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# Sturmian Theorems, Asymptotic Behavior for Eigenvalues, and Spectral Theory for Ordinary Differential Equations With Distributional Coefficients

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# STURMIAN THEOREMS, ASYMPTOTIC BEHAVIOR FOR EIGENVALUES, AND SPECTRAL THEORY FOR ORDINARY DIFFERENTIAL EQUATIONS WITH DISTRIBUTIONAL COEFFICIENTS


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STURMIAN THEOREMS, ASYMPTOTIC BEHAVIOR FOR EIGENVALUES, AND SPECTRAL THEORY FOR ORDINARY DIFFERENTIAL EQUATIONS WITH DISTRIBUTIONAL COEFFICIENTS

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ABSTRACT

We study the theory of ordinary differential equations whose coefficients are distributions of order 0.

In the first chapter of the thesis, we present some of the basic facts about the theory of distributions of order 0, the first-order ordinary differential equation with distributional coefficients, and the theory of linear relation associated with these differential equations.

In the second chapter, we study the spectral theory of the first order equation with distributional coefficients. In particular, we prove that the adjoint of the minimal relation is equal to the maximal relation, which is a cornerstone for a spectral theory since it ensures the existence and classification of self-adjoint linear relations via the well-known von Neumann theorem about Friedrich extensions.

The next chapter of the thesis deals with a collection of Sturm-Liouville type theorems regarding the close connection between the counting number of eigenvalues and the zero set of their associating eigenfunctions.

In the last chapter, we establish a result about the existence and asymptotic formula for the eigenvalues of a regular two-point boundary value problem.

## DEDICATION

*To mom and dad,*

*my beloved sister,*

*my advisor,*

*and my friends.*

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# CHAPTER 1

# INTRODUCTION

## 1. Functions Of Bounded Variation, Distributions Of Order 0, And Radon Measures

In this section we shall cover some of the basic facts about functions of locally bounded variation, distributions of order 0, and Radon measures. We shall also cover the Riesz Representation theorem, a fundamental result connecting these three mathematical objects.

Throughout the introduction section we let $I = (a, b)$, $-\infty \leq a < b \leq \infty$, to be an interval on the real line on which a complex-valued function $f$ is defined.

For any subinterval $J$ of $I$, we define the *variation* of $f$ over $J$ as

$$\mathrm{Var}_J(f) = \sup_{\mathcal{P}} \sum_{j=0}^{n} |Q(x_{j+1}) - Q(x_j)|, \tag{1.1}$$

where the supremum is taken over all possible finite partition $\mathcal{P} = \{x_0 < x_1 < \ldots < x_n : x_j \in I\}$ of $J$.

The function $f$ is said to be of *bounded variation* on $I$ if $\mathrm{Var}_I(f) < \infty$. It is said to be *of locally bounded variation* on $I$ if $\mathrm{Var}_K(f) < \infty$ for any compact subinterval $K$ of $I$. We denote the set of all functions of bounded variation and locally bounded variation on $(a, b)$ by $\mathrm{BV}(I)$ and $\mathrm{BV}_{\mathrm{loc}}(I)$, respectively. It is obvious that $\mathrm{BV}(I) \subset \mathrm{BV}_{\mathrm{loc}}(I)$

If $f \in \mathrm{BV}_{\mathrm{loc}}(I)$ then $f^-(x) = \lim_{t\uparrow x} f(t)$ and $f^+(x) = \lim_{t\downarrow x} f(t)$ exist for any $x \in I$. We call the functions $f^-$, $f^+$ defined over $I$ as the the *left continuous* and the *right continuous* version of $f$, respectively. Moreover, $f$ has at most countably many *discontinuities of the first type*, i.e., any $x \in (a, b)$ such that $f^+(x) \neq f^-(x)$. The function $f^\#$ defined as $f^\#(x) = (f^+(x) + f^-(x))/2$ is said to be the *balanced* version of $f$. $f$ is said to be a *balanced* function if $f = f^\#$.

