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On the Spectral Theory for Ordinary Differential Equations with Distributional Coefficients with and Emphasis on the Periodic Case

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ON THE SPECTRAL THEORY FOR ORDINARY DIFFERENTIAL EQUATIONS WITH DISTRIBUTIONAL COEFFICIENTS WITH AN EMPHASIS ON THE PERIODIC CASE

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ABSTRACT

In this thesis, we consider differential equations with distributional matrix coefficients giving particular care to periodic distributions. In pursuit of this, we first examine some cases in which there is no existence and uniqueness theorem for a given initial value problem, and then verify some properties of the minimal and maximal operators.

We then focus on periodic systems and establish a Floquet Theory for such systems as well as extend several fundamental properties which are well known for periodic functions to periodic distributions. We also examine stability sets for differential equations with periodic distributional coefficients in one and two dimensions, and the relationship between stability sets and the spectrum of the differential equation.

DEDICATION

To my parents who have guided me to this point, allowing me the freedom to explore, my siblings who have supported my endeavors, and my ever-evolving cohort which has helped me beyond measure.

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TABLE OF CONTENTS

ABSTRACT ii

DEDICATION iii

ACKNOWLEDGEMENTS iv

CHAPTER 1. INTRODUCTION 1
REFERENCES 7

CHAPTER 2. ON THE SPECTRAL THEORY FOR FIRST-ORDER SYSTEMS WITHOUT THE UNIQUE CONTINUATION PROPERTY . 8
1. Introduction 9
2. Obtaining solutions 11
3. Minimal and maximal relations 16
Acknowledgements 19
REFERENCES 20

CHAPTER 3. ON THE SPECTRAL THEORY OF SYSTEMS OF FIRST ORDER EQUATIONS WITH PERIODIC DISTRIBUTIONAL COEFFICIENTS 21
1. Introduction 22
2. Basic properties of periodic distributions 23
3. Floquet theory 27
4. Spectral theory 30
5. The case $n = 1$ 32
6. The case $n = 2$, real coefficients 34
7. Examples 47
REFERENCES 49

CHAPTER 1
INTRODUCTION

Modern approaches to the spectral theory ordinary differential equations can be traced to a book by Joseph Fourier, titled "Théorie analytique de la chaleur" in which he motivates, and models, the heat equation

$$u_t = u_{xx}.$$

In this work he establishes equations similar to those which had existed for fluid dynamics and establishes a way to solve for separable variables, see [7]. Using properties of thermal equilibrium, he obtained a formula which allow the equality:

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} = -\lambda.$$

Here we rely on expressing our solution to the heat equation as by $u(x,t) = X(x)T(t)$. In this way we express our temperature function as a separable equation in terms of time and position. These λ values will be eigenvalues of the differential equation, and the associated u function given as XT will be our eigenfunction. He further expressed these solutions as a sum or difference of sine and cosine terms multiplied by an exponential.

This originated as a boundary value problem and was expanded upon as studies of physical phenomena continued by Sturm and Liouville for problems of the form $-(pu')' + (q - \lambda w)u = 0$, where u satisfies the boundary conditions posed at both endpoints. In this case, q represents the potential, and w is a weight allowing for more varied studies of physical properties beyond temperature functions. They further went on to find the eigenfunctions associated with the differential equation. Large developments have been made on this topic, in particular, given p and w non-negative and if w, q and $\frac{1}{p}$ all locally Lebesgue integrable then solutions are unique. This will be a particularly important property for many of the assumptions on finding spectrum. Sturm-Liouville will be a primary concern for this paper.

Periodic second order differential equations became popular largely due to the influence of George William Hill in 1886 [3] when he used them to describe the motion of the orbit of the moon. This led to equations of the form $\{P(x)y'(x)\}' + Q(x)y(x) = 0$ being referred to as Hill's equations when both $P(x)$ and $Q(x)$ are periodic with coincident periods. He considered equations using the mean angular motion of the sun, while ignoring solar eccentricity.

Naturally, periodic ordinary differential equations had been previously considered. In 1883, Gaston Floquet published "Sur les équations différentielles linéaires à coefficients périodiques" [2] in which he considered homogeneous periodic differential equations with shared period amongst the coefficients. He then established a generalized form of solutions to the system, a determinant, which is closely related to stability, and various fundamental properties of the solutions and their anti-derivatives. Throughout this paper we will be following a similar path to establish Floquet theory for differential equations with periodic distributional coefficients with coincident periods. Liapunov in 1907 further built upon this, as well as the work done by Poincaré which examined stability, instability, and equilibrium of particles in motion more generally, see [1]. A portion of his motivation was to remove a priori assumptions which allowed for simplification, as well as extending the theory which existed for only a small selection of cases.

The true motivation for this dissertation though, lies not in the foundational material of periodic differential equations, but instead with periodic potentials. This leads to time-independent Schrödinger equation developed in 1926. This equation is used to model the behavior of electrons in atoms using wave equations to find the behavior of the electron's orbits, and while it is not originally a system with periodic potential, these studies of the behavior of the electrons of the hydrogen atom laid foundational work. The homogeneous time-independent Schrödinger equation takes the form $-\psi(x)'' + v(x)\psi(x) = \lambda r(x)\psi(x)$, where v is the potential given by the electromagnetic field. In the case of a crystalline solid, this potential becomes periodic due to the repeated atomic nature of crystals. This equation is closely tied to de Broglie's hypothesis that all matter has an associated wavelength. He created this theory as an attempt to bridge classical matter based Newtonian physics and an emerging field of radiation physics based in light and electromagnetic propagation, see [4]. This was based off of Planck's assertion that, unlike in classical assumptions, radiation emissions cannot be continuous, but must emit finite quantities, further referred to as quanta.

With the introduction of quanta, and further quantum mechanics, it became necessary to consider eigenvalues, eigenfunctions, and the spectrum associated with the Schrödinger equation. To this end, we find it useful to rewrite the Schrödinger equation as $Ju' + qu = \lambda wu$ with distributional coefficients, which will be expounded upon shortly.

First, we must discuss some fundamentals of the spectrum. We will do so following the lead of Ghatasheh and Weikard [6]. Given a Hilbert space $\mathcal{H}$, and a closed relation mapping $\mathcal{H}$ onto itself, denoted by T we can then discuss the spectrum of T. A number $\lambda \in \mathbb{C}$ is an eigenvalue of T if there is a $u \in \mathcal{H}$ such that $(u, \lambda u) \in T$. Similar to in the case of matrices, the associated u is the eigenfunction associated with the eigenvector λ. We then can also consider the solvable space, $S_\lambda = \{u \in \mathcal{H} : (v, \lambda v + u) \in T\}$ where $v \in \mathcal{D}(T)$, the domain of T. In this case we are not requiring that λ be an eigenvalue. Combining both of these concepts, the resolvent set $\rho(T)$ is the set of λ such that λ is not an eigenvalue, and $S_\lambda = \mathcal{H}$. The spectrum, which we are concerned with, is the complement of $\rho(T)$, which is to say the collection of all λ such that they are eigenvalues, or such that their solvable space is not all of $\mathcal{H}$.

We will be particularly concerned with spectral theory for self adjoint relations. This concept requires that we establish equivalence classes in the space $L^2(w)$ such that f and g are in the same equivalency class if and only if $fw = gw$. Using this equivalence relation we can make the following subspaces of $L^2(w) \times L^2(w)$:

$$T_{\max} = \{([u],[f]) \in L^2(w) \times L^2(w) : u \in BV^{\#}_{\mathrm{loc}}((a,b))^n, Ju' + qu = wf\}$$

and,

$$T_{\min} = \{([u],[f]) \in L^2(w) \times L^2(w) : u \in T_{\max} \text{ and } \operatorname{supp}(u) \text{ is compact}\}.$$

We then say T is self adjoint if $T_{\min} = T_{\max}$.

We will be looking at establishing this relationship with distributional coefficients. To understand distributions we must first define test functions. A test function is an infinitely often differentiable and compactly supported function. We denote the space of test functions as C_0^∞. The set of distributions is a subset of the linear forms from C_0^∞ to $\mathbb{C}$ such that for all compact sets K, and all test functions, ϕ, with support in K, there exists a C_K and $k_K \in \mathbb{N}_0$ such that:

$$|u(\phi)| \leq C_K \sum_{\nu=0}^{k_K} \sup |\phi^{(\nu)}|.$$

Distributions can further be categorized by their order, the order of a distribution is finite if the same $k \in \mathbb{N}_0$ can be used for all compact sets K, the smallest non-negative k such that the above inequality holds is then called the order of u.

There will be several known facts about distributions that will be used throughout without proof with all of them further elaborated by Bennewitz, Brown, and Weikard [5]. The first, and possibly most often employed, is the definition of a distributional derivative: $u'(\phi) = -u(\phi')$. We will also frequently use that every positive distribution is a positive Radon measure. We will also use the following two theorems heavily, which appear as C.11 and C.12 in [5]:

(1) (du Bois-Reymond) Suppose $u \in D'$ and that $u^{(k)} = 0$, then u is a polynomial of degree at most k.

(2) A function of locally bounded variation has a distributional derivative which is of order zero, i.e., it is a Radon measure, in fact a Stieltjes integral generated by the function. Conversely, any primitive of a distribution of order zero is a function of locally bounded variation.

This last theorem gives us the definition that, for a distribution of order zero, w with primitive dW, and a test function, ϕ, $w(\phi) := \int \phi dW$. We will also use that if $f \in \mathcal{L}^2(w)$, then it is in $\mathcal{L}^1_{loc}(w)$. To show this, let $f \in \mathcal{L}^2(w)$ and K compact, we have $\left|\int_K f dW\right| \leq \left(\int_K |f|^2 dW\right)^{\frac{1}{2}} \left(\int_K dW\right)^{\frac{1}{2}} \leq \left(\int |f|^2 dW\right)^{\frac{1}{2}} \left(\int_K dW\right)^{\frac{1}{2}}$, which is finite as $f \in \mathcal{L}^2(w)$.

In our differential equation, we will, at times, need the product of a non-negative distribution of order zero, w, and a function, f, in $\mathcal{L}^1_{loc}(w)$ as a distribution. We define $(wf)(\phi) := \int f\phi dW$. To establish that wf is a distribution, we will use the compact set K, and ϕ, a test function with support in K, to obtain:

$$\begin{aligned} |wf(\phi)| &= \left|\int_K f\phi dW\right| \\ &\leq \sup_K(|\phi|) \int_K |f| dW. \end{aligned}$$

The second factor is the constant C_K determined by K, as $f \in \mathcal{L}^1_{\text{loc}}$. As such we see this is, indeed, a distribution.

My dissertation has two main portions. The first focuses on establishing a novel approach to solving the differential equation with measures that are not absolutely continuous with respect to Lebesgue measure through solutions on each interval for with the measure is continuous. After doing so we establish $T^*_{\min} = T_{\max}$ for the differential equation $Ju' + qu = wf$ with J a constant, invertable, skew-Hermitian matrix, and q and w matrices of distributions of order zero, with q Hermitian and w non-negative. By establishing that $T^*_{\min} = T_{\max}$ we have many further conclusions as shown by Ghatasheh and Weikard in [6].

The second portion focuses on periodic distributional coefficients and establishing stability and instability intervals as well as an extension of Floquet theory, a monodromy matrix, and a Greens function for this case. In pursuit of this we first establish a definition of periodicity for distributions, while also verifying some fundamental characteristics of periodic distributions. Following this we also look at specialized cases of first and second order differential systems. In our considerations, particular attention is given to the extension of stability sets and their relationship with the spectrum of the associated systems.

This portion in particular, the establishment of bounded solutions, has implications for stable physical solutions avoiding either solutions of uncontrolled decay or growth. We see also that there is no point spectrum, i.e. there are no eigenvalues, but instead that the spectrum is purely continuous which is consistent with the observations made by de Broglie in [4]. We also explore the relationship between the Floquet discriminant and $T_{\max}$ seeing that the behavior of the system may be understood through the Monodromy matrix and the Floquet discriminant.

The second portion also ends with several examples which both highlight some differences between this and the classical cases as well as looking at some cases which return familiar results.

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ON THE SPECTRAL THEORY FOR FIRST-ORDER SYSTEMS WITHOUT THE UNIQUE CONTINUATION PROPERTY

by

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ON THE SPECTRAL THEORY FOR FIRST-ORDER SYSTEMS WITHOUT THE UNIQUE CONTINUATION PROPERTY

Abstract. We consider the differential equation $Ju' + qu = wf$ on the real interval (a, b) when J is a constant, invertible skew-Hermitian matrix and q and w are matrices whose entries are distributions of order zero with q Hermitian and w nonnegative. In this situation it may happen that there is no existence and uniqueness theorem for balanced solutions of a given initial value problem. We describe the set of solutions the equation does have and establish that the adjoint of the minimal operator is still the maximal operator, even though unique continuation of balanced solutions fails.

1. Introduction

Ghatasheh and Weikard [4] investigated the spectral theory for the first-order system

$$Ju' + qu = wf$$

of differential equations on the real interval (a, b) assuming that J is a constant, invertible skew-Hermitian matrix and q and w are matrices whose entries are distributions of order zero[1] with q Hermitian and w nonnegative. Crucially, [4] requires that initial value problems for this equation have unique balanced[2] solutions. Indeed, unique continuation of a solution across a point where Q, the anti-derivative of q, has a discontinuity may fail. With the aid of linear algebra we were able to overcome this obstacle, describe the set of solutions of the differential equation, and establish

[1]Recall that distributions of order 0 are distributional derivatives of functions of locally bounded variation and hence may be thought of, on compact subintervals of (a, b), as measures. For simplicity we might use the word measure instead of distribution of order 0 below.
[2]The concept of balanced solutions is defined below and the rationale of using it is explained in [4].

that the adjoint of the minimal operator is still the maximal operator, even if unique continuation of balanced solutions fails.

