can write our parallelogram number as $P = 2kh$ to account for the missing tiles.

Corollary 2.2. *A number is a parallelogram number if and only if it is even.*

Proof. Since all parallelogram numbers P can be written as $P = 2kh$, this follows directly from 2.1. □

When determining the multiplicity of parallelogram numbers, we look to the given number's prime factorization.

Definition 2.3 (Prime Factorization)**.** The prime factorization of any natural number N is $N = 2^{k_0} p_1^{k_1} p_2^{k_2} \dots p_n^{k_n}$, where each p_i is a distinct prime factor of N.

Theorem 2.4. *There are two closed form formulas that calculate the multiplicity of a parallelogram number.*

1. *If the given parallelogram number is also the double of a square number, then the formula to find the multiplicity is* $m(P) = \frac{(k_0)(k_1+1)(k_2+1)\cdots(k_n+1)+1}{2}$.

2. *If the given parallelogram number is not the double of a square number, then the formula to find the multiplicity is* $m(P) = \frac{(k_0)(k_1+1)(k_2+1)\cdots(k_n+1)}{2}$.

Proof. When considering the multiplicity of a parallelogram number, we see that the parameterization of a parallelogram number is comprised of two factors, its width k and height h. Thus, we can look at the number of combinations of prime factors that make up the height and width of the parallelogram. Each prime factor can either go in the height, or end up in the width of the parallelogram. Thus, there are $k_i + 1$ ways for a prime factor to be used. Note, however, from 2.2, the factor 2 must always be used at least once. Thus, there are exactly k_0 ways for the factor 2 to be used to yield a parallelogram. Of course, this method of counting does not differentiate between height and width, so we must divide by two to account for geometrically congruent parallelograms. In other words, our method of counting will yield a parallelogram of height a and width b, as well as a parallelogram with height b and width a. These two parallelograms are algebraically non-congruent, but geometrically congruent. Finally, in the case of our parallelogram number P being a square number, we will only count the square case once. That is, the case where height and width are equal will only occur once, so we forcibly double count this case in the numerator and divide by 2 as usual. □

2.2 Trapezoid Numbers

To parameterize a trapezoid number, we can view a trapezoid as being tiled from either an equilateral triangle or a parallelogram.

Theorem 2.5. *A trapezoid number* T *can be written as* $T = h^2 + 2kh$.

Proof. As we can see in Figure 5, we can tile a trapezoid within a tiled triangle. It is easy to see the trapezoid is the difference of tiled triangles, meaning $T = (h + k)^2 - k^2$, which simplifies to $T = h^2 + 2kh$. □

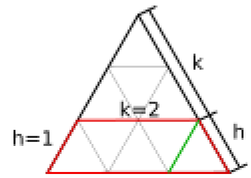

Figure 5: An example of a trapezoid tiled from an already tiled triangle, or from an already tiled parallelogram (green line).

A much more helpful way to look at trapezoid number parameterization, as stated in [6], is $T = h(2k + h)$.

Theorem 2.6. *A natural number N is a trapezoid number if and only if $N \neq 2 \pmod 4$, except $N = 1$ and $N = 4$.*

Proof. If N is a trapezoid number, then $N = h(2k + h)$ for some height and width $h, k > 0$. If $N \equiv 2 \pmod 4$, then $N = 4a + 2, a \in \mathbb{Z}$. Thus, $h(2k + h) = 4a + 2$. Now, h can be either even or odd. If h is even, then $2k + h$ is even, meaning $h = 2x, x \in \mathbb{Z}$, and $2k + h = 2y, y \in \mathbb{Z}$.

$$h(2k + h) = 4a + 2$$

$$2x(2y) = 4a + 2$$

$$4xy = 4a + 2$$

Since $4xy$ is divisible by 4, but $4a + 2$ is not, this equality is impossible. Therefore, h cannot be even and must be odd. If h is odd, then $2k + h$ is also odd, and we can write $h = 2x + 1, x \in \mathbb{Z}$, and $2k + h = 2y + 1, y \in \mathbb{Z}$.

$$h(2k + h) = 4a + 2$$

$$(2x + 1)(2y + 1) = 4a + 2$$

$$4xy + 2x + 2y + 1 = 4a + 2$$

$$4xy + 2(x + y) + 1 = 4a + 2$$

Since the left hand side is odd, and the right hand side is even, once again this equality is impossible. Thus, a trapezoid number N cannot be congruent to $2 \pmod 4$. □

Theorem 2.7. *There are four closed form formulas that calculate the multiplicity of a trapezoid number.*

1. *If the given trapezoid number is both an odd number and a square number, then the formula to find the multiplicity is* $m(T) = \frac{(k_1+1)(k_2+1)\cdots(k_n+1)-1}{2}$.