A set function $\mu$ defined on the set $\mathcal{B}(I)$ of Borel subsets of $I$ is said to be a *Radon measure* on $I$ if for any compact subset $K \subset I$, $\mu$ is a measure on the measure space $(K, \mathcal{B}(K))$. If such a set function $\mu$ is indeed a measure on $(I, \mathcal{B}(I))$ then it is said to be a *finite Radon measure.*

For any function $f \in \mathrm{BV}_{\mathrm{loc}}(I)$, there is a Radon measure $df$ on $I$ such that $df([c,d)) = f^-(d) - f^-(c)$ for any finite interval $[c,d)$ as long as $[c,d] \subset (a,b)$. This measure $df$ is called the *Lebesgue-Stiltjes measure* associated with $f$. We refer to Folland ([3], Chapter 1) for the construction of the measure $df$. We shall oftentimes interchangeably use the two notations $df$ and $f'$ to denote the same measure.

We denote $C_c^\infty(I)$ to be the set of function $\phi$ defined on $I$ such that it is infinitely often continuously differentiable and $\operatorname{supp}\phi \subset I$. Such a function $\phi$ is called a *test function.* We shall also oftentimes use the notation $\mathcal{D}(I)$ to denote $C_c^\infty(I)$, the space of test functions.

A linear functional $p : C_c^\infty(I) \to \mathbb{C}$ is called a *distribution* if for every compact subset $K$ of $I$ there are numbers $C \geq 0$ and $k \in \mathbb{N}_0$ such that

$$|p(\phi)| \leq C \sum_{j=0}^{k} \|\phi^{(j)}\|_\infty$$

for all test functions $\phi$ such that $\operatorname{supp}\phi \subset K$.

The distribution $p$ is called a *distribution of order* 0 if the number $k$ can be chosen as 0 independently from $K$. We denote $\mathcal{D}'(I)$ and $\mathcal{D}'^0(I)$ to be the space of distributions and distributions of order 0 on $I$, respectively.

For a distribution $p$ its *derivative* $p'$ is a distribution defined as $p'(\phi) = -p(\phi')$ for any test function $\phi$.

If there is a distribution $q$ such that $q' = p$ then we say $q$ is an *antiderivative* of $p$. It is a fact that antiderivatives of $p$ exist. And the difference of any two of such antiderivatives is a *constant distribution*, i.e., the distribution $\phi \mapsto C \int \phi \, dx$, for some constant $C$. This is the result of the following lemma.

LEMMA 1.1 (Du Bois-Reymond). *If $p \in \mathcal{D}'(I)$ and $p' = 0$ then $p$ is a constant distribution.*

Let $P \in \mathrm{BV}_{\mathrm{loc}}(I)$, $dP$ to be the Lebesgue-Stieltjes measure associated with $P$, and m to be the Lebesgue measure on $I$. The two linear functionals $\phi \mapsto \int \phi P \, d\mathrm{m}$ and $\phi \mapsto \int \phi \, dP$ are two distributions of order 0. We shall oftentimes identify the function $P$ with the former distribution.

It is an important fact that the antiderivatives of a distribution of order 0 are functions of locally bounded variation. This follows from the following version of the Riesz's Representation theorem.

THEOREM 1.2 (Riesz's Representation Theorem). *If $p \in \mathcal{D}'^0(I)$ then there exits $P \in \mathrm{BV}_{\mathrm{loc}}(I)$ such that $p(\phi) = \int \phi p = \int \phi \, dP$.*

PROOF. We recall that $C_c^\infty(I)$ is dense in $C_c(I)$ (see e.g., Theorem 3.12, [5]), and $C_c(I)$ is dense in $C_0(I)$ (see e.g., Theorem 3.17, [6]), both with respect to the uniform norm $||\phi||_u := \sup_{x \in U} |\phi(x)|$.

Fix a point $x_0 \in (a, b)$, and let $\{U_j := (c_j, d_j)\}_{j=1}^\infty$ be a strictly-increasing sequence of finite subintervals covering $I$ and adopting $x_0$ as the common point.