The relationship between minimal and maximal operators is a cornerstone for the spectral theory for differential equations. Of course, this topic is very well studied when the coefficients are locally integrable functions but in the case of measure coefficients much less is known. The first to consider an equation with a measure coefficient was Krein [5] in 1952 when he modeled a vibrating string. Also motivated by physical applications were Gesztesy and Holden [3] in 1987 who described Schrödinger equations with point interactions, specifically δ'-interactions. In 1999 Savchuk and Shkalikov [6] treated Schrödinger equations with potentials in the Sobolev space $W^{-1,2}_{\rm loc}$, a paper which spurred many further developments. With the help of quasi-derivatives Eckhardt et al. [1] showed in 2013 that such equations can be cast as first-order 2×2-systems with locally integrable coefficients. Eckhardt and Teschl [2] considered a system where the coefficients are measures, viz. $Ju' + qu = wf$ where $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $q = \left(\begin{smallmatrix} \chi & 0 \\ 0 & -\varsigma \end{smallmatrix}\right)$, and $w = \left(\begin{smallmatrix} \rho & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Their approach covers both the Krein string ($\chi = 0$ and $\varsigma = 1$) as well as the δ'-interaction ($\chi = 0$, $\varsigma = 1 + \beta\delta_0$, and $\rho = 1$ in the simplest case). Crucially, they require that the support of the discrete part of ς does not intersect the corresponding sets for χ or ρ, a condition which guarantees unique continuation. Both [1] and [2] are also excellent sources for a more thorough history of the subject.

Let us add a few words about notation. $\mathcal{D}'^0((a,b))$ is the space of distributions of order 0, i.e., the space of distributional derivatives of functions of locally bounded variation. Any function u of locally bounded variation has left- and right-hand limits denoted by u^- and u^+, respectively. Also, u is called balanced if $u = u^\# = (u^+ + u^-)/2$. We use $\mathbb{1}$ to denote an identity matrix of appropriate size and superscripts $^\top$ and * indicate transposition and adjoint, respectively. The orthogonal complement of a subspace S of a Hilbert space H is denoted by $H \ominus S$ or by $S^\perp$. For $c_1, ..., c_N \in \mathbb{C}^n$ we abbreviate the vector $(c_1^\top, ..., c_N^\top)^\top \in \mathbb{C}^{nN}$ by $(c_1, ..., c_N)^\diamond$. Finally we note that, generally, our differential equations are represented by linear relations rather than linear operators. Consequently we work with graphs of such relations (even when they are operators).

2. Obtaining solutions

We begin by describing the set of solutions of the first-order system

$$Ju' + qu = wf$$

on the interval (a, b) assuming that the coefficients satisfy the following hypothesis.

HYPOTHESIS 2.1. *J is a constant, invertible and skew-Hermitian $n \times n$-matrix. Both q and w are in $\mathcal{D}'^0((a,b))^{n\times n}$, w is nonnegative, and q Hermitian.*

Associated with a nonnegative distribution $w \in \mathcal{D}'^0((a,b))^{n\times n}$ is a Hilbert space $L^2(w)$ with inner product $\langle u, v\rangle = \int u^* w v$ (recall that positive distributions are positive measures). Its elements are equivalence classes of functions $[f]$ satisfying $\|f\|^2 = \int f^* w f < \infty$ and, as usual, two functions f and g are equivalent, if $\|f - g\| = 0$.

We denote the left-continuous anti-derivatives of q and w by Q and W, respectively. $\Delta_q(x) = Q^+(x) - Q^-(x)$ stands for the jump of Q at a point x. Similarly, $\Delta_w(x) = W^+(x) - W^-(x)$.

Suppose $\xi_0 \in (\xi_1, \xi_2) \subset (a, b)$. When $f \in L^2(w)$ (as we shall henceforth assume) it was shown in [4] that the initial value problem $Ju' + qu = wf$, $u(\xi_0) = u_0 \in \mathbb{C}^n$ has a unique balanced solution of locally bounded variation in (ξ_1, ξ_2) provided that the matrices $B_\pm(x) = J \pm \Delta_q(x)/2$ are invertible for all $x \in (\xi_1, \xi_2)$. At a point x of discontinuity of Q or W, the differential equation requires that

$$J(u^+(x) - u^-(x)) + \Delta_q(x)u(x) = \Delta_w(x)f(x)$$

where, u being balanced, $u(x) = (u^+(x) + u^-(x))/2$. This is equivalent to

$$B_+(x)u^+(x) - B_-(x)u^-(x) = \Delta_w(x)f(x). \tag{2.1}$$

From this it is obvious that we may not be able to continue a solution across x from left to right (or from right to left), if $B_+(x)$ (or $B_-(x)$) fails to be invertible. In our particular situation, where $J^* = -J$ and $\Delta_q(x)^* = \Delta_q(x)$, we have $B_-(x) = -B_+(x)^*$ and hence that $B_-(x)$ is invertible if and only if $B_+(x)$ is.

On account of the fact that Q is locally of bounded variation, it is clear that the set of points x where $B_\pm(x)$ are not invertible is discrete and finite on compact subintervals of (a, b) even though the set of all jumps of Q may be dense.

Let us now fix an interval $[\xi_1, \xi_2] \subset (a, b)$ assuming that the points in (ξ_1, ξ_2) where $B_\pm$ are not invertible are among the points $x_1 < ... < x_N$. Normally one would choose these to be precisely the points where $B_\pm$ are not invertible but it is advantageous to avoid the case $N = 1$. For convenience let us also set $x_0 = \xi_1$ and $x_{N+1} = \xi_2$. As mentioned above we do have unique solutions of initial value problems and, indeed, a variation of constants formula in any of the intervals (x_j, x_{j+1}). These solutions have limits at the endpoints of the interval and we may even use, for instance, the left endpoint to pose an initial condition. Therefore the general solution of $Ju' + qu = wf$ in (x_j, x_{j+1}) is represented by

$$u^-(x) = U_j^-(x)(c_j + J^{-1} \int_{(x_j,x)} U_j^* wf) \tag{2.2}$$

where c_j is an arbitrary element of $\mathbb{C}^n$ and U_j a balanced fundamental matrix for $Ju' + qu = 0$ in (x_j, x_{j+1}) which we may choose so that $\lim_{x \downarrow x_j} U_j(x) = \mathbb{1}$. We then define $U_j(x_{j+1}) = \lim_{x \uparrow x_{j+1}} U_j^-(x)$. For u to be a solution of $Ju' + qu = wf$ on (x_0, x_{N+1}) we need u to be determined by (2.2) (for appropriate choices of the c_j) in the respective intervals. Moreover, according to equation (2.1), u must satisfy

$$B_+(x_j)u^+(x_j) - B_-(x_j)u^-(x_j) = \Delta_w(x_j)f(x_j) \quad \text{for } j = 1, ..., N. \tag{2.3}$$

Note that $u^+(x_j) = c_j$ and $u^-(x_j) = U_{j-1}(x_j)(c_{j-1} + J^{-1}I_{j-1}(f))$ where $I_{j-1}(f) = \int_{(x_{j-1},x_j)} U_{j-1}^* wf$. Thus we may rewrite equation (2.3) as

$$(-B_-(x_j)U_{j-1}(x_j), B_+(x_j))\left(\begin{smallmatrix} c_{j-1} \\ c_j \end{smallmatrix}\right) = \Delta_w(x_j)f(x_j) + B_-(x_j)U_{j-1}(x_j)J^{-1}I_{j-1}(f).$$

At this point it appears helpful to introduce the following notation. Let

$$\begin{aligned}
\mathcal{B} &= \operatorname{diag}(B_+(x_1), ..., B_+(x_N)), \\
\mathcal{U} &= \operatorname{diag}(U_0(x_1), ..., U_{N-1}(x_N)), \\
\mathcal{J} &= \operatorname{diag}(J, ..., J),
\end{aligned}$$

and $E_\top = (0, \mathbb{1})$ and $E_\perp = (\mathbb{1}, 0)$, two $nN \times n(N+1)$-matrices which, respectively, strip the first and the last n coordinates off a vector. Then we have

$$\begin{aligned}
B &= \mathcal{B}^*\mathcal{U}E_\perp + \mathcal{B}E_\top \\
&= \begin{pmatrix} -B_-(x_1)U_0(x_1) & B_+(x_1) & 0 & \cdots & 0 \\ 0 & -B_-(x_2)U_1(x_2) & B_+(x_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & -B_-(x_N)U_{N-1}(x_N) & B_+(x_N) \end{pmatrix}.
\end{aligned}$$

If we now introduce the abbreviations

$$\begin{aligned}
\tilde{u} &= (c_0, ..., c_N)^\diamond, \\
\mathcal{R}(f) &= (\Delta_w(x_1)f(x_1), ..., \Delta_w(x_N)f(x_N))^\diamond, \\
\mathcal{I}(f) &= (I_0(f), ..., I_{N-1}(f))^\diamond,
\end{aligned}$$

and, for later purposes,

$$\tilde{\mathcal{I}}(f) = (0, ..., 0, I_N(f))^\diamond \in \mathbb{C}^{nN},$$

equations (2.3) may be written as

$$B\tilde{u} = \mathcal{R}(f) - \mathcal{B}^*\mathcal{U}\mathcal{J}^{-1}\mathcal{I}(f). \tag{2.4}$$

We have proved the following result.

THEOREM 2.1. If u is any solution of $Ju' + qu = wf$ on (x_0, x_{N+1}) then $\tilde{u} = (u^+(x_0), ..., u^+(x_N))^\diamond$ is a solution of equation (2.4). Conversely, a solution $\tilde{u} = (c_0, ..., c_N)^\diamond$ of equation (2.4) provides a solution of $Ju' + qu = wf$ on (x_0, x_{N+1}) given by (2.2) for $j = 0, ..., N$.

It is clear that the rank of B is no larger than nN and hence the kernel of B has at least dimension n. Note, however, that it is possible for the dimension of the kernel of B to be larger than n, i.e., to have more than n independent solutions of the homogeneous differential equation $Ju' + qu = 0$.

When we consider balanced solutions of $Ju'+qu = 0$, the relationship between $\tilde{u}$ and the vector of values of u at the points x_1, ..., x_N, i.e., the vector $\hat{u} = (u(x_1), ..., u(x_N))^\diamond$ is given by $\hat{u} = C\tilde{u}$ where

$$C = \frac{1}{2}(\mathcal{U}E_\perp + E_\top) = \frac{1}{2}\begin{pmatrix} U_0(x_1) & \mathbb{1} & 0 & \cdots & 0 \\ 0 & U_1(x_2) & \mathbb{1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & U_{N-1}(x_N) & \mathbb{1} \end{pmatrix}.$$

We also introduce the matrices B_m and C_m which are obtained from B and C respectively by removing the first and last n columns. Earlier we chose, without loss of generality, $N \geq 2$ to avoid the case when B_m and C_m have no columns.

We have the following relationship between B and C.

LEMMA 2.2. $C^*B - B^*C = \operatorname{diag}(-J, 0, ..., 0, J)$. In particular, $C_m^*B - B_m^*C = 0$.

PROOF. Since $\mathcal{B} - \mathcal{B}^* = 2\mathcal{J}$ we get

$$C^*B - B^*C = (E_\top^*\mathcal{J}E_\top - E_\perp^*\mathcal{U}^*\mathcal{J}\mathcal{U}E_\perp).$$

It was shown in [4] that $u^{-*}Jv^{-}$ is constant on any interval on which $B_\pm$ are everywhere invertible when u and v are solutions of $Ju' + qu = 0$. In particular, $U_j(x_{j+1})^* J U_j(x_{j+1}) = \lim_{x\uparrow x_{j+1}} U_j^-(x)^* J U_j^-(x) = J$. This implies that $\mathcal{U}^*\mathcal{J}\mathcal{U} = \mathcal{J}$ and hence the claim. □

LEMMA 2.3. Suppose $\hat{u} \in \ker B_m^*$. Then there exists a unique vector $\tilde{u}$ such that $B\tilde{u} = 0$ and $C\tilde{u} = \hat{u}$. Moreover, if $\hat{u} \in \ker B^* \subset \ker B_m^*$, then the first and the last n components of $\tilde{u}$ are equal to 0.

PROOF. If a solution $\tilde{u}$ indeed exists, it must satisfy $\mathcal{U}E_\perp\tilde{u} = 2\hat{u} - E_\top\tilde{u}$ and hence $0 = 2\mathcal{B}^*\hat{u} + (\mathcal{B} - \mathcal{B}^*)E_\top\tilde{u}$. This implies $E_\top\tilde{u} = -\mathcal{J}^{-1}\mathcal{B}^*\hat{u}$. Similarly, using $E_\top\tilde{u} = 2\hat{u} - \mathcal{U}E_\perp\tilde{u}$, we get $E_\perp\tilde{u} = \mathcal{J}^{-1}\mathcal{U}^*\mathcal{B}\hat{u}$. Thus a solution is unique.

To prove existence note that $\hat{u} = (\hat{u}_1, ..., \hat{u}_N)^\diamond \in \ker B_m^*$ implies

$$B_+(x_k)^*\hat{u}_k + U_k(x_{k+1})^* B_+(x_{k+1})\hat{u}_{k+1} = 0$$

for $k = 1, ..., N-1$. Hence the assignments $E_\top\tilde{u} = -\mathcal{J}^{-1}\mathcal{B}^*\hat{u}$ and $E_\perp\tilde{u} = \mathcal{J}^{-1}\mathcal{U}^*\mathcal{B}\hat{u}$ define $\tilde{u}$ unambiguously. Also $\tilde{u}$ satisfies $B\tilde{u} = 0$ and $C\tilde{u} = \hat{u}$.