2. *If the given trapezoid number is an odd number but not a square number, then the formula to find the multiplicity is* $m(T) = \frac{(k_1+1)(k_2+1)\cdots(k_n+1)}{2}$.

3. *If the given trapezoid number is both an even number and a square number, then the formula to find the multiplicity is* $m(T) = \frac{(k_0-1)(k_1+1)(k_2+1)\cdots(k_n+1)-1}{2}$.

4. *If the given trapezoid number is an even number but not a square number, then the formula to find the multiplicity is* $m(T) = \frac{(k_0-1)(k_1+1)(k_2+1)\cdots(k_n+1)}{2}$.

Proof. This proof follows very closely to the proof of 2.4. The important thing to notice, however, are the geometric implications of this new parameterization. When counting parallelogram numbers, we divide by two to account for doubling up the height and the width. Here, our height and width are not independent, but rather our width depends on our height. Thus, a geometric constraint is introduced, where the height must be less than or equal to the width. Otherwise, we will end up computing a negative width, which makes no sense geometrically. Luckily, we can still count algebraically the same way, choosing which prime factors go into the height and the others into the second factor $2k + h$. When we divide by two, we will then discount all the options where $h > k$. If a trapezoid number T is a square number, we must account for the square case of factors, where $h = 2k+h$. When this occurs, $k = 0$, yielding a triangle rather than a trapezoid. Thus, instead of forcibly double counting the square case as we did for parallelogram numbers, we discount it entirely. If a trapezoid number T is odd, the count k_0 is not included as there is no factor of 2 in T. If T is even, we must have a factor of 2 in both h and $2k + h$. If h is even (contains a factor of 2), then the factor $2k + h$ will also be even. If the factor $2k + h$ is even, that similarly forces h to be even. Thus, when counting how many way we can put factors of 2 into each factor of T, we must subtract the 2 factors that are set, one in each factor, yielding the count $k_0 - 1$. Putting both constraints together, we get the four cases listed for trapezoid number multiplicity. If T is odd and square, we subtract the square case in the numerator and don't include k_0. If T is odd and non-square, we only don't include k_0. If T is even and square, we include $k_0 - 1$ and subtract the square case in the numerator. If T is even and non-square, we include $k_0 - 1$ in the count. □

3 HEXAGON NUMBERS

We move onto the hexagon case because we are able to use prime factorization and some other prior techniques to find information about the multiplicities of hexagon numbers. Pentagon numbers will be discussed later, as there was no clear formula for finding the multiplicity of pentagon numbers based on prime factorization.

Theorem 3.1. *A hexagon number H can be written as $H = 2kh + h^2 - (x^2 + y^2)$, for height h, width k, and $0 < x, y < h$.*

Proof. First, we construct a trapezoid with top width k and height h. Then, we can cut off equilateral triangles of heights x and y respectively from the lower corners (60 deg angles) of the trapezoid. We have the restraint that $0 < x, y < h$ to make sure we are cutting off both triangles (otherwise we would have a pentagon), and that we are not cutting off too much to end up with a parallelogram or some other polygon. □

The next two sections are taken directly from Cameron et al [3].

3.1 Numerical Evidence

When the investigation of polygon numbers began, Cameron Hale, Jonathan Kelleher, and Caleb Falcione all worked together to discover information about hexagon numbers with prime heights congruent to 1 (mod 4). A spreadsheet of heights up to 3000 was created, which gave strong numerical evidence for both prime numbers congruent to 1 (mod 4) and composite heights of a particular prime factor composition. The following propositions and theorems are found in [3].