We extend by zero any test function $\phi \in C_c^\infty(U_j)$ into a function $\phi \in C_c^\infty(a, b)$. Since $p$ is a distribution of order 0 there are constants $C_{[c_j, d_j]}$ such that

$$|p(\phi)| \le C_{[c_j,d_j]} \sup_{x \in [c_j,d_j]} |\phi(x)| = C_{[c_j,d_j]} \sup_{x \in (c_j,d_j)} |\phi(x)| = C_{[c_j,d_j]} ||\phi||_u.$$

This means the restriction of $p$ to $C_c^\infty(U_j)$ is a bounded linear functional, and hence can be extended to a linear functional on $C_0(U_j)$. By the Riesz-Markov theorem (see e.g., Theorem 6.19, [6]), there exists a unique (regular) complex Borel measure

$\mu_j : \mathcal{B}(U_j) \to \mathbb{C}$ such that

$$p(\phi) = \int_{\mu_j} \phi \chi_{U_j}, \text{ for any } \phi \in C_0(U_j).$$

Let $F_{\mu_j}$ be the *cumulative distribution function* of $\mu_j$ with respect to the fixed point $x_0$. We note that so far $F_{\mu_j}$ is only defined on $U_j$. If $\phi \in C_0(U_j)$ then $\phi \in C_0(U_{j+1})$, thus

$$\int_{\mu_j} \phi \chi_{U_j} = p(\phi) = \int_{\mu_{j+1}} \phi \chi_{U_{j+1}} = \int_{\mu_{j+1}} \phi \chi_{U_j}.$$

This means $p$ is being represented by both $\mu_j$ and $\mu_{j+1}$ on $C_0(U_j)$. The uniqueness property in the Riesz-Markov theorem confirms that $\mu_j$ is a restriction of $\mu_{j+1}$, and thus $F_j$ is a restriction of $F_{j+1}$. Since $\{U_j\}$ covers $(a, b)$ we can get a function $P$ from $\{F_j\}$. Since each $F_j$ is of locally bounded variation function, so is $P$. □

Let $p \in \mathcal{D}'^0(I)$ and $P \in \mathrm{BV}_{\mathrm{loc}}(I)$ as in Theorem 1.2. Then, as previously mentioned, $P$ is an antiderivative of $p$. Moreover, the function $P$ can be chosen to be left-continuous since $dP = dP^-$ and $P^-$ is a left-continuous function.

We note that the measure $dP$ is independent of the choice of $x_0$ and the sequence $\{U_j\}$, and the antiderivative $P$ is also independent up to a constant.

## 2. The Hilbert Space $L^2(w)$

In this section we shall define the space $L^2(w)$ associated with a given nonnegative matrix-valued distribution $w$ of order 0.

For any distribution $p \in \mathcal{D}'(I)$, we define its *conjugate* $\overline{p}$ as a distribution defined by $\overline{p}(\phi) = \overline{p(\overline{\phi})}$. A distribution $p$ is said to be *nonnegative* if $p(\phi) \geq 0$ whenever $\phi \geq 0$. Similarly, $p$ is said to be *real* if $p(\phi) \in \mathbb{R}$ whenever $\phi$ is real-valued. Moreover, $p$ is real if and only if $p = \overline{p}$.

We denote $\mathcal{D}'^0(I)^{n\times n}$ to be the space of $n\times n$-matrices whose entries are distributions of order 0 on $I$.

A matrix-valued distribution $w \in \mathcal{D}'^0(I)^{n\times n}$ is said to be *hermitian* if $\overline{w_{jk}} = w_{kj}$. Equivalently, $w$ is hermitian if and only if $z^* w z$ is real for all $z \in \mathbb{C}^n$. $w$ is said to be *nonnegative*, write $w \geq 0$, if $z^* w z$ is nonnegative for all $z \in \mathbb{C}^n$. Nonnegative matrix-valued distributions are hermitian and all of their diagonal entries are nonnegative.