For the last claim notice that $(C^*B - B^*C)\tilde{u} = 0$. If $\tilde{u} = (c_0, ..., c_N)^\diamond$, Lemma 2.2 gives $-Jc_0 = Jc_N = 0$ and hence $c_0 = c_N = 0$ as claimed. □

THEOREM 2.4. If $\hat{u} \in \ker B_m^*$, then $Ju' + qu = 0$ has a unique solution u on the interval (x_0, x_{N+1}) such that $\tilde{u} = (u^+(x_0), ..., u^+(x_N))^\diamond$ satisfies $B\tilde{u} = 0$ and $C\tilde{u} = (u(x_1), ..., u(x_N))^\diamond = \hat{u}$. If $\hat{u} \in \ker B^* \subset \ker B_m^*$ then, additionally, $u^+(x_0) = u^-(x_{N+1}) = 0$ so that $\operatorname{supp} u \in [x_1, x_N]$.

PROOF. This is an immediate consequence of Theorem 2.1 and Lemma 2.3. □

LEMMA 2.5. Suppose $f \in L^2(w)$ and $\hat{u} \in \ker B_m^*$. Then there is a function u satisfying $Ju' + qu = 0$ on (x_0, x_{N+1}), $(u(x_1), ..., u(x_N))^\diamond = \hat{u}$, and $\hat{u}^*\mathcal{F}(f) = \int_{(x_0,x_{N+1})} u^* w f$, where

$$\mathcal{F}(f) = \mathcal{R}(f) - \mathcal{B}^*\mathcal{U}\mathcal{J}^{-1}\mathcal{I}(f) + \mathcal{B}\mathcal{J}^{-1}\tilde{\mathcal{I}}(f).$$

PROOF. Let u be the function furnished by Theorem 2.4 and let $\tilde{u}$ be the vector $(u^+(x_0), ..., u^+(x_N))^\diamond$. Then $\mathcal{B}E_\top\tilde{u} = -\mathcal{B}^*\mathcal{U}E_\perp\tilde{u}$ since $B\tilde{u} = 0$. We also have $2C\tilde{u} = \mathcal{U}E_\perp\tilde{u} + E_\top\tilde{u}$. Using $\mathcal{B} - \mathcal{B}^* = 2\mathcal{J}$ gives us that $\mathcal{B}C\tilde{u} = \mathcal{J}\mathcal{U}E_\perp\tilde{u}$ and $\mathcal{B}^*C\tilde{u} = -\mathcal{J}E_\top\tilde{u}$ or, taking adjoints,

$$\tilde{u}^*C^*\mathcal{B}^* = -\tilde{u}^*E_\perp^*\mathcal{U}^*\mathcal{J} \quad \text{and} \quad \tilde{u}^*C^*\mathcal{B} = \tilde{u}^*E_\top^*\mathcal{J}.$$

These and $\mathcal{U}^*\mathcal{J}\mathcal{U} = \mathcal{J}$ imply

$$\begin{aligned}\hat{u}^*\mathcal{F}(f) &= \hat{u}^*\mathcal{R}(f) - \tilde{u}^*C^*\mathcal{B}^*\mathcal{U}\mathcal{J}^{-1}\mathcal{I}(f) + \tilde{u}^*C^*\mathcal{B}\mathcal{J}^{-1}\tilde{\mathcal{I}}(f) \\ &= \hat{u}^*\mathcal{R}(f) + \tilde{u}^*E_\perp^*\mathcal{I}(f) + \tilde{u}^*E_\top^*\tilde{\mathcal{I}}(f) \\ &= \hat{u}^*\mathcal{R}(f) + \tilde{u}^*(I_0(f), ..., I_N(f))^\diamond = \int_{(x_0,x_{N+1})} u^*wf\end{aligned}$$

using that $u(x) = U_j(x)u^+(x_j)$ for $x \in (x_j, x_{j+1})$. □

3. Minimal and maximal relations

Our differential equation $Ju' + qu = wf$ gives rise to the following two linear relations. $T_{\max}$ is the set of all pairs $([u], [f]) \in L^2(w) \times L^2(w)$ for which there are representatives $u \in [u]$ and $f \in [f]$ such that $Ju' + qu = wf$ (in particular, u is a balanced function of locally bounded variation). $T_{\min}$ is the set of those elements in $T_{\max}$ for which the solution $u \in [u]$ may be chosen with compact support.

Recall that the adjoint of a linear relation $S \subset L^2(w) \times L^2(w)$ is defined to be

$$S^* = \{(v, g) \in L^2(w) \times L^2(w) : \forall (u, f) \in S : \langle g, u\rangle = \langle v, f\rangle\}.$$

Our main result in this paper is the following theorem.

THEOREM 3.1. $T_{\min}^* = T_{\max}$.

Our proof requires a little preparation with which we begin. Suppose $[\xi_1, \xi_2] \subset (a, b)$ and consider the relation $\breve{T}_{\max}$ associated with J, q, and w but restricted to the interval

(ξ_1, ξ_2). Of course, solutions of $Ju' + qu = wf$ have limits at ξ_1 and ξ_2. We denote the restriction of w to (ξ_1, ξ_2) by $\breve{w}$ and set $T_0 = \{([u],[f]) \in \breve{T}_{\max} : u^+(\xi_1) = u^-(\xi_2) = 0\}$ and $K_0 = \ker \breve{T}_{\max}$.

Lemma 3.2. $\operatorname{ran} T_0 = L^2(\breve{w}) \ominus K_0$.

Proof. Let $[f] \in \operatorname{ran} T_0$ and $[r] \in K_0$. Then $Ju' + qu = \breve{w}f$ for some u which vanishes at ξ_1 and ξ_2 and r (chosen appropriately in $[r]$) satisfies $Jr' + qr = 0$. Integration by parts shows then

$$\int f^* \breve{w} r = \int u^*(Jr' + qr) = 0.$$

Hence $\operatorname{ran} T_0 \subset L^2(\breve{w}) \ominus K_0$.

Conversely, suppose $[f] \in L^2(\breve{w}) \ominus K_0$. We want to show the existence of a balanced function u of bounded variation defined on (ξ_1, ξ_2), vanishing at the endpoints, and satisfying $Ju' + qu = \breve{w}f$. Using the notation established in Section 2 and, in particular, Theorem 2.1 we have to show the existence of a solution $\tilde{u} = (\gamma_0, ..., \gamma_N)^\diamond$ of equation (2.4) satisfying $\gamma_0 = 0$ and $\gamma_N = -J^{-1} I_N(f)$ (so that $u^+(\xi_1) = u^-(\xi_2) = 0$). Thus we need to find $\tilde{u}_0 = (\gamma_1, ..., \gamma_{N-1})^\diamond$ such that $B_m \tilde{u}_0 = \mathcal{F}(f)$ where, as in Lemma 2.5,

$$\mathcal{F}(f) = \mathcal{R}(f) - \mathcal{B}^* \mathcal{U} \mathcal{J}^{-1} \mathcal{I}(f) + \mathcal{B} \mathcal{J}^{-1} \tilde{\mathcal{I}}(f).$$

This system has a solution precisely when $\mathcal{F}(f)$ is in $\operatorname{ran} B_m = (\ker B_m^*)^\perp$.

Hence suppose $\hat{r} \in \ker B_m^*$. The function r associated with $\hat{r}$ according to Theorem 2.4 is a representative of an element in K_0 so that $\int r^* \breve{w} f = 0$. But Lemma 2.5 shows that $\hat{r}^* \mathcal{F}(f) = \int r^* \breve{w} f$ guaranteeing the existence of u. □

Lemma 3.3. If $g \in \operatorname{ran} T_{\min}^*$, then the differential equation $Ju' + qu = wg$ has at least one solution on (a, b).

Proof. Let τ_n, $n \in \mathbb{Z}$, be an enumeration of points in (a, b) which include all points where the matrices $J \pm \Delta_q(x)/2$ are not invertible. The labeling is such that

$\tau_n < \tau_{n+1}$ and we may arrange things so that a and b are accumulation points, and the only ones, of the sequence τ_n.

According to Theorem 2.1 there is a balanced solution v_j of $Ju' + qu = wg$ on $(\xi_1, \xi_2) = (\tau_{-j}, \tau_j)$ (at least when $j > 1$) provided $B\tilde{v}_j = G$ where

$$G = \mathcal{R}(g) - \mathcal{B}^* \mathcal{U} \mathcal{J}^{-1} \mathcal{I}(g) = \mathcal{F}(g) - \mathcal{B} \mathcal{J}^{-1} (0, ..., 0, I_N(g))^\diamond.$$

This, in turn, happens if and only if $G \in \operatorname{ran} B = (\ker B^*)^\perp$ which we show next.

For any $\hat{r} \in \ker B^*$ Theorem 2.4 and Lemma 2.5 show the existence of a solution r of $Ju' + qu = 0$ on (ξ_1, ξ_2) such that $\tilde{r} = (r^+(x_0), ..., r^+(x_N))^\diamond$ satisfies $B\tilde{r} = 0$, $r^+(x_0) = r^+(x_N) = 0$, and $\hat{r}^* \mathcal{F}(g) = \int_{(\xi_1, \xi_2)} r^* wg$ (here $N = 2j - 1$ and $x_\ell = \tau_{-j+\ell}$). In fact, since r vanishes near x_0 and x_{N+1}, we may extend it by 0 to obtain a solution of $Ju' + qu = 0$ on all of (a, b). Thus $\langle r, g \rangle = \hat{r}^* \mathcal{F}(g)$ and it follows, as in the proof of Lemma 2.5, that $\hat{r}^* G = \langle r, g \rangle - \tilde{r}^* E_\top^* (0, ..., 0, I_N(g))^\diamond$. Since $([r], 0) \in T_{\min}$ we have $\langle r, g \rangle = \langle 0, v \rangle = 0$ and since $r^+(x_N) = 0$ we also have $\tilde{r}^* E_\top^* (0, ..., 0, I_N(g))^\diamond = 0$. Thus $G \in \operatorname{ran} B$ and this guarantees the existence of v_j.

Now define, for any $j \geq k \geq 2$, the set $A_{k,j}$ to be the collection of restrictions to (τ_{-k}, τ_k) of solutions of $Ju' + qu = wg$ on (τ_{-j}, τ_j). According to the above the $A_{k,j}$ are nonempty and nested in the sense that $A_{k,j+1} \subset A_{k,j}$. Each $A_{k,j}$ is an affine subspace of, say, the space of all functions defined on (τ_{-k}, τ_k) and their dimensions form, in j, a nonincreasing sequence of nonnegative integers which must eventually be constant (possibly zero). Hence, for a sufficiently large m, we have that $B_k = \bigcap_{j \geq k} A_{k,j} = A_{k,m}$. We now define inductively a sequence of functions v_k whose pointwise limit is a solution of $Ju' + qu = wg$ on (a, b). For v_2 we choose any element of B_2. Then suppose we had constructed a sequence $(v_2, ..., v_k)$ such that $v_j \in B_j$ and $v_{j-1} = v_j|_{(\tau_{-j+1}, \tau_{j-1})}$. Note that the elements of B_k are restrictions of elements in B_{k+1} to (τ_{-k}, τ_k) (and vice versa). Thus we may choose for v_{k+1} an element of B_{k+1} which extends v_k and this completes our definition of the sequence v_k, except that we extend each of its elements arbitrarily to (a,b). Now v, the pointwise limit of the v_k, is the desired solution of $Ju'+qu=wg$ on (a,b). □

Proof of Theorem 3.1. If $([v],[g])$ and $([u],[f])$ are in $T_{\max}$ Lagrange's identity (cf. [4]) states that

$$\langle v,f\rangle - \langle g,u\rangle = (v^*Ju)^-(b) - (v^*Ju)^+(a). \tag{3.1}$$

Therefore, if $([u],[f])\in T_{\min}$, so that u has compact support, we get $\langle v,f\rangle=\langle g,u\rangle$ which proves that $T_{\max}\subset T_{\min}^*$.

To prove $T_{\min}^*\subset T_{\max}$ assume that $([v],[g])\in T_{\min}^*$ and let v_0 be a solution of $Ju'+qu=wg$ on (a,b) as constructed by Lemma 3.3. Next we will employ Lemma 3.2. We consider an interval $[\xi_1,\xi_2]\subset(a,b)$ and define $\breve{w}$, T_0 and K_0 as we did there. Given $([u],[f])\in T_0$ extend both u and f by 0 to all of (a,b) (denoting the extensions also by u and f). We then have $Ju'+qu=wf$ so that $([u],[f])\in T_{\min}$ and $\langle f,v\rangle=\langle u,g\rangle$. To establish a relationship between v and v_0 we apply integration by parts to obtain

$$\begin{aligned}\int_{(\xi_1,\xi_2)} f^*\breve{w}v = \int_{(a,b)} u^*wg = \int_{(a,b)} u^*(Jv_0'+qv_0)\\ = \int_{(a,b)}(Ju'+qu)^*v_0 = \int_{(\xi_1,\xi_2)} f^*\breve{w}v_0.\end{aligned}$$

Thus $\int f^*\breve{w}(v-v_0)=0$ so that, by Lemma 3.2, $[v-v_0]\in K_0$ showing that $[v]$ has a representative $v=v_0+k_0$ where $Jk_0'+qk_0=0$ and hence $Jv'+qv=wg$ on (ξ_1,ξ_2). We obtain a solution on all of (a,b) in a similar way as we did in the proof of Lemma 3.3. We only have to modify the definition of the sets $A_{k,j}$ to specify that the solutions u considered are locally representatives of v, i.e, that $\int_{(\tau_{-j},\tau_j)}(u-v)^*w(u-v)=0$. □

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ON THE SPECTRAL THEORY OF SYSTEMS OF FIRST ORDER EQUATIONS WITH PERIODIC DISTRIBUTIONAL COEFFICIENTS

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ON THE SPECTRAL THEORY OF SYSTEMS OF FIRST ORDER EQUATIONS WITH PERIODIC DISTRIBUTIONAL COEFFICIENTS

Abstract. We establish a Floquet theorem for a first-order system of differential equations $u' = ru$ where r is an $n \times n$-matrix whose entries are periodic distributions of order 0. Then we investigate, when $n = 1$ and $n = 2$, the spectral theory for the equation $Ju' + qu = wf$ on $\mathbb{R}$ when J is a real, constant, invertible, skew-symmetric matrix and q and w are periodic matrices whose entries are real distributions of order 0 with q symmetric and w non-negative.