Proposition 3.2. *Adding $2h$ triangles to the two adjacent sides of a hexagon with height h does not change the residue class of the hexagon modulo $2h$.*

Proposition 3.3. *Fixing height h and side k and varying a and b, if we obtain all residues modulo $2h$, then, consequently, there exists H_h so that for every $H \geq H_h$, H is a hexagon number.*

The reason we've been focusing so much on residue classes of hexagon numbers modulo $2h$ is to help find multiplicities. When approaching the other polygon numbers, it is easy to take a particular polygon number and calculate its multiplicity. With hexagon numbers, however, it is much more difficult to do it this way.

Instead, we look at hexagon heights. If we can find a complete residue class for a particular hexagon height h, then we know that at some z in the integers, every integer past z will create a hexagon with that particular height h. So, we choose k large enough such that every $z \in \mathbb{Z}^+ \geq H$ is a hexagon number with height h.

Theorem 3.4. *Height $h = 13$ of a hexagon is the minimum height that produces all of the residues modulo $2h$.*

Proposition 3.5. *No even height h can produce all residues modulo $2h$.*

Theorem 3.6. *Given a positive integer n, there exists a hexagon number H_h of geometric multiplicity $m(H_n) \geq n$.*

3.2 Prime Heights Congruent to $1 \pmod 4$

Theorem 3.7. *Let $p > 5$ be a prime and $0 \neq a \in \mathbb{Z}_p$. Then $x^2 + y^2 = a$ is solvable in $\mathbb{Z}_p$.*

Theorem 3.8. *Let p be an odd prime. Then $p \equiv 1 \pmod 4$ iff $x^2 + y^2 \equiv 0 \pmod p$ is solvable with $0 < x, y < p$.*

Theorem 3.9. *$x^2 + y^2 \equiv 0 \pmod p$ is solvable with $0 < x, y < p$ iff either $x^2 + y^2 \equiv 0 \pmod{2p}$ or $x^2 + y^2 \equiv p \pmod{2p}$ is solvable with $0 < x, y < p$.*

Theorem 3.10. *Let $p \equiv 1 \pmod 4$, $p \geq 13$, be a prime. Then $x^2 + y^2 \equiv a \pmod{2p}$ is solvable with $0 < x, y < p$.*

Theorem 3.11. *Prime height $p \geq 13$ produces all residues for $H = 2kp + p^2 - (a^2 + b^2)$ modulo $2p$ with $0 < a, b < p$ iff $p \equiv 1 \pmod 4$. Moreover, no height $h < 13$ produces all residues modulo $2h$ with $0 < a, b < h$.*

3.3 Composite Heights

Definition 3.12 (Composite Height)**.** For a hexagon with height h, we can write h as a product pq where $p > 5$ is the product of primes congruent to $1 \pmod 4$ and q is either 1 or the product of distinct primes congruent to $3 \pmod 4$.

Here are some things we need to know about $\pmod{2pq}$:

Proposition 3.13. $(pq)^2 \equiv pq \pmod{2pq}$

Proof. $(pq)^2 - pq \equiv pq(pq - 1) \pmod{2pq}$

We know pq is an odd number from Definition 3.12. So, $pq - 1$ is an even number and can be written as some $2k$ for $k \in \mathbb{Z}^+$. This implies

$pq(pq - 1) \equiv 2kpq \equiv 0 \pmod{2pq}$

$(pq)^2 - pq \equiv 0 \pmod{2pq}$

$(pq)^2 \equiv pq \pmod{2pq}$ □

Our work relies heavily on Theorem 3.2 in Harrington's paper, stated here [4]:

Theorem 3.14. *Let $n \geq 2$ be an integer. Then, for every $z \in \mathbb{Z}_n$, $x^2 + y^2 \equiv z$ has a nontrivial solution if and only if*

1. *$n \not\equiv 0 \pmod{q^2}$ for any prime $q \equiv 3 \pmod 4$ with $n \equiv 0 \pmod q$*
2. *$n \not\equiv 0 \pmod 4$*
3. *$n \equiv 0 \pmod p$ for some prime $p \equiv 1 \pmod 4$*
4. *Also, when $n \equiv 1 \pmod 2$, we have the following additional conditions. Write $n = 5^k m$, where $m \not\equiv 0 \pmod 5$. Then, either*

(a) $k \geq 3$, *with no further restrictions on* m, *or*

(b) $k < 3$ *and* $m \equiv 0 \pmod{p}$ *for some prime* $p \equiv 1 \pmod{4}$.