Let $w \in \mathcal{D}'^0(I)^{n\times n}$ be a nonnegative matrix-valued distribution, $W$ be the matrix whose entries are the left-continuous antiderivatives of the entries of $w$, and $dW$ be the matrix whose entries are the corresponding Lebesgue-Stieltjes measure.

The trace $\operatorname{tr} w = \sum w_{jj}$ is then a nonnegative distribution of order 0, and hence its antiderivative $\operatorname{tr} W$ is a nondecreasing function. This is so since for any $z \in \mathbb{C}^n$, and any $x, y \in I$ such that $y \geq x$ we have

$$\begin{aligned}\operatorname{tr} W(y) - \operatorname{tr} W(x) &= \int_{\mathrm{m}} \chi_{(x,y)} \operatorname{tr} W' = -\int_{\mathrm{m}} \chi'_{(x,y)} \operatorname{tr} W \\ &= \operatorname{tr} W'(\chi_{(x,y)}) = \operatorname{tr} w(\chi_{(x,y)}) \geq 0,\end{aligned}$$

Thence the induced Lebesgue-Stieltjes measure $d \operatorname{tr} W$ is a positive Borel measure.

Let $\widetilde{W} = (dW_{j,k}/d \operatorname{tr} W)$ be the matrix whose entries are the Radon-Nikodym derivative of $dW_{j,k}$ with respect to the positive measure $d \operatorname{tr} W$.

We have $0 \leq dW \leq (\operatorname{tr} dW) I_n = d(\operatorname{tr} W) I_n$. And thus each $dW_{jk}$ is absolutely continuous with respect to $d(\operatorname{tr} W)$. In addition, since $0 \leq \widetilde{W}(x) \leq 1$ a.e. with respect to $d(\operatorname{tr} W)$, thus $\widetilde{W}(x)$ is Hermitian a.e. with respect to $d(\operatorname{tr} W)$.

So, the square root $\sqrt{\widetilde{W}}(x)$ is defined a.e. and measurable with respect to $d(\operatorname{tr} W)$.

We define

$$\begin{aligned}\mathcal{L}^1_{\mathrm{loc}}(w) &:= \{f : I \to \mathbb{C}^n : \widetilde{W} f \in L^1_{\mathrm{loc}}(d(\operatorname{tr} W))^n\} \\ &= \{f : I \to \mathbb{C}^n;\ \text{components of } \widetilde{W} f \text{ are locally } d(\operatorname{tr} W)\text{-integrable}\}, \\ \mathcal{L}^2(w) &:= \{f : I \to \mathbb{C}^n : \sqrt{\widetilde{W}} f \in L^2(d(\operatorname{tr} W))^n\} \\ &= \{f : I \to \mathbb{C}^n :\ \text{components of } \sqrt{\widetilde{W}} f \text{ are } d(\operatorname{tr} W)\text{-square integrable}\} \\ &= \{f : I \to \mathbb{C}^n : \int f^* \widetilde{W} f \, d(\operatorname{tr} W) < \infty\},\end{aligned}$$

For any $f, g \in \mathcal{L}^2(w)$, define their semi-inner product and semi-norm by

$$\langle f, g\rangle = \int f^* \widetilde{W} g \, d(\operatorname{tr} W), \text{ and } ||f||^2 := \int f^* \widetilde{W} f \, d(\operatorname{tr} W).$$

We define the $L^2(w)$ as the quotient space of $\mathcal{L}^2(w)$ via the equivalent relation $f \sim g$ iff $||f - g|| = 0$. It is an important fact that $(L^2(w), ||\cdot||)$ is a Hilbert space.

## 3. Ordinary Differential Equations Whose Coefficients Are Distribution Of Order 0

We recall that a function $f$ defined on $I$ is said to be *balanced* if $f = f^\# = (f^- + f^+)/2$. We denote $\mathrm{BV}^\#_{\mathrm{loc}}(I)^n$ to be the space of $n \times 1$-matrices whose entries are balanced functions of locally bounded variation on $I$.