1. Introduction

Periodic structures and periodic phenomena have always played a large role in the sciences and in mathematics. In 1883 Floquet [4] gave a canonical form of the solutions of an m-th order homogeneous differential equation with periodic coefficients. Later his result turned out to be instrumental in the understanding of the associated spectral theory. Such results are now classical even in the somewhat more general case of a first order system $u' = Au$ with a periodic locally integrable matrix A. There are many excellent sources for these matters but we have benefited most from the books by Eastham [3] and Brown, Eastham, Schmidt [1]. Both have extensive lists of references to further literature on the subject.

In this paper we generalize some of these classical results by allowing the coefficients of the differential equation to be periodic distributions of order 0.[1] In Section 2 we recall the concept of periodicity for distributions and state its most important properties. This includes the relationship of distributions of order 0 with measures. In Section 3 we state and prove the generalization of Floquet's theorem. In Section 4 we recall some

[1] Recall that distributions of order 0 are distributional derivatives of functions of locally bounded variation and hence may be thought of, on compact subintervals of $\mathbb{R}$, as measures. For simplicity we might use the word measure instead of distribution of order 0 below.

basic facts of the spectral theory for the case of distributional coefficients. These are taken from Ghatasheh and Weikard [5] where one may also find additional information on the history and background of the subject. In Sections 5 and 6 we specialize to the cases of first and second order systems, respectively. Some simple examples are briefly considered in Section 7.

We end this introduction with a few words on notation. The set of complex-valued functions of locally bounded variation on $\mathbb{R}$ is represented by $(\mathbb{R})$. Any $f \in (\mathbb{R})$ has right- and left-hand limits denoted by $f^{\pm}$, respectively. We also use $f^{\#}$ for the function $(f^+ + f^-)/2$ which we call *balanced*. Corresponding to these kinds of functions we have the subspaces $^{\pm}(\mathbb{R})$ and $^{\#}(\mathbb{R})$ of $(\mathbb{R})$. Identity operators are denoted by $\mathbb{1}$ and χ_E is the characteristic function associated with the set E. A function $R \in (\mathbb{R})$ generates a complex measure dR at least on compact subsets of $\mathbb{R}$. The corresponding total variation measure is denoted by $|dR|$. In particular, for Lebesgue measure we use the symbol dx, regarding the symbol x as the identity function. We will often write $\int f$ in place of $\int f dx$, i.e., , when an integral does not explicitly specify a measure it may be taken for granted that integration is with respect to Lebesgue measure. Similarly, if a range for the integration is not specified, integration is over the whole real axis.

2. Basic properties of periodic distributions

A distribution r is a linear functional on the set of test functions, i.e., , the set of compactly supported infinitely often differentiable functions from $\mathbb{R}$ to $\mathbb{C}$, satisfying the following property: for every compact subset K of $\mathbb{R}$ there are numbers $C \geq 0$ and $k \in \mathbb{N}_0$ such that

$$|r(\phi)| \leq C \sum_{j=0}^{k} \|\phi^{(j)}\|_\infty \tag{2.1}$$

whenever ϕ is a test function whose support is contained in K.

For example, if $x_0 \in \mathbb{R}$ is fixed, we have the Dirac distribution defined by $\delta_{x_0} : \phi \mapsto \phi(x_0)$. Also, for any $f \in L^1_{\text{loc}}(dx)$ a distribution $\mathfrak{f}$ is defined by $\phi \mapsto \int f(x)\phi(x)dx$.

Below we will often follow the ubiquitous convention to use the same symbol for $\mathfrak{f}$ and f. Context will serve to distinguish the two meanings.

If we define $r'(\phi) = -r(\phi')$ it follows that r' is again a distribution, called the derivative of r. Distributions also have antiderivatives and we have the following important lemma.

Lemma 2.1 (Du Bois-Reymond). Suppose the derivative of the distribution r is zero. Then r is the constant distribution, i.e., , there is a complex number C such that $r(\phi) = C \int \phi$ for every test function ϕ.

The number $\omega \in \mathbb{R}$ is called a *period* of the distribution r, if $r(\phi) = r(\phi(\cdot + \omega))$ for every test function ϕ. The distribution r is called *periodic* (or, more specifically, ω-periodic) if it has a non-zero period ω. Of course, 0 is a period of any distribution.

Theorem 2.2. A periodic distribution r has the following properties:

(1) If ω_1 and ω_2 are periods of r, then so are $\omega_1 \pm \omega_2$.

(2) If ω is a period of r, then any integer multiple of ω is also a period.

(3) If r is constant, then all real numbers are periods of r.

(4) If the set of periods of r has a finite limit point, then r is constant.

(5) The infimum ω_0 of the set of positive periods of r is itself a period. If $\omega_0 > 0$, every period of r is an integer multiple of ω_0. The period ω_0 is then called the fundamental period of r.

(6) If R is an antiderivative of r, then there is a complex number α and a periodic distribution P with the same periods as r, such that $R(\phi) = \alpha \int x\phi(x)\, dx + P(\phi)$.

Proof. Properties (1) – (3) are trivial.

To prove (4) notice first that 0 must also be a limit point of the periods of r. We show that $r' = 0$, since our claim follows then from du Bois-Reymond's lemma. Suppose ω is a period of r with $0 < \omega < 1$ and let ϕ be a test function. Pick a compact interval K such that the supports of both ϕ and $\phi(\cdot+\omega)$ are in K. Then

$$0 = \frac{r(\phi(\cdot+\omega) - r(\phi))}{\omega} = r\left(\frac{\phi(\cdot+\omega)-\phi}{\omega}\right)$$

and hence, for appropriate numbers C and k,

$$|r'(\phi)| = |r(\phi')| = |r(\psi)| \leq C \sum_{j=0}^{k} \|\psi^{(j)}\|_\infty$$

where

$$\psi = \frac{\phi(\cdot+\omega)-\phi}{\omega} - \phi'.$$

Using the mean value theorem twice, we obtain

$$\psi^{(j)}(x) = \phi^{(j+1)}(c) - \phi^{(j+1)}(x) = (c-x)\phi^{(j+2)}(\tilde{c})$$

for some $c \in (x, x+\omega)$ and some $\tilde{c} \in (x, c)$. Hence $|r'(\phi)| \leq C(k+1)M\omega$ where $M = \max\{\|\phi^{(j+2)}\|_\infty : 0 \leq j \leq k\}$. Since ω may be arbitrarily small, we find $r'(\phi) = 0$ and since ϕ was arbitrary, we get $r' = 0$ as promised.

Property (5) is clear when the infimum ω_0 is 0, so assume it is not. Then r is not constant and the previous result shows that ω_0 is not a limit point of the set of positive periods. Instead it must be a period itself. Now suppose ω is any other period of r. Then $\omega = n\omega_0 + b$ for some $n \in \mathbb{Z}$ and $b \in [0, \omega_0)$. Since b is also a period it must be equal to 0.

Finally, for property (6) we first obtain from du Bois-Reymond's lemma and the periodicity of r that $R(\phi(\cdot+\omega)) - R(\phi) = C\int\phi$ for some number C. Now define the distribution P by setting

$$P(\phi) = R(\phi) + \frac{C}{\omega}\int x\phi(x)dx.$$

Since $\int x\phi(x+\omega)dx = \int(x-\omega)\phi(x)dx$ we find that P is periodic. Hence the claim follows if we set $\alpha = -C/\omega$. □

REMARK 2.1. *A function* $f \in L^1_{\text{loc}}(dx)$ *is called periodic with period* ω*, if* $f(x+\omega) = f(x)$ *for almost all (with respect to Lebesgue measure)* $x \in \mathbb{R}$*. As mentioned earlier, such a function gives rise to a distribution* $\mathfrak{f}$ *by setting* $\mathfrak{f}(\phi) = \int f\phi$ *for any test function* ϕ*. Since*

$$\mathfrak{f}(\phi(\cdot+\omega)) = \int f\phi(\cdot+\omega) = \int f(\cdot-\omega)\phi = \int f\phi = \mathfrak{f}(\phi)$$

we see that $\mathfrak{f}$ *is a periodic distribution with the same periods as* f*.*

In the following we will be concerned only with distributions of order 0, i.e., , those for which one may choose $k = 0$ in inequality (2.1) regardless of K. They are in close correspondence with functions of locally bounded variation. Specifically, if $R \in (\mathbb{R})$, then it generates a (Borel) measure dR on compact subsets of $\mathbb{R}$. It follows that $\phi \mapsto \int \phi dR$ is a distribution of order 0, in fact, it is the derivative of the distribution $\phi \mapsto \int R\phi dx$. Conversely, if r is a distribution of order 0, then Riesz's representation theorem shows that there is a function $R \in (\mathbb{R})$ yielding $r(\phi) = \int \phi dR$. For brevity we will frequently identify the distribution r and the local measure dR. We will also identify the antiderivative of r with the corresponding function R in $(\mathbb{R})$. In particular, we use the designations $r(\phi)$, $\int r\phi$, and $\int \phi dR$ interchangeably.

If $f \in L^1_{\text{loc}}(|dR|)$ and r is a distribution of order 0 we may define the product of r and f (or f and r) by setting

$$\phi \mapsto (rf)(\phi) = (fr)(\phi) = \int f\phi dR = \int rf\phi.$$

rf is again a distribution of order 0.

We need the following substitution rule when dealing with integrals.

LEMMA 2.3. Suppose (a, b) and (α, β) are real intervals and $R : (\alpha, \beta) \to \mathbb{C}$ is left-continuous and of bounded variation. If $T : (a, b) \to (\alpha, \beta)$ is continuous and bijective (and hence strictly monotone), then $R \circ T : (a, b) \to \mathbb{C}$ is also left-continuous and of bounded variation. Moreover, if $g \in L^1(|dR|)$, then

$$\int g dR = \pm \int g \circ T \, d(R \circ T)$$

where one has to choose the positive sign if T is strictly increasing and the negative sign if it is strictly decreasing.

This lemma has the following consequence in the context of periodic distributions.

THEOREM 2.4. Suppose w is a periodic distribution of order 0 with period ω and $f \in L^1(|w|)$. Then $\int wf = \int wf(\cdot + \omega)$.

PROOF. Let $T : \mathbb{R} \to \mathbb{R} : x \mapsto x + \omega$ and let W be an anti-derivative of w. Note that by property (6) of Theorem 2.2 we have $W(x) = \alpha x + P(x)$ for some periodic function (of locally bounded variation) P. Then $W(T(x)) = W(x) + \alpha\omega$ and hence $dW = \alpha + dP = d(W \circ T)$. $\square$

3. Floquet theory

In this section we shall develop a Floquet theory for the differential equation

$$u' = ru$$

where r is an $n \times n$-matrix whose entries are periodic distributions of order 0 all of which have a common period ω (in this case we call r periodic with period ω). We seek solutions among *balanced* $\mathbb{C}^n$-valued functions of locally bounded variation.

THEOREM 3.1. Suppose u is a balanced function of locally bounded variation such that $u' = ru$. If v is defined by $v(x) = u(x + \omega)$, then we also have $v' = rv$.

PROOF. Let $T(x) = x - \omega$, let ϕ be a test function, and set $\psi = \phi \circ T = \phi(\cdot - \omega)$. This and Theorem 2.4 give

$$(rv)(\phi) = \int rv\phi = \int r(v\phi) \circ T = \int ru\psi = (ru)(\psi).$$

We also have, by the translation invariance of Lebesgue measure,

$$v'(\phi) = -\int v(x)\phi'(x)dx = -\int u(x)\psi'(x)dx = u'(\psi).$$

Since the rightmost expressions are the same so are the leftmost. □

Thus the operator which assigns $u(\cdot + \omega)$ to u is a map from the space of solutions of $u' = ru$ to itself. It is called the *monodromy operator*.

The examples $r = \left(\begin{smallmatrix} 2 & 0 \\ 0 & -2 \end{smallmatrix}\right) \sum_{k\in\mathbb{Z}} \delta_k$ and $r = \left(\begin{smallmatrix} 2 & 0 \\ 0 & -2 \end{smallmatrix}\right) \sum_{k\in\mathbb{Z}} (\delta_{2k+1} - \delta_{2k})$, which are periodic distribution with period 1 and 2, respectively, show that the solution space of $u' = ru$ may not be n-dimensional. Indeed, in the former case the solution space is trivial while, in the latter case it is infinite-dimensional. This is due to the fact, that the existence and uniqueness theorem for initial value problems fails for these equations.