Theorem 3.15. *For any hexagon height* pq*, as in Definition 3.12, for every* $z \in \mathbb{Z}_{pq}$*,* $z = x^2 + y^2$ *is solvable for* $x, y \in \mathbb{Z}_{pq}$*,* $0 < x, y < pq$.

Proof. **Case 1**. Assume 5 is not a factor of p. Using Theorem 3.14, we show our height pq holds from Definition 3.12. First, we know pq is set up so that p is a non-empty product of primes congruent to $1 \pmod{4}$, and q is either 1 or is a product of distinct primes congruent to $3 \pmod{4}$. Thus, parts 1 through 3 of Theorem 3.14 hold. When checking part 4, we write $pq = 5^0 m$. Then, it is evident part 4b holds, as $m = pq$ and p is non-empty.

Case 2. Assume $p = 5^k m$, $k \neq 0$, $m \not\equiv 0 \pmod{5}$. Using the same logic as Case 1, we know the first 3 parts of Theorem 3.14 hold from our composition of pq as in Definition 3.12. To show part 4 holds, we have the following sub-cases:

Sub-case 2a. $k = 1$. From Definition 3.12, we know $p > 5$, which means there must be another prime factor congruent to $1 \pmod{4}$. This satisfies part 4b of Theorem 3.14. Thus, this case holds for pq.

Sub-case 2b. $k = 2$. The reason this case fails in Theorem 3.14 is because $z = x^2 + y^2$ is unsolvable for $x^2, x^2 \neq 0$. Harrington in fact calls this a trivial solution, and throws it out.

However, this is perfect example of the difference between meaningful algebraic multiplicity and geometric multiplicity. Algebraically, $x^2, y^2 = 0$ are trivial solutions, but not so geometrically. Remember, our x and y values represent two side lengths of our hexagon. So, as long as $0 < x, y < h$, we count every solution to $z = x^2 + y^2$ as nontrivial.

Since the only reason this case fails is by throwing out algebraically trivial solutions, we know by counting these solutions as significant allows $z = x^2 + y^2$ to be solvable within the ring.

Sub-case 2c. $k \geq 3$. From Theorem 3.14, this sub-case passes part 4b, which means this case holds for pq. □

Theorem 3.16. *For any hexagon height* pq *as in Definition 3.12,* $x^2 + y^2 \equiv a \pmod{2pq}$ *is solvable for* $0 < x, y < pq$.

Proof. From Theorem 3.15, we know $x^2 + y^2 \equiv a \pmod{pq}$ is solvable for $0 < x, y < pq$. This means $x^2 + y^2 = a + kpq$ for some $k \in \mathbb{Z}^+ \cup \{0\}, x, y \neq 0$. Now we have two cases:

1. If k is even: $x^2 + y^2 = a + 2mpq$ for some $m \in \mathbb{Z}^+ \cup 0$ So, $x^2 + y^2 \equiv a \pmod{2pq}$.

2. If k is odd:

   $x^2 + y^2 = a + (2m + 1)pq$ for some $m \in \mathbb{Z}^+ \cup \{0\}$

   $x^2 + y^2 = a + 2mpq + pq$

   $x^2 + 2pqx + (pq)^2 + y^2 = a + 2mpq + pq + 2pqx + (pq)^2$

   $(x + pq)^2 + y^2 \equiv a + 0 + pq + 0 + (pq)^2 \pmod{2pq}$

$(x + pq)^2 + y^2 \equiv a + pq + pq \pmod{2pq}$ from Theorem 3.13

$(x + pq)^2 + y^2 \equiv a \pmod{2pq}$

Since we know $pq \not\equiv 0 \pmod{2pq}$, we know $x+pq \not\equiv 0 \pmod{2pq}$. Additionally, either $0 < x+pq \pmod{2pq} < pq$, or $pq < x+pq \pmod{2pq} < 2pq$. If the prior occurs, we're done. If the latter occurs, we can instead write a as $(-(x+pq))^2 + y^2 \equiv a \pmod{2pq}$. In other words, we now have $0 < -(x + pq) \pmod{2pq} < pq$.