Let $r \in \mathcal{D}'^0(I)^{n\times n}$ and $g \in \mathcal{D}'^0(I)^n$. We consider the first order ordinary differential equation

$$u' = ru + g \tag{3.1}$$

whose solutions would be sought in the space $\mathrm{BV}^\#_{\mathrm{loc}}(I)^n$.

We shall interpret the equation (3.1) in the sense of distributions, i.e., considering both sides of this equation as distributions. Indeed, if $u \in \mathrm{BV}^\#_{\mathrm{loc}}(I)^n$ then the entries of $u'$ are a distribution of order 0. The entries of $ru$ are also distributions of order 0 since the multiplication of a distribution of order 0 with a function of locally bounded variation is again a distribution of order 0.

For any $p \in \mathcal{D}'^0(I)^{m\times n}$, we define the function $\Delta_p : I \to \mathbb{C}^{m\times n}$ as

$$\Delta_p(x) = P^+(x) - P^-(x) \tag{3.2}$$

where $P$ is the antiderivative of $p$.

Assuming that the matrices $\mathbb{1} \pm \Delta_r(x)/2$ is invertible for all $x \in I$ then the existence and uniqueness problem associated with the equation (3.1) has been studied by Ghatasheh and Weikard [4]. Without this assumption, there may be more than one solution for a given initial condition or none at all.

## 4. The Equation $Ju' + qu = wf$ And Its Associated Linear Relations

Suppose $\mathcal{H}_1$, $\mathcal{H}_2$ are two Hilbert spaces. We equip the set $\mathcal{H}_1 \times \mathcal{H}_2$ with the scalar product $\langle (u,f),(v,g)\rangle = \langle u,v\rangle + \langle f,g\rangle$ for all $(u,f),(v,g) \in \mathcal{H}_1 \times \mathcal{H}_2$.

A *linear relation* $T$ on $\mathcal{H}_1 \times \mathcal{H}_2$ is a linear subspace of $\mathcal{H}_1 \times \mathcal{H}_2$. The *domain*, the *range*, and the *kernel* of $T$ are defined as the subspaces

$$\operatorname{dom}(T) = \{u \in \mathcal{H}_1 : \exists f \in \mathcal{H}_2, (u,f) \in T\},$$

$$\operatorname{ran}(T) = \{f \in \mathcal{H}_2 : \exists u \in \mathcal{H}_1, (u,f) \in T\},$$

$$\ker(T) = \{u \in \mathcal{H}_1 : (u,0) \in T\}.$$

The *adjoint* of a linear relation $T$ on $\mathcal{H}_1 \times \mathcal{H}_2$ is defined to be

$$T^* = \{(v,g) \in \mathcal{H}_2 \times \mathcal{H}_1 : \forall (u,f) \in T : \langle g,u\rangle = \langle v,f\rangle\}.$$

We refer to the book by Bennewitz, Brown, and Weikard [2] for further details about the theory of linear relations.

We shall now switch gear by considering the main equation that is studied in Chapter 2 and 3 of this thesis. Consider the first-order equation

$$Ju' + qu = wf \tag{4.1}$$

on the interval $I = (a,b)$ where $J$ is a constant, invertible and skew-Hermitian $n \times n$-matrix. Both $q$ and $w$ are in $\mathcal{D}'^0(I)^{n\times n}$, while $w$ is nonnegative and $q$ is Hermitian. It is easy to see that equation (4.1) is a special case of equation (3.1) where $r = -J^{-1}qu$ and $g = J^{-1}wf$.