To proceed we define the matrix $\Delta_r(x) = R^+(x) - R^-(x)$ and add the following hypothesis.

Hypothesis 3.2. Let $\omega > 0$. Assume r is an $n \times n$-matrix of ω-periodic distributions of order 0 such that the matrices $\mathbb{1} \pm \frac{1}{2}\Delta_r(x)$ are invertible for every $x \in \mathbb{R}$.

It was shown in [5] that, under this hypothesis, existence and uniqueness of balanced solutions of initial value problems holds. It follows immediately that the solution space of $u' = ru$ is n-dimensional and hence that we have a fundamental matrix U of solutions. The determinant of $U(x)$ is different from 0 for any $x \in \mathbb{R}$. Theorem 3.1 shows that $U(\cdot + \omega)$ is also a fundamental matrix of solutions. Hence there is a constant matrix M such that

$$U(x + \omega) = U(x)M.$$

The matrix M, called a *monodromy matrix*, depends on the choice of U. If V is another fundamental matrix of solutions so that $V = US$ for a constant invertible matrix S and $\tilde{M}$ is the associated monodromy matrix, then $M = S\tilde{M}S^{-1}$, i.e., , M and $\tilde{M}$ are similar matrices. In particular, they have the same eigenvalues.

It is known from Linear Algebra that we may choose S so that $\tilde{M}$ is a matrix in Jordan normal form, i.e., , $\tilde{M}$ is a block diagonal matrix where the diagonal blocks, called Jordan blocks, are square matrices of the form

$$\tilde{M}_k = \begin{pmatrix} \rho_k & 1 & 0 & \cdots & 0 \\ 0 & \rho_k & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \rho_k & 1 \\ 0 & 0 & \cdots & 0 & \rho_k \end{pmatrix}.$$

Here ρ_k is an eigenvalue of $\tilde{M}$ (and of M) and hence non-zero. If the number of Jordan blocks is s and if their size is μ_k, $k = 1, ..., s$, we have, of course $\sum_{k=1}^{s} \mu_k = n$.

THEOREM 3.3. Suppose r satisfies Hypothesis 3.2. The differential equation $u' = ru$ has a fundamental system of solutions of the form

$$\mathrm{e}^{\alpha_k x} \sum_{j=0}^{\ell} q_{k,j}(x) p_{k,\ell-j}(x) \text{ for } \ell = 0, ..., \mu_k - 1 \text{ and } k = 1, ..., s$$

where $\mathrm{e}^{\alpha_k \omega} = \rho_k$, $q_{k,0}(x) = 1$, $q_{k,j+1}(x) = q_{k,j}(x) \frac{x - j\omega}{(j+1)\rho_k \omega}$, and the $p_{k,j}$ are balanced, periodic $\mathbb{C}^n$-valued function with period ω.

PROOF. We are adapting Hochstadt's proof in [6] which avoids introducing the logarithm of M.[2] Let v_m be the m-th column of V, the fundamental matrix whose monodromy matrix is in Jordan normal form. There are unique numbers $k \in \{1, ..., s\}$ and $\ell \in \{0, ..., \mu_k - 1\}$ such that $m = 1 + \ell + \sum_{h=1}^{k-1} \mu_h$. Let $m_0 = 1 + \sum_{h=1}^{k-1} \mu_h$. Then we will prove, by induction over ℓ, that there are ω-periodic functions $p_{k,j}$ such that

$$v_{m_0+\ell}(x) = \mathrm{e}^{\alpha_k x} \sum_{j=0}^{\ell} q_{k,j}(x) p_{k,\ell-j}(x) \tag{3.1}$$

[2]There appears to be a flaw in Hochstadt's reasoning which we tried to circumvent.

for $\ell = 0, ..., \mu_k - 1$. If $\ell = 0$ define $p_{k,0}$ by $p_{k,0}(x) = v_{m_0}(x)\,\mathrm{e}^{-\alpha_k x}$. Then the identity $v_{m_0}(x+\omega) = \rho_k v_{m_0}(x)$ shows that $p_{k,0}$ is ω-periodic, i.e., , v_{m_0} has the required form. Now assume that (3.1) has been established for $\ell = 0, ..., r-1$ including the periodicity of $p_{k,0}$, ..., $p_{k,r-1}$ for some $0 < r < \mu_k$. Let $\ell = r$ and note that $v_{m_0+\ell}$ defines the yet undetermined function $p_{k,\ell}$. It is only left to prove that $p_{k,\ell}$ is ω-periodic. Since $v_{m_0+\ell}(x+\omega) - \rho_k v_{m_0+\ell}(x) - v_{m_0+\ell-1}(x) = 0$ we get, using the periodicity of $p_{k,0}$, ..., $p_{k,\ell-1}$,

$$p_{k,\ell}(x+\omega) - p_{k,\ell}(x) = \sum_{j=1}^{\ell}\left(\frac{1}{\rho}q_{k,j-1}(x) + q_{k,j}(x) - q_{k,j}(x+\omega)\right)p_{k,\ell-j}(x).$$

For our choice of the polynomials $q_{k,j}$ induction shows that each term on the right-hand side is 0 proving the periodicity of $p_{k,\ell}$. □

The eigenvalues ρ_k of a monodromy matrix are called *Floquet multipliers* while the numbers α_k are called *Floquet exponents*. The associated (generalized) eigenfunctions are called (generalized) *Floquet solutions* of $u' = ru$.

4. Spectral theory

The main goal of this paper is to investigate the spectral theory associated with a periodic first order 2×2-system of differential equations. To this end we will recall some basic definitions and results from [5] and [2].

If w is a non-negative $n \times n$-matrix whose entries are distributions of order 0, $\operatorname{tr} w$ represents a positive scalar measure. By $\mathcal{L}^2(w)$ we denote the collection of $\mathbb{C}^n$-valued functions f whose components are measurable with respect to $\operatorname{tr} w$ and which satisfy $\|f\|^2 = \int f^* w f < \infty$. Then $L^2(w)$ designates the corresponding Hilbert space, i.e., , the quotient of $\mathcal{L}^2(w)$ by the kernel of $\|\cdot\|$. The inner product of $L^2(w)$ is, of course, given by $\langle f, g\rangle = \int f^* w g$. Now consider the differential equation

$$Ju' + qu = wf \tag{4.1}$$

where J is a constant, invertible and skew-hermitian $n \times n$-matrix and q is a hermitian $n \times n$-matrix whose entries are distributions of order 0. Define the linear relations

$$\mathcal{T}_{\max} = \{(u, f) \in \mathcal{L}^2(w) \times \mathcal{L}^2(w) : u \in^{\#} (\mathbb{R})^n, Ju' + qu = wf\}$$

and

$$\mathcal{T}_{\min} = \{(u, f) \in \mathcal{T}_{\max} : \operatorname{supp} u \text{ is compact in } \mathbb{R}\}.$$

Then, in the Hilbert space setting, we represent our differential equation by the relations

$$T_{\max} = \{([u], [f]) \in L^2(w) \times L^2(w) : (u, f) \in \mathcal{T}_{\max}\}$$

and

$$T_{\min} = \{([u], [f]) \in L^2(w) \times L^2(w) : (u, f) \in \mathcal{T}_{\min}\}.$$

The cornerstone of spectral theory is the result that $T_{\min}^* = T_{\max}$, i.e., , that $T_{\min}$ is a symmetric relation (see [2]). As a consequence we have that

$$T_{\max} = \overline{T_{\min}} \oplus D_{\mathrm{i}} \oplus D_{-\mathrm{i}}$$

where D_λ is defined as $\{(u, \lambda u) \in T_{\max}\}$. These are called *deficiency spaces* if $\lambda \notin \mathbb{R}$. The numbers $n_\pm = \dim D_{\pm \mathrm{i}}$ are called *deficiency indices*. It is important to recall that $\dim D(\lambda)$ is independent of λ as long as λ varies in either the upper or the lower half of the complex plane. The deficiency indices are finite (see [7]) and if they are identical then there exist self-adjoint restrictions of $T_{\max}$ (possibly $T_{\max}$ itself).

Even in the case of constant coefficients two complications arise. Firstly, the space $\mathcal{L}_0 = \{u : Ju' + qu = 0 \text{ and } wu = 0\}$ may be non-trivial (note here that $wu = 0$ if and only if $\|u\| = 0$). If this happens the problem is called non-definite. The other issue is that $T_{\max}$ may indeed not be a linear operator as our introduction of linear relations already insinuates.

If T is a self-adjoint restriction of $T_{\max}$ one defines the resolvent set of T by

$$\varrho(T) = \{\lambda \in \mathbb{C} : \ker(T - \lambda) = \{0\}, \operatorname{ran}(T - \lambda) = L^2(w)\}$$

and the spectrum of T by $\sigma(T) = \varrho(T)^{\mathrm{c}}$, the complement of $\varrho(T)$. For $\lambda \in \varrho(T)$ the linear relation $(T - \lambda)^{-1}$ is, in fact, a linear operator from $L^2(w)$ to the domain of T.

Define the space $\mathcal{H}_\infty = \{f \in L^2(w) : (0, f) \in T\}$ and $\mathcal{H}_0$ to be its orthogonal complement. Then the domain of T is a dense subset of $\mathcal{H}_0$ and if $(u, f) \in T$, then, of course, $f = f_0 + f_\infty$ with $f_0 \in \mathcal{H}_0$ and $f_\infty \in \mathcal{H}_\infty$. But since $(0, f_\infty)$ is in T so is (u, f_0). Since f_0 is uniquely determined by u we have that $T_0 = T \cap (\mathcal{H}_0 \times \mathcal{H}_0)$ is a densely defined self-adjoint linear operator, called the operator part of T. In particular, $\mathcal{H}_0 = \{0\}$ if and only if the spectrum of T is empty.

5. The case $n = 1$

HYPOTHESIS 5.1. Throughout this section we assume that ω is a positive real number and J a non-zero purely imaginary number. Moreover, q is a real distribution of order 0 while w is a non-negative but non-zero distribution of order 0. Both q and w are periodic with period ω.

When λ and x are in $\mathbb{R}$ the imaginary part of the number

$$B_\pm(x, \lambda) = J \pm \frac{1}{2}(\Delta_q(x) - \lambda \Delta_w(x))$$

is equal to that of J and hence non-zero. Hypothesis 3.2 is therefore satisfied for $n = 1$.

Let $U(\cdot, \lambda)$ be the unique solution of the homogeneous equation $Ju' + qu = \lambda wu$ satisfying the initial condition $U^+(0, \lambda) = 1$. Then $\rho(\lambda) = U^+(\omega, \lambda)$ is the Floquet multiplier. Thus $U(x + n\omega, \lambda) = \rho(\lambda)^n U(x, \lambda)$ whenever $n \in \mathbb{Z}$. Recall from Lemma 3.2 in [5] that

$$U^+(x, \overline{\lambda})^* J U^+(x, \lambda) = U^-(x, \overline{\lambda})^* J U^-(x, \lambda) = J.$$

Hence $U^+(x, \overline{\lambda})^* U^+(x, \lambda) = 1$ and, in particular, $\overline{\rho(\overline{\lambda})} \rho(\lambda) = 1$.

THEOREM 5.2. Assume that Hypothesis 5.1 holds. Then the linear relation $T_{\max}$ is self-adjoint. Its spectrum is purely continuous and fills the entire real line.

PROOF. Fix λ in $\mathbb{C}$ and note that $\rho(\lambda) \neq 0$. Using Theorem 2.4 we obtain, for any $n \in \mathbb{Z}$,

$$\int_{[n\omega,(n+1)\omega)} |U(\cdot,\lambda)|^2 w = |\rho(\lambda)|^{-2n} \int_{[0,\omega)} |U(\cdot,\lambda)|^2 w.$$

Hence

$$\|U(\cdot,\lambda)\|^2 = \sum_{n\in\mathbb{Z}} |\rho(\lambda)|^{-2n} \int_{[0,\omega)} |U(\cdot,\lambda)|^2 w$$

which is finite only when $w = 0$, a case we have excluded for being trivial. Hence no λ can be an eigenvalue of $T_{\max}$, i.e., , the deficiency spaces are trivial and $T_{\max}$ is self-adjoint.

Now assume that λ is real. Since $\overline{\rho(\overline{\lambda})}\rho(\lambda) = 1$ this shows that $|\rho(\lambda)| = 1$. It follows that λ is an element of the so called stability set S, the set of those λ for which $U(\cdot,\lambda)$ is bounded. We will prove that $S \subset \sigma$, the spectrum of $T_{\max}$. Then

$$\mathbb{R} \subset S \subset \sigma \subset \mathbb{R}$$

which entails that $S = \sigma = \mathbb{R}$.

Thus assume now that $\lambda \in S$ and, by way of contradiction, that it is also in the resolvent set of $T_{\max}$. If we can construct a sequence $n \mapsto (\phi_n, \lambda\phi_n + f_n) \in T_{\max}$ such that $\|\phi_n\| = 1$ and $\lim_{n\to\infty} \|f_n\| = 0$, then $\phi_n = R_\lambda f_n$. Since R_λ, the resolvent operator for $T_{\max}$ at λ, is bounded[3], we get $1 = \|\phi_n\| \leq \|R_\lambda\|\|f_n\| \to 0$, a contradiction.