□

Theorem 3.17. *For any hexagon height $h = pq$ as in Definition 3.12, h produces all residues for $H = 2kh + h^2 - (x^2 + y^2)$ modulo $2h$, with $0 < x, y < h$.*

Proof. We know $H \equiv 2kh + h^2 - (x^2 + y^2) \equiv h^2 - (x^2 + y^2) \pmod{2h}$. Additionally, we know $h^2 \equiv h \pmod{2h}$ from Theorem 3.13. Thus, to produce all residues $\pmod{2h}$, we need to show that $h - (x^2 + y^2) \equiv a \pmod{2h}$ is solvable for all $a \in \mathbb{Z}_h$. From Theorem 3.15, we know $x^2 + y^2 \equiv a \pmod{2h}$ is solvable to all $a \in \mathbb{Z}_h$. Since addition is closed in the ring, we then know $h - (x^2 + y^2) \equiv a \pmod{2h}$ is solvable. Thus, we can produce all residues $\pmod{2h}$. □

Theorem 3.18. *For any hexagon height $h \neq pq$ as in Definition 3.12, h will not produce all residues for $H = 2kh + h^2 - (x^2 + y^2)$ modulo $2h$, with $0 < x, y < h$.*

Proof. If $h \neq pq$ as in Definition 3.12, then either h is even, or h contains non-distinct prime factors congruent to $3 \pmod 4$. If h is even, then $2h \equiv 0 \pmod 4$, which then fails part 2 of Theorem 3.14, making it impossible to produce all residues for $H = 2kh + h^2 - (x^2 + y^2) \pmod{2h}$. If h contains non-distinct prime factors congruent to $3 \pmod 4$, then $2h \equiv 0 \pmod{c^2}$ for some prime $c \equiv 3 \pmod 4$. This fails part 1 of Theorem 3.14, once again making it impossible to produce all the residues. □

As we can see, the proof for hexagon numbers with composite height pq encompasses hexagons with prime heights congruent to $1 \pmod 4$. The method of proving these theorems was different, and in fact stronger, than that of subsection 3.2, Prime Heights.

4 PENTAGON NUMBERS

We look at parameterizing pentagon numbers from triangle numbers and parallelogram numbers.

Theorem 4.1. *A pentagon number G can be written $G = 2kh - a^2$, for top width k, height h, and side length a.*

Proof. We can create a pentagon from a parallelogram. We start with some parallelogram with height h and width k. Then, we cut off the equilateral triangle with side length a from the lower left hand corner of the parallelogram. The restraints of $0 < a < h, k$ is to make sure we don't cut off too much and end up with another parallelogram or some other polygon. □

Theorem 4.2. *For a pentagon number G, the maximum value of top width k is $\lceil \frac{G+1}{4} \rceil$.*

Proof. If we think of creating pentagons by subtracting an equilateral triangle from one side of a parallelogram, it is easy to see using Theorems 1.7 and 2.1 that we can write a pentagon number G as $G = 2kh - a^2$. Since we know in order to have a pentagon, we must have a minimum height of 2, we can see our maximum value for k will be dependent upon the size of G.

If we set $h = 2$ and $a = 1$, we can then calculate what our k value would be, if we allowed for partial equilateral triangles:

$G = 2 * k * 2 - (1)^2$

$G = 4k - 1$

$k = \frac{G+1}{4}$

However, we cannot allow for partial equilateral triangles when tiling a polygon. Thus, we take the ceiling of this value to find the maximum value k can be for a given pentagon number G. □

While we were unable to find a closed form formula to describe the multiplicities of pentagon numbers, we were able to find an algorithm to exhaustively check the multiplicity of any given pentagon number.

Definition 4.3 (Pentagon Number Multiplicity Algorithm)**.** We start off by minimizing the height and maximizing the top width of a possible pentagon. We will be iterating through a combination of height, top width, and values for a, the height of the triangle taken off the parallelogram to create the pentagon, and calculating the resulting pentagon number. We set our variables:

$a = 1$, where a is the height of the equilateral triangle subtracted from our parallelogram.

$m(G) = 0$, where $m(G)$ stand for the multiplicity of pentagon number G.