The assumption that $w$ to be a nonnegative distribution here is crucial since then we are equipped with the Hilbert space $L^2(w)$. We define

$$\mathcal{T}_{\max} = \{(u,f) \in \mathcal{L}^2(w) \times \mathcal{L}^2(w) : u \in \mathrm{BV}^{\#}_{\mathrm{loc}}(I)^n, Ju' + qu = wf\},$$

and

$$\mathcal{T}_{\min} = \{(u,f) \in \mathcal{T}_{\max} : \operatorname{supp} u \text{ is compact in } I\}.$$

Finally, we can now define the maximal and minimal relations in $L^2(w) \times L^2(w)$ associated with the equation (4.1) to be the two following quotient spaces

$$\begin{aligned} T_{\max} &= \{([u],[f]) \in L^2(w) \times L^2(w) : (u,f) \in \mathcal{T}_{\max}\} \\ &= \{([u],[f]) \in L^2(w) \times L^2(w) : \exists u \in [u], \exists f \in [f], Ju' + qu = wf\}, \end{aligned}$$

and

$$\begin{aligned} T_{\min} &= \{([u],[f]) \in L^2(w) \times L^2(w) : (u,f) \in \mathcal{T}_{\min}\} \\ &= \{([u],[f]) \in L^2(w) \times L^2(w) : \exists u \in [u], \operatorname{supp} u \text{ is compact in } I, \\ &\qquad \exists f \in [f], Ju' + qu = wf\}. \end{aligned}$$

## A. The Vol'pert Chain Rule For BV functions

The goal of this section is to present a proof of the chain rule by Aizik I. Vol'pert (Theorem A.3) for the distributional derivative $(f \circ u)'$ where $u$ is a BV-function and $f$ is a $C^1$ function. In order to do this, we first need the following result from Helly, which is taken from the textbook by Bennewitz, Brown, and Weikard [2].

THEOREM A.1 (Helly). *(1) If $j \mapsto u_j$ is a uniformly bounded sequence of nondecreasing functions on an interval $I$, then there is a subsequence $k \mapsto u_{j_k}$ converging pointwise to some nondecreasing function $u$ on $I$.*

*(2) Let $I$ be a compact interval. If $j \mapsto u_j$ is a uniformly bounded sequence of nondecreasing functions on $I$ that is pointwise converging to a function $u$ then*

$$\lim_{j\to\infty}\int_I \phi du_j = \int_I \phi du,$$

*for any $\phi \in C(I)$.*

*(3) Let $I$ be an open interval. If $j \mapsto u_j$ is a sequence in $\mathrm{BV}(I)$, and if there are $c \in I$ and $M > 0$ so that the sequence $j \mapsto u_j(c)$ is bounded and $\operatorname{Var}_I u_j \le M$ for all $j$, then there is a subsequence $j \mapsto u_{j_k}$ of $j \mapsto u_j$ converging pointwise to some $u \in \mathrm{BV}_{\mathrm{loc}}(I)$ and that $\operatorname{Var}_I u \le M$.*

*(4) Let $I$ be a compact interval. If the sequence $j \mapsto u_j$ in $\mathrm{BV}(I)$ converges pointwise to a function $f$, and $\operatorname{Var}_I u_j \le M$ for some $M > 0$, then $u \in \mathrm{BV}(I)$ and*

$$\lim_{j\to\infty}\int_I \phi df_j = \int_I \phi df,$$

*for any $\phi \in C(I)$, and in particular for all test function $\phi \in \mathcal{D}(I)$.*

Proof. The proofs of (1) and (2) can be found in Bennewitz, Brown, and Weikard [2] (Lemma E.1.4). To prove (3), observe that the sequence $j \mapsto u_j$ is uniformly bounded since

$$|u_j(x) - u_1(x)| \le |u_j(x) - u_j(c)| + |u_j(c) - u_1(c)| + |u_1(c) + u_1(x)|$$
$$\le \operatorname{Var}_I u_j + \operatorname{Var}_I u_1 + |u_j(c) - u_1(c)| \le 2M + |u_j(c) - u_1(c)|.$$

To prove (4), observe that we can write $u = \sum_{k=0}^{3} i^k u_k$, where $u_k$ are nondecreasing functions. □

Lemma A.2. *If $u$ is a function of bounded variation on a finite interval $(c, d)$ then there is a sequence of step functions $n \mapsto u_n$ converging uniformly to $u$. Furthermore, $\operatorname{Var}_{(c,d)} u_n \le \operatorname{Var}_{(c,d)} u$.*

PROOF. Consider a decomposition $u = u_c + u_d$ where $u_c$, $u_d$ are the continuous part and discrete part of $u$.