Let us construct the required sequence in $T_{\max}$. Since λ is fixed we will, in the course of this proof, simply write U in place of $U(\cdot,\lambda)$. Let W be the left-continuous anti-derivative of w which vanishes at 0. For $n \in \mathbb{Z}$ let $I_n = [n\omega,(n+1)\omega)$ and for $n \in \mathbb{N}$ define

$$h_n = (W + (n+1)W(\omega))\chi_{I_{-n-1}} + W(\omega)\chi_{[-n\omega,n\omega)} - (W - (n+1)W(\omega))\chi_{I_n}$$

[3]Even in the case of a relation the resolvent is necessarily an operator.

and $\phi_n = a_n(h_nU)^\#$ where the numbers a_n are chosen so that $\|\phi_n\| = 1$. Note that the functions h_n are left-continuous, i.e, $h_n = h_n^-$.

Recall the product rule for functions of locally bounded variation, i.e., , $(fg)' = f^+g' + f'g^- = f^-g' + f'g^+$. Hence $\phi_n' = a_nh_n'U^+ + a_nh_nU'$. Since $(h_nU)^\# = h_nU + \frac{1}{2}(h_n^+ - h_n^-)U^+$ we get

$$J\phi_n' + (q-\lambda w)\phi_n$$
$$= a_nh_n(JU' + (q-\lambda w)U) + a_n\big(Jh_n' + \frac{1}{2}(q-\lambda w)(h_n^+ - h_n^-)\big)U^+.$$

The first term on the right is 0 while the second is a measure supported on the closure of $I_{-n-1} \cup I_n$. On those intervals we have $h_n' = \pm w$ and $h_n^+ - h_n^- = \pm\Delta_w$ where the upper sign is to be used for the interval on the left while the lower sign is to be used for the interval on the right. Now observe that the discrete measures $q\Delta_w$ and $\Delta_q w$ are identical so that we get

$$J\phi_n' + (q-\lambda w)\phi_n = wa_n(J + \frac{1}{2}(\Delta_q - \lambda\Delta_w))U^+(\chi_{I_{-n-1}} - \chi_{I_n}) = wf_n$$

thereby defining f_n. It follows that $(\phi_n, \lambda\phi_n + f_n)$ is an element of $T_{\min} \subset T_{\max}$.

Next we show that the norming constants a_n tend to 0. Indeed, using again Theorem 2.4 and the fact that $|\rho(\lambda)| = 1$,

$$\|\phi_n\|^2 \geq |a_n|^2 W(\omega)^2 \int_{[-n\omega,n\omega)} |U|^2 w = 2n|a_n|^2 C$$

where $C = W(\omega)^2 \int_{[0,\omega)} |U|^2 w$ does not depend on n. Hence, $|a_n| \leq 1/\sqrt{2nC}$. It is now also clear that $\|f_n\|$ tends to 0 so that our proof is finished. $\square$

6. The case $n = 2$, real coefficients

The following assumptions are in force throughout this section.

HYPOTHESIS 6.1. The 2×2-matrix J is real, skew-hermitian, and invertible. The entries of the 2×2-matrices q and w are real distributions of order 0 with q hermitian (symmetric) and w non-negative. Both q and w are ω-periodic where ω is a positive real number. Moreover, the matrices

$$B_{\pm}(x, \lambda) = J \pm \frac{1}{2}(\Delta_q(x) - \lambda \Delta_w(x))$$

are invertible for all $\lambda, x \in \mathbb{R}$.

We observe that for all $x \in \mathbb{R}$ and all $\lambda \in \mathbb{C}$ we have $\det B_+(x, \lambda) = \det B_-(x, \lambda)$. Let Λ be the set of all those $\lambda \in \mathbb{C}$ such that, for some $x \in \mathbb{R}$, $\det B_{\pm}(x, \lambda) = 0$. Our assumptions guarantee that Λ does not intersect $\mathbb{R}$ and that there are only finitely many elements of Λ in any disk of finite radius. Hence Λ is a discrete set. It is also symmetric with respect to the real axis. See [7] for more details.

Furthermore, our condition on J implies that $J = r\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ for some number $r \in \mathbb{R} \setminus \{0\}$. The motivation to restrict our attention to the real case only, is that $v = \overline{u}$ satisfies $Jv' + (q - \overline{\lambda} w)v = 0$, if u satisfies $Ju' + (q - \lambda w)u = 0$. In particular, if $U(\cdot, \lambda)$, for $\lambda \notin \Lambda$, is a fundamental matrix for $Ju' + (q - \lambda w)u = 0$, then $\overline{U(\cdot, \lambda)}$ is a fundamental matrix for $Ju' + (q - \overline{\lambda} w)u = 0$. It follows that, for $\lambda \in \mathbb{R}$, we may choose the fundamental matrix to be real-valued. Moreover, $r \det U^{\pm}(\cdot, \lambda) = -U_1^{\pm}(\cdot, \overline{\lambda})^* J U_2^{\pm}(\cdot, \lambda)$ when U_1 and U_2 denote the first and second column of U, respectively. Lemma 3.2 in [5] implies therefore that $\det U^+(\cdot, \lambda) = \det U^-(\cdot, \lambda)$ is constant. Thus any monodromy matrix $M(\lambda)$ has determinant 1 and this, in turn implies that the Floquet multipliers are reciprocals of each other. Therefore the Floquet multipliers are uniquely determined (up to transposition) by their sum, the *Floquet discriminant*

$$D(\lambda) = \operatorname{tr} M(\lambda).$$

Note that $\operatorname{tr} M(\lambda)$, just like $\det M(\lambda)$, is invariant under similarity transforms, i.e., , it is independent of the specific fundamental matrix used to find $M(\lambda)$. Since we may choose $U(\cdot, \lambda)$ real for $\lambda \in \mathbb{R}$, it follows that $D(\lambda)$ is then also real. Moreover, by Theorem 2.7 of [5] the entries of $U(x,\cdot)$ are analytic in $\mathbb{C}\setminus\Lambda$ for any $x\in\mathbb{R}$. Consequently, the Floquet discriminant D is analytic, too.

Our first major goal is to show the absence of eigenvalues of $T_{\max}$. First we show that no point in Λ can be an eigenvalue.

THEOREM 6.2. If $\lambda\in\Lambda$, then any solution of $Ju'+qu=\lambda wu$ is identically equal to 0.

PROOF. Since $\lambda\in\Lambda$ there is a point x_0 for which $B_\pm(x_0,\lambda)$ are not invertible. Of course $x_0+\omega$ is then also such a point. First we show that $\operatorname{ran} B_+(x_0,\lambda)$ and $\operatorname{ran} B_-(x_0,\lambda)$ intersect only trivially. Let us abbreviate $B_\pm(x_0,\lambda)$ simply by $B_\pm$. Neither B_+ nor B_- is 0. Hence there are two vectors v_+ and v_- spanning, respectively, their kernels. Moreover, $0\neq 2Jv_+=B_-v_+$ and $0\neq 2Jv_-=B_+v_-$. Assuming now, by way of contradiction and without loss of generality that $B_-v_+=B_+v_-$ we get $Jv_+=Jv_-$ which contradicts the fact that J is injective and shows that $\operatorname{ran} B_+\cap\operatorname{ran} B_-=\{0\}$.

If u solves the equation $Ju'+qu=\lambda wu$ we must have $B_+(x_0,\lambda)u^+(x_0)=B_-(x_0,\lambda)u^-(x_0)$ and hence, by the above, that $B_\pm(x_0,\lambda)u^\pm(x_0)=0$. By the same argument we get $B_\pm(x_0+\omega,\lambda)u^\pm(x_0+\omega)=0$. It follows now that $v=(u\chi_{(x_0,x_0+\omega)})^\#$ is also a solution of $Ju'+qu=\lambda wu$, in fact a solution of finite norm. If the norm were positive we would have a complex eigenvalue of $T_{\min}$ which is impossible. Therefore $wv=0$ so that v also solves $Ju'+qu=\lambda wu$ for all $\lambda\in\mathbb{C}$ including $\lambda=0$. However, for $\lambda=0$ solutions of initial value problems are unique. Hence $\mathcal{L}_0$ is trivial and u is identically equal to 0. □

LEMMA 6.3. Fix $\lambda\in\mathbb{C}\setminus\Lambda$. For a solution u of $Ju'+qu=\lambda wu$ and $n\in\mathbb{Z}$ define

$$I_n(u)=\int_{[n\omega,(n+1)\omega)} u^*wu.$$

If $I_n(u) = 0$ for two consecutive non-zero integers, then $\|u\| = 0$. Otherwise u has infinite norm. If, in the former case, u is not a Floquet solution and not equal to 0, then all Floquet solutions of $Ju' + qu = \lambda wu$ also have norm 0.

Proof. Suppose first that we have two linearly independent Floquet solutions ψ_1 and ψ_2, the former with multiplier ρ and the latter with multiplier $1/\rho$. Define $N_j^2 = \int_{[0,\omega)} \psi_j^* w \psi_j$ with $N_j \geq 0$ and $A = \int_{[0,\omega)} \psi_1^* w \psi_2$. Note that $|A| \leq N_1 N_2$. If $u = \alpha\psi_1 + \beta\psi_2$ we obtain

$$I_n(u) = |\alpha|^2 |\rho|^{2n} N_1^2 + 2\operatorname{Re}(\overline{\alpha}\beta(\overline{\rho}/\rho)^n A) + |\beta|^2 |\rho|^{-2n} N_2^2 \\ \geq (|\alpha| N_1 |\rho|^n - |\beta| N_2 |\rho|^{-n})^2.$$

If $u = 0$ there is nothing to prove and thus we may assume that α and β are not both 0. If one of α and β is 0, then u is a Floquet solution and either all of the $I_n(u)$ are 0 or all of them are positive. Correspondingly, $\|u\| = 0$ or $\|u\| = \infty$. If neither α nor β is 0 let us assume that $I_n(u) = I_{n+1}(u) = 0$ for some $n \in \mathbb{Z}$. This implies that either $N_1 = N_2 = 0$ or else $|\rho| = 1$. In the former case all solutions of $Ju' + qu = \lambda wu$ have norm 0. In the latter case we have

$$I_n(u) = 2N^2(1 + \operatorname{Re}(z\rho^{-2n}))$$

where we put $N = |\alpha| N_1 = |\beta| N_2 > 0$ and $z = \overline{\alpha}\beta A/N^2$. Since $|z\rho^{-2n}| \leq 1$ and $I_n(u) = I_{n+1}(u) = 0$ we get $z\rho^{-2n} = z\rho^{-2n-2} = -1$ which implies that $\rho^2 = 1$. But this means that ψ_1 and ψ_2 have the same Floquet multiplier. Hence u itself is a Floquet solution and must have norm 0.

We now consider the case when there is only one linearly independent Floquet solution. Then $\rho = \pm 1$ and we treat the case when $\rho = 1$; the other one is similar. The general solution of $Ju' + qu = \lambda wu$, according to Theorem 3.3, is

$$u(x) = \frac{\alpha x}{\omega}\psi(x) + \alpha p(x) + \beta\psi(x),$$

where ψ is a Floquet solution and p is some ω-periodic function. We confine ourselves to the case $\alpha = 1$ and define $\varphi = p + \beta\psi$, which is ω-periodic. For $n \geq 1$ we find $I_n(u) \geq (F_n - G)^2$ where $F_n^2 = \int_{[0,\omega)} \frac{(x+n\omega)^2}{\omega^2}\psi^* w\psi$ and $G^2 = \int_{[0,\omega)} \varphi^* w\varphi$. Now $I_n(u) = I_{n+1}(u) = 0$ implies that $F_n = G = F_{n+1}$ and hence $0 = \int_{[0,\omega)}(2n + 1 + 2x/\omega)\psi^* w\psi$. From this we conclude $\psi^* w\psi = 0$ so that $\|\psi\| = 0$, $F_n = G = 0$ and $\|u\| = 0$.

Now the only thing left to prove is that a solution u which does not have norm 0 has infinite norm. This is already known for Floquet solutions so we assume that u is not a Floquet solution. For u to have finite norm it is necessary that $I_n(u)$ tends to 0 as n tends to ∞ or $-\infty$. In the presence of two independent Floquet solutions it is thus necessary that $|\alpha|N_1 = |\beta|N_2$ and that $|z| = |\overline{\alpha}\beta A|/N^2 = 1$. In this case we have $I_n(u) = 2N^2(1 + \cos(t_0 + 2nt))$ where we set $z = \mathrm{e}^{it_0}$ and $\rho = \mathrm{e}^{-it}$ for some $t_0, t \in \mathbb{R}$. However, the sequence $n \mapsto I_n(u)$ converges to 0 only if it is identically equal to 0. If only one linearly independent Floquet solution exists and u is not a Floquet solution $I_n(u)$ cannot converge to 0 unless $F_n = G = 0$. □

An immediate consequence of the previous lemma is the next theorem.

Theorem 6.4. The linear relation $T_{\max}$ has no eigenvalues. In particular, it is self-adjoint.

Lemma 6.5. Fix $\lambda \in \mathbb{C} \setminus \Lambda$. Let B be the set of the vectors $\int U(\cdot, \overline{\lambda})^* wf$ when f varies among the functions in $L^2(w)$ with support in $[0, 2\omega)$. If $\mathcal{L}_0$ is trivial, then $B = \mathbb{C}^2$.