$h = 2$, where h is the height of our pentagon

$go = True$

Now, we calculate the biggest value our top width k can be using Theorem 4.2 and call it $kMax$. We will be using $kMax$ to keep track of where we started iterating.

We initialize the first value of k we start at:

$kInitial = kMax$

$k = kInitial$

Now, we loop through changing h, k, and a to find each possible pentagon number, and count the number of matches we get to the given pentagon number G.

We set go to $True$, and run until we stop.

We calculate $g = 2kh - a^2$, and set $testA$ to $False$. $testA$ tells us when to move on from changing k values and to instead change either a or move on to a different value of h. Once g is calculated, we see if it is equal to or greater than our objective, pentagon number G. If $g > G$, we look at k. If $k = h$, we immediately go to change a values, since to avoid counting geometric rotations of the same pentagon, we keep $k \geq h$. If $k \neq h$, we subtract 1 from k and restart in the while loop. Now, if $g = G$, then we have found a set of parameters that yield pentagon number G, so we increase the multiplicity of G by 1. Then, we move on to set $testA$ to $True$. If neither of the above two conditions occur, we set $testA$ to $True$.

If $testA$ is set to $True$, we then decide whether to further manipulate a, or to increase h and start over with a new $kInit$. First, if $a \neq h - 1$, we know since we start at $a = 1$, we can increase a and test a new pentagon. So, we increase a by 1, and we reset k to our $kInit$. We must reset k in case we tested for a lower k value earlier.

If $a = h - 1$, this means that we cannot increase a anymore without taking too big of a triangle away from our parallelogram, yielding a trapezoid instead of a pentagon. Thus, we either need to change h or end the while loop. We know if $h < kInit - 1$, then we have room to increase h and reset our $kInit$. Recall, h is the height of our pentagon, and $kInit$ is the current max top width for a given height. By going through our algorithm this way, we can guarantee that we only test for $k \geq h$, therefore eliminating any rotations or mirror images of pentagons.

So, if $a = h - 1$ and $h < kInit - 1$, we decrease $kInit$ by 1, increase h by 1, reset $a = 1$, and reset $k = kInit$ (the new $kInit$ that is). Now, if $a = h - 1$ but $h \nless kInit - 1$, then we have nothing left to test. So, we set go to $False$, thereby ending the while loop.

Once out of the while loop, we access the variable that has been storing the multiplicity of G and can print the multiplicity for any given pentagon number G.

To see the code in full, please see Figure 6.

```python
testTo=100
i=7

while i<=testTo:

    a = 1
    mP = 0
    h = 2
    p = i
    go = True

    kMax = (p+1-(p%4))/4


    kInit = kMax
    k = kInit

    while(go):
        temp = 2*k*h-(a*a)
        testA = False

        if(temp == p):
            print("(" + str(k) + ", " + str(h) + ", " + str(a) + ")")
            mP += 1
            testA = True
        elif(temp > p):
            if(k==h):
                testA = True
            else:
                k -= 1
        else:
            testA = True


        if(testA):
            if(a==h-1):
                if(h<kInit-1):
                    kInit -= 1
                    h += 1
                    a = 1
                    k=kInit
                else:
                    go = False
            else:
                a += 1
                k = kInit
    if(mP != 0):
        print("The multiplicity of " + str(p) + " is " + str(mP)) # + ", with the above")
        #print(mP)
    #print("triples representing different values of (k,h,a).")

    i+=1
```

Figure 6: PYTHON CODE TO GENERATE THE MULTIPLICITY OF A GIVEN PENTAGON NUMBER

5 FUTURE WORK

There are still several unanswered questions surrounding the nature of various polygon numbers.

Conjecture 5.1. For prime $p \equiv 1 \pmod 4$, the minimum number of geometrically non-congruent hexagons of height p at every residue modulo $2p$ is the integer part of $\frac{p}{8}$.

There seems to be a large amount of numerical evidence to support this claim. When originally investigating this conjecture, we wrote a quick program in python to investigate how hexagon parameters are counted to see if this proof could be done by showing double counting, but unfortunately did not have enough data to show this.

Question 1. Is it the case that for every natural number n, there is a natural number J_n, such that every pentagon number $P \geq J_n$ has $m(P) \geq n$?

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