Let $k \mapsto P_k = \{c = x_{0,k}, x_{1,k}, \ldots, x_{m,k}, x_{m(k)+1,k} = d\}$ be a sequence of partitions of $(c, d)$ such that $P_{k'}$ is finer than $P_k$ if $k' > k$, and $\max_j |x_{j+1,k} - x_{j,k}| < 1/k$. Let $I_{j,k} = [x_{j,k}, x_{j+1,k})$, and for any $x \in (c, d)$ define

$$u_{c,k}(x) = \sum_{j=0}^{m(k)} u_c(x_{j,k})\chi_{I_{j,k}}(x).$$

Let $\varepsilon > 0$. Since $u_c$ is uniformly continuous, there is $\delta$ such that $|u_c(x) - u_c(y)| \leq \varepsilon$ if $|x - y| \leq \delta$. Suppose $x \in [x_{l,k'}, x_{l+1,k'}) \subset [x_{j,k}, x_{j+1,k})$, fix $k_0$ such that $k_0\delta \geq 1$, then

$$|u_{c,k}(x) - u_{c,k'}(x)| = |u_c(x_{j,k}) - u_c(x_{l,k'})| \leq \varepsilon,$$

for all $k' > k \geq k_0$. Thus the sequence of step functions $k \mapsto u_{c,k}$ converges uniformly to $u_c$. It is also easy to see that $\operatorname{Var} u_{c,k} \leq \operatorname{Var} u_c$.

Suppose $(u_d)' = \sum_{j=0}^{\infty} \alpha_j \delta_{x_j}$ where $x_j$'s are jump points of $u_d$ and the series $\sum_{j=0}^{\infty} \alpha_j$ converges absolutely. Thus $u_d$ can be written as a series of Heaviside functions $u_d = \sum_j \alpha_j \chi_{I_j}$, where each $I_j$ is one amongst $(c, x_j]$ or $(c, x_j)$. For any $x \in (c, d)$, define

$$u_{d,k}(x) = \sum_{j=1}^{k} \alpha_j \chi_{I_j}.$$

Since $\sum_{j=0}^{\infty} |\alpha_j| < \infty$, the sequence of step functions $k \mapsto u_{d,k}$ also converges uniformly to $u_d$. Also, $\operatorname{Var} u_{d,k} \leq \operatorname{Var} u_d$.

The sequence of step functions $k \mapsto u_k = u_{c,k} + u_{d,k}$ satisfies all stated conditions since $\operatorname{Var} u_k \leq \operatorname{Var} u_{c,k} + \operatorname{Var} u_{d,k} \leq \operatorname{Var} u_c + \operatorname{Var} u_d = \operatorname{Var} u$. □

THEOREM A.3 (A. I. Vol'pert, [7]). *If $u \in \mathrm{BV}_{\mathrm{loc}}((a, b))$ and $f \in C^1(J)$ for some interval $J$ contains the range of $u$, then $f \circ u \in \mathrm{BV}_{\mathrm{loc}}((a, b))$ and*

(A.1) $$(f \circ u)' = \left[\int_0^1 f'(su^+(\cdot) + (1 - s)u^-(\cdot))ds\right] u'.$$

*Here the equation (A.1) is understood in the sense of distributions, $(f \circ u)'$, $u'$ are the distributional derivatives while $f'$ is the regular pointwise derivative.*

Proof. Let $\phi$ be a test function on $(a,b)$ with $\operatorname{supp}\phi = [c,d]$, we need to prove that

$$\int \phi (f \circ u)' = \int \phi \left[ \int_0^1 f'(su^+(\cdot) + (1-s)u^-(\cdot)) ds \right] u'. \tag{A.2}$$

<u>Case I:</u> $f'$ is Lipschitz.

Suppose $u$ is a step function on $[c,d]$, then there is a partition $c = x_0 < x_1 < \ldots < x_