Proof. Assume, by way of contradiction, that B is less than two-dimensional. Since $B = \{0\}$ would imply $w = 0$ it follows that, in fact, B is one-dimensional. Denote the subspaces of B obtained by restricting f to those functions which are supported only on $I_1 = [0, \omega)$ or $I_2 = [\omega, 2\omega)$ by B_1 and B_2, respectively. These are also one-dimensional and hence $B_1 = B_2 = B$. Choosing $0 \neq \alpha \in B^\perp$ and $f \in L^2(w)$

with support in I_1 we get, using Theorem 2.4 and the periodicity of w,

$$0 = \alpha^* \int_{I_2} U(\cdot, \overline{\lambda})^* w f(\cdot - \omega) = \alpha^* \int_{I_1} U(\cdot + \omega, \overline{\lambda})^* w f$$
$$= \alpha^* M(\overline{\lambda})^* \int_{I_1} U(\cdot, \overline{\lambda})^* w f.$$

This shows $M(\overline{\lambda})\alpha$ is perpendicular to B_1 and hence a multiple of α. Therefore $u = U(\cdot, \overline{\lambda})\alpha$ is a Floquet solution of $Ju' + qu = \overline{\lambda} w u$ with norm 0, i.e., , $u \in \mathcal{L}_0$ which contradicts our hypothesis. □

If our problem is non-definite $T_{\max}$ is a particularly simple relation as our next theorem shows.

THEOREM 6.6. For the relation $T_{\max}$ to be equal to $\{0\} \times L^2(w)$ it is necessary and sufficient that $\mathcal{L}_0 \neq \{0\}$. In this situation the spectrum of $T_{\max}$ is empty.

PROOF. To show necessity assume that $T_{\max} = \{0\} \times L^2(w)$ and, by way of contradiction, that $\mathcal{L}_0 = \{0\}$. For any $f \in L^2(w)$ with support in $[0, 2\omega)$ there is a function u of locally bounded variation such that $Ju' + qu = wf$ and $wu = 0$. In $(-\infty, 0)$ we have that u must be equal to a linear combination of (generalized) Floquet solutions which we call $\tilde{u}$. Since then, in the terminology of Lemma 6.3, $I_n(\tilde{u}) = 0$ for all negative integers n and since $\mathcal{L}_0 = \{0\}$ we have, in fact, $\tilde{u} = 0$ and hence $u = 0$ on $(-\infty, 0)$. Similarly we can show that $u = 0$ on $[2\omega, \infty)$. For $x > 0$ the variation of constants formula (see Lemma 3.3 in [5]) gives

$$u^-(x) = U^-(x, 0) J^{-1} \int_{[0,x)} U(\cdot, 0)^* w f.$$

In particular, $\int_{[0,2\omega)} U(\cdot, 0)^* w f = 0$ regardless of f, contradicting the findings of Lemma 6.5.

To show sufficiency we note first that we may assume $w \neq 0$ since otherwise $L^2(w) = \{0\}$. The conclusion will follow from Theorem 7.3 in [5] once we establish its hypotheses. We have already shown that $n_\pm = 0$ and it is clear that $0 < \dim \mathcal{L}_0 < 2$, the former inequality following from the hypothesis that $\mathcal{L}_0$ is not trivial and the latter since $w \neq 0$. It remains to show that $\ker \Delta_w(x) \subset \ker \Delta_w(x) J^{-1} \Delta_q(x)$ for all $x \in \mathbb{R}$. Thus fix $x \in \mathbb{R}$ and let $\Delta_q(x) = \left(\begin{smallmatrix} a & b \\ b & d \end{smallmatrix}\right)$ and $\Delta_w(x) = \left(\begin{smallmatrix} \alpha & \beta \\ \beta & \delta \end{smallmatrix}\right)$. If the rank of $\Delta_w(x)$ is 0 or 2, then the needed inclusion holds trivially. Hence we assume that $\alpha\delta = \beta^2$ with at least one of those numbers non-zero. In this case $\det B_\pm(x, \lambda)$ is a real polynomial in λ of degree at most 1. Since $\det B_\pm(x, \cdot)$ must not have a real zero, it follows that the coefficient of λ must be 0, i.e., , $a\delta + \alpha d = 2b\beta$. Under that condition it turns out that $\Delta_w(x) J^{-1} \Delta_q(x)$ is symmetric and hence equal to $-\Delta_q(x) J^{-1} \Delta_w(x)$ which proves the desired inclusion and hence the lemma.

For the last claim we simply note that the operator part of $T_{\max}$ is $\{([0], [0])\}$ which has no spectrum. □

We now define the conditional stability set S to be the set of those $\lambda \in \mathbb{C}$ such that the equation $Ju' + qu = \lambda wu$ has a non-trivial bounded solution. Note that Theorem 6.2 shows that S and Λ do not intersect. It follows that the conditional stability set is the set of those $\lambda \in \mathbb{C} \setminus \Lambda$ for which the monodromy matrix $M(\lambda)$ has an eigenvalue of absolute value 1.

Lemma 6.7. If λ is not in $S \cup \Lambda$, then $M(\lambda)$ has distinct eigenvalues $\rho_1 = \mathrm{e}^{\tilde{m}\omega}$ and $\rho_2 = \mathrm{e}^{-\tilde{m}\omega}$ where $m = \operatorname{Re} \tilde{m} > 0$. The associated eigenfunctions $\psi_{1,2}$ may be chosen so that $(\psi_1^\pm)^\top J \psi_2^\pm = 1$. If, for a given $f \in L^2(w)$, we define

$$u^-(x) = \psi_2^-(x) \int_{(-\infty,x)} \psi_1^\top wf + \psi_1^-(x) \int_{[x,\infty)} \psi_2^\top wf,$$

then $u = u^\#$ satisfies the differential equation $Ju' + qu = w(\lambda u + f)$.

Proof. The first two statements follow from Theorem 3.3 and Lemma 3.2 in [5].

As before we shall write $B_\pm$ in place of $B_\pm(\cdot, \lambda)$. Since $\psi_j = (\psi_j^+ + \psi_j^-)/2$, $B_+\psi_j^+ = B_-\psi_j^-$, and $B_+ + B_- = 2J$, our hypothesis on the normalization of $\psi_{1,2}$ shows that

$$B_+(\psi_2^+\psi_1^\top - \psi_1^+\psi_2^\top) = \mathbb{1}. \tag{6.1}$$

Since

$$u^+(x) = \psi_2^+(x) \int_{(-\infty,x]} \psi_1^\top wf + \psi_1^+(x) \int_{(x,\infty)} \psi_2^\top wf$$

equation (6.1) gives $B_+u^+ - B_-u^- = \Delta_w f$ and hence

$$Ju^- = B_+u^\# - \frac{1}{2}\Delta_w f. \tag{6.2}$$

Now recall that, by the definition of Lebesgue-Stieltjes measures, the derivatives of u^- and u are the same. Hence the product rule for functions of locally bounded variation gives

$$Ju' = J\psi_2' \int_{(-\infty,\cdot)} \psi_1^\top wf + J\psi_1' \int_{[\cdot,\infty)} \psi_2^\top wf + J(\psi_2^+\psi_1^\top - \psi_1^+\psi_2^\top)wf.$$

Using the identity $J\psi_k' = (\lambda w - q)\psi_k = (\lambda w - q)B_+^{-1}J\psi_k^-$ and equations (6.1) and (6.2) we obtain now

$$\begin{aligned} Ju' &= (\lambda w - q)B_+^{-1}Ju^- + JB_+^{-1}wf \\ &= (\lambda w - q)u^\# + \frac{1}{2}(q - \lambda w)B_+^{-1}\Delta_w f + JB_+^{-1}wf. \end{aligned}$$

Since

$$\frac{1}{2}(q - \lambda w)B_+^{-1}\Delta_w f = \frac{1}{2}(\Delta_q - \lambda\Delta_w)B_+^{-1}wf = (\mathbb{1} - JB_+^{-1})wf$$

we finally find that $Ju' + (q - \lambda w)u^\# = wf$. □

Next we show that the function u just constructed is actually an element of $\mathcal{L}^2(w)$. In the following we use the notation of Lemma 6.7 and its proof freely. First note that averaging u^+ and u^- gives

$$\begin{aligned} u(x) = \psi_2(x) &\int^x \psi_1^\top wf + \psi_1(x) \int_x \psi_2^\top wf \\ &+ \frac{1}{2}(\psi_2^+(x)\psi_1(x)^\top + \psi_1^-(x)\psi_2(x)^\top)\Delta_w(x)f(x) \end{aligned} \tag{6.3}$$

where we introduced the notation $\int^x$ and $\int_x$ as abbreviations for $\int_{(-\infty,x)}$ and $\int_{(x,\infty)}$, respectively.

Introduce $\tilde{W}$, the Radon-Nikodym derivative of w with respect to the positive scalar measure $\operatorname{tr} w$. Then $\tilde{W}$ is a real symmetric matrix and $0 \le \tilde{W} \le \mathbb{1}$ pointwise almost everywhere with respect to $\operatorname{tr} w$ and we may assume it holds indeed everywhere. Therefore $f^* \tilde{W}(x) g$ is a semi-inner product in $\mathbb{C}^2$ which we denote by $\langle f, g\rangle_x$. The corresponding semi-norm is $|f|_x = \sqrt{\langle f, f\rangle_x}$.

Recall that $m = \operatorname{Re} \tilde{m} > 0$. Since $\psi_1(x) = \mathrm{e}^{\tilde{m}x}\, p_1(x)$ we get $|\psi_1(x)|_x = \mathrm{e}^{mx}\, |p_1(x)|_x$. Similarly $|\psi_2(x)|_x = \mathrm{e}^{-mx}\, |p_2(x)|_x$. Since p_1 and p_2 are periodic we may find a constant K such that $|\psi_j(x)|_x \le K\, \mathrm{e}^{\pm mx}$ where the sign in the exponent is $(-1)^{j+1}$. The norm of a rank one matrix $ab^\top$ is the product of the norms of a and b. Hence we may estimate the $|\cdot|_x$-norm of the last term in (6.3) by $K^2 |\Delta_w(x) f(x)|_x$. However, the matrices $\Delta_w(x)$ have to be uniformly bounded by, say, a number $D \ge 1$ since w is periodic and locally finite. Thus our estimate on the last term in (6.3) becomes $K^2 D |f(x)|_x$.

To deal with the other terms in (6.3) we define

$$\phi_1(x) = \int^x \psi_1^\top w f \quad \text{and} \quad \phi_2(x) = \int_x \psi_2^\top w f.$$

Upon using Cauchy's inequality for the $\langle\cdot,\cdot\rangle_x$-inner product we obtain the estimates $|\phi_j(x)| \le K H_j(x)$ where

$$H_1(x) = \int^x \mathrm{e}^{my}\, |f(y)|_y \operatorname{tr} w(y) \quad \text{and} \quad H_2(x) = \int_x \mathrm{e}^{-my}\, |f(y)|_y \operatorname{tr} w(y).$$

Therefore we find

$$|u(x)|_x \le K^2\, \mathrm{e}^{-mx}\, H_1(x) + K^2\, \mathrm{e}^{mx}\, H_2(x) + K^2 D |f(x)|_x.$$

Since $\|u\|^2 = \int |u(x)|_x^2 \operatorname{tr} w(x)$ and $H_j^2 \le H_j^{2\#}$ we obtain

$$\|u\|^2 \le 3K^4 D^2 \int \left(\mathrm{e}^{-2mx}\, H_1^{2\#}(x) + \mathrm{e}^{2mx}\, H_2^{2\#}(x) + |f(x)|_x^2 \right) \operatorname{tr} w(x).$$

Next we estimate the integrals $I_1(a,b) = \int_{(a,b)} \mathrm{e}^{-2mx} H_1^{2\#}(x) \operatorname{tr} w(x)$ and $I_2(a,b) = \int_{(a,b)} \mathrm{e}^{2mx} H_2^{2\#}(x) \operatorname{tr} w(x)$ assuming that a and b are points of continuity of W, the antiderivative of w. To this end we define

$$R_-(x) = \int_x \mathrm{e}^{-2my} \operatorname{tr} w(y) \quad \text{and} \quad R_+(x) = \int^x \mathrm{e}^{2my} \operatorname{tr} w(y).$$

We emphasize that $R_\pm$ are positive functions and that R_+ is non-decreasing while R_- is non-increasing. Hence $R_\pm \leq R_\pm^\#$. Using the periodicity of w and Theorem 2.4 we find $R_+(x) \leq C_m \mathrm{e}^{2mx}$ and $R_-(x) \leq C_{-m} \mathrm{e}^{-2mx}$ for appropriate constants $C_{\pm m}$. The continuity of these upper bounds implies that even $R_+^\#(x) \leq C_m \mathrm{e}^{2mx}$ and $R_-^\#(x) \leq C_{-m} \mathrm{e}^{-2mx}$. Now we see that, using integration by parts,

$$I_1(a,b) = -\int_{(a,b)} H_1^{2\#} dR_- = -H_1(b)^2 R_-(b) + H_1(a)^2 R_-(a) + \int_{(a,b)} R_-^\# dH_1^2.$$

The first term on the right is negative and may be omitted. To deal with the second term we use Cauchy's inequality to show that $H_1(a)^2 \leq R_+(a) \int^a |f|_y^2 \operatorname{tr} w(y)$. Hence $\lim_{a\to-\infty} H_1(a)^2 R_-(a) = 0$. In the last term we use the chain rule $dH_1^2 = 2H_1^\# dH_1$ (see Vol′pert [8], Section 13.2 or Vol′pert and Hudjaev [9], Chapter 4, §6.3). Thus

$$\int_{(a,b)} R_-^\# dH_1^2 = 2\int_{(a,b)} R_-^\# H_1^\# dH_1$$
$$\leq 2C_{-m} \int_{(a,b)} \mathrm{e}^{-mx} H_1^\#(x) |f(x)|_x \operatorname{tr} w(x).$$

Cauchy's inequality and the fact that $H_j^{\#2} \leq H_j^{2\#}$ give next that

$$I_1(a,b) \leq H_1(a)^2 R_-(a) + 2C_{-m} I_1(a,b)^{1/2} \|f\|.$$

Taking now the limit as a tends to $-\infty$ and b to ∞ shows that $I_1(-\infty,\infty) \leq 4C_{-m}^2 \|f\|^2$. Treating I_2 in a similar way we finally obtain

$$\|u\|^2 \leq 3K^4 D^2 (4C_{-m}^2 + 4C_m^2 + 1) \|f\|^2.$$

This proves that $(u, \lambda u + f) \in T_{\max}$, i.e., , $\lambda \in \varrho$. Equation (6.3) means that the Green's function for $T_{\max}$ is given by

$$G(x,y) = \begin{cases} \psi_2(x)\psi_1(y)^\top & \text{if } y < x \\ \frac{1}{2}(\psi_2(x)\psi_1(x)^\top + \psi_1(x)\psi_2(x)^\top) & \text{if } y = x \\ \psi_1(x)\psi_2(y)^\top & \text{if } x < y. \end{cases}$$

We have now shown that $(S \cup \Lambda)^{\mathrm{c}} \subset \varrho$. Since Λ is also a subset of ϱ we have established the following lemma.

Lemma 6.8. The complement S^{c} of the conditional stability set S is contained in ϱ, the resolvent set of $T_{\max}$.

Next we will address the converse statement.

Lemma 6.9. If $\mathcal{L}_0$ is trivial the conditional stability set S is contained in σ, the spectrum of $T_{\max}$.

Proof. Suppose $\lambda \in S$ and, by way of contradiction, that it is also in ϱ. Theorem 6.2 shows that $\lambda \notin \Lambda$. Hence there is a balanced, bounded Floquet solution ψ of the differential equation $J\psi' + (q - \lambda w)\psi = 0$ with Floquet multiplier ρ. Of course, $\psi = U(\cdot, \lambda)c_0$ for some vector c_0 which is an eigenvector of $M(\lambda)$ associated with ρ. Lemma 3.2 in [5] shows that Ju_0 is an eigenvector of $M(\overline{\lambda})^*$ associated with the eigenvalue $1/\rho$. Lemma 6.5 provides the existence of a function g_0 compactly supported in $[0, 2\omega)$ satisfying

$$\int_{[0,2\omega)} U(\cdot, \overline{\lambda})^* w g_0 = -Jc_0.$$

For $n \in \mathbb{N}$ we put $g_n = \rho^n g_0(\cdot - n\omega)$, a function which is supported in $[n\omega, (n+2)\omega)$, and use the variation of constants formula to obtain a solution v_n of the equation

$Ju' + qu = w(\lambda u + g_n)$, i.e., ,

$$v_n^-(x) = U^-(x,\lambda)\big(c_0 + J^{-1}\int_{[0,x)} U(\cdot,\overline{\lambda})^* w g_n\big).$$

The function v_n coincides with ψ on $(-\infty, n\omega)$. Since

$$\int_{[0,(n+2)\omega)} U(\cdot,\overline{\lambda})^* w g_n = \rho^n M(\overline{\lambda})^{*n} \int_{[0,2\omega)} U(\cdot,\overline{\lambda})^* w g_0$$

$$= -\rho^n M(\overline{\lambda})^{*n} J c_0 = -J c_0$$

it follows that v_n is identically equal to 0 beyond $(n+2)\omega$. With a similar device for the negative half-line we may now construct a sequence $(u_n, \lambda u_n + f_n) \in T_{\max}$ with the following properties: (i) $u_n = 0$ outside $[-(n+2)\omega, (n+2)\omega]$, (ii) $u_n = \psi$ on $(-n\omega, n\omega)$, and (iii) $\|f_n\|$ is independent of n.

Since $|\rho| = 1$ we have that

$$\|u_n\|^2 \geq \int_{[-n\omega,n\omega)} \psi^* w \psi = 2nC$$

where $C = \int_{[0,\omega)} \psi^* w \psi$ does not depend on n.

Arguing similarly to the conclusion in the proof of Theorem 5.2 we now have $\|u_n\| \leq \|(T-\lambda)^{-1}\|\|f_n\|$ where the left-hand side tends to infinity while the right-hand side is constant, the desired contradiction. ☐

Combining Theorem 6.4, Lemma 6.8, and Lemma 6.9 we have now the following result.

THEOREM 6.10. Assume that $\mathcal{L}_0$ is trivial. Then the spectrum σ of $T_{\max}$ is purely continuous and coincides with the conditional stability set S.

We end this section by an investigation of the properties of the Floquet discriminant D. Suppose x_0 is a point of continuity for Q and W, the antiderivatives of q and w, and $U(\cdot,\lambda)$ is a fundamental matrix for $Ju' + qu = \lambda w u$ such that $U(x_0,\lambda) = \mathbb{1}$.

Observe that

$$J\dot{U}'(\cdot,\lambda) + (q - \lambda w)\dot{U}(\cdot,\lambda) = wU(\cdot,\lambda)$$

where the accent $\dot{}$ denotes the derivative with respect to λ. Since $M = U(x_0 + \omega, \cdot)$ and $\dot{U}(x_0, \cdot) = 0$ the variation of constants formula shows that

$$\dot{M}(\lambda) = U(x_0 + \omega, \lambda)J^{-1} \int_{(x_0, x_0+\omega)} U(\cdot, \overline{\lambda})^* w U(\cdot, \lambda) = M(\lambda)J^{-1}T(\lambda)$$

where the last equation defines the matrix T.

Assume that $\lambda \in \mathbb{R}$. Then $M(\lambda)$ and $D(\lambda)$ are real and $T(\lambda)$ is real and positive semi-definite. Since $\det M = 1$ one may now check that $D^2 - 4 = (M_{11} - M_{22})^2 + 4M_{12}M_{21}$. Moreover, setting $a = (M_{11} - M_{22}, 2M_{21})^\top$ and $b = (-2M_{12}, M_{11} - M_{22})^\top$ one obtains

$$4rM_{21}\dot{D} = -T_{11}(D^2 - 4) + a^*Ta \text{ and } 4rM_{12}\dot{D} = T_{22}(D^2 - 4) - b^*Tb. \tag{6.4}$$

Now suppose that $D(\lambda)^2 - 4 < 0$. It is then clear that $M_{12}(\lambda)$ and $M_{21}(\lambda)$ are different from 0 and thus the above identities show that $\dot{D}(\lambda) \neq 0$ provided that at least one of $T_{11}(\lambda)$ and $T_{22}(\lambda)$ is different from 0. But $T_{11}(\lambda) = T_{22}(\lambda) = 0$ may only happen when $w = 0$, a case excluded by our hypothesis.

LEMMA 6.11. The Floquet discriminant D is constant if and only if $T_{\max}$ equals $\{0\} \times L^2(w)$. Moreover, if $\dim \mathcal{L}_0 = 1$, D does not take its value in $(-2, 2)$.

PROOF. Suppose D is constant. If $D^2 > 4$, then $\mathbb{R} \subset S^c \subset \varrho$, the resolvent set of $T_{\max}$. Since the spectrum is empty the domain of $T_{\max}$ must be $\{[0]\}$. Since $T_{\max}$ is self-adjoint its range must be $L^2(w)$. If $D^2 \leq 4$, then $S = \mathbb{C} \setminus \Lambda$ in view of the analyticity of D. If $\mathcal{L}_0$ were trivial, Lemma 6.9 would show that $\sigma = \mathbb{C}$. Now Theorem 6.6 shows that $T_{\max} = \{0\} \times L^2(w)$.

Conversely, if $T_{\max} = \{0\} \times L^2(w)$ and, consequently, $\mathcal{L}_0 \neq \{0\}$, there exists a non-trivial solution of $Ju' + qu = 0$ with norm 0. Lemma 6.3 shows that then at least one of the Floquet solutions must have norm 0. Such a solution is then a solution of

$Ju' + qu = \lambda wu$ for all λ. If ρ is the associated Floquet multiplier, then $D(\lambda) = \rho + 1/\rho$ for all $\lambda \in \mathbb{C}$.

Finally, suppose $\dim \mathcal{L}_0 = 1$ and, by way of contradiction, that D is in $(-2, 2)$. Then at least one of the diagonal entries of the matrix T in (6.4) is positive. Thus $\dot{D}(\lambda) \neq 0$ (for $\lambda \in \mathbb{R}$), a contradiction. □

THEOREM 6.12. Suppose $\mathcal{L}_0$ is trivial. Then, the Floquet discriminant D, restricted to the real line, is strictly monotone in intervals where $D^2 < 4$. If $D(\lambda)^2 = 4$, we have $\dot{D}(\lambda) = 0$ if and only if $M(\lambda) = \pm\mathbb{1}$. Such a point is not a strict local minimum if $D(\lambda) = 2$ nor a strict local maximum if $D(\lambda) = -2$.

PROOF. We already mentioned above that $\lambda \mapsto D(\lambda)$ is real-valued and analytic as λ varies in $\mathbb{R}$. We have also proved that $\dot{D}(\lambda) \neq 0$ when $D(\lambda)^2 < 4$.

Now, assume $D(\lambda)^2 = 4$ and $\dot{D}(\lambda) = 0$. Then (6.4) shows that $a^* T a = b^* T b = 0$. Since $\mathcal{L}_0$ is trivial T is positive definite and hence a and b are 0 proving $M(\lambda) = \pm\mathbb{1}$. Conversely, if $M(\lambda) = \pm\mathbb{1}$, then $\dot{D}(\lambda) = \pm \operatorname{tr}(J^{-1} T(\lambda)) = 0$. Finally, if D has a strict local minimum of 2 at a point λ, then λ is an isolated point in the spectrum of $T_{\max}$ and hence an eigenvalue which is impossible. Of course, a similar argument prevents D to have a local maximum of -2 anywhere. □

7. Examples

In all examples presented below we assume that $n = 2$, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\omega = 1$ (except in our last example).

Our first three examples have constant coefficients.

- $q = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$ and $w = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Here we have $\mathcal{L}_0 = \{0\}$ and $D(\lambda) = 2\cos(\sqrt{\lambda}) = 2\cosh\sqrt{-\lambda}$. Hence $S = \sigma = [0, \infty)$. This is the system describing the second order equation $-y'' = \lambda y$.
- $q = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$ and $w = 0$. We get $D = 2\cosh\sqrt{b^2 - ad} = 2\cos\sqrt{ad - b^2}$. Hence D is constant (as anticipated) and may take any value in $[-2, \infty)$. If $ad - b^2 > 0$ we have $S = \mathbb{R}$ but $\sigma = \emptyset$.

- $q = \left(\begin{smallmatrix} a & b \\ b & 0 \end{smallmatrix}\right)$ and $w = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Now $\mathcal{L}_0$ is spanned by the vector $(0, \mathrm{e}^{bx})^\top$. We have $D = 2\cosh b$. Note that D cannot be in $(-2, 2)$.

These last two examples show that the hypothesis of a trivial $\mathcal{L}_0$ is necessary in Lemma 6.9.

Our next three examples involve discrete measures as coefficients. Let $\mu = \sum_{k\in\mathbb{Z}} \delta_k$ where δ_k is the Dirac measure concentrated on $\{k\}$.

- $q = \left(\begin{smallmatrix} a\mu & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $w = \left(\begin{smallmatrix} \alpha\mu & 0 \\ 0 & 0 \end{smallmatrix}\right)$. If $\alpha > 0$ this problem is definite, i.e., , $\mathcal{L}_0$ is trivial. The discriminant is $D(\lambda) = 2 + a - \alpha\lambda$. Hence $\sigma = [a/\alpha, (4 + a)/\alpha]$. However, if $\alpha = 0$, we have $\dim \mathcal{L}_0 = 2$. In this case D is constant and the constant may take any real value.
- $q = \left(\begin{smallmatrix} a & b \\ b & 0 \end{smallmatrix}\right)\mu$ and $w = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\mu$ where $b^2 \neq 4$ to avoid an intersection of Λ and $\mathbb{R}$. Here $\mathcal{L}_0$ is one dimensional. We get $D = 2(4 + b^2)/(4 - b^2)$ which may attain any value outside $[-2, 2)$.
- $q = \left(\begin{smallmatrix} a & b \\ b & d \end{smallmatrix}\right)\mu$ and $w = \mathbb{1}\mu$ where we require $(a - d)^2 + 4b^2 < 16$ to avoid that Λ intersects $\mathbb{R}$. Then

$$D(\lambda) = \frac{16}{\lambda^2 - (a + d)\lambda + ad - b^2 + 4} - 2.$$

  Our condition on a, b and d guarantees that the denominator is always positive. Thus $D(\lambda) > -2$ for all $\lambda \in \mathbb{R}$ but $D(\lambda)$ approaches -2 as λ tends to $\pm\infty$. There is one maximum for D at $\lambda = (a + d)/2$. Its value is at least 2 and is equal to 2 precisely when $a = d$ and $b = 0$. Except for this special situation the spectrum consists of two rays separated by one gap.

Finally we mention one example which violates our condition that $\Lambda \cap \mathbb{R} = \emptyset$. The example is taken from [7] where more details may be found.

- $q = \left(\begin{smallmatrix} 0 & 2 \\ 2 & 0 \end{smallmatrix}\right) \sum_{k\in\mathbb{Z}}(\delta_{2k} - \delta_{2k+1})$ and $w = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 0 \end{smallmatrix}\right) \sum_{k\in\mathbb{Z}} \delta_k$. Note that here $\omega = 2$. In this case $\Lambda = \mathbb{C}$, i.e., , unique continuation of solutions fails for any λ. Since there are compactly supported solutions of $Ju' + qu = 0$, it turns out that 0 is an eigenvalue of infinite multiplicity. The resolvent set is $\mathbb{C} \setminus \{0\}$.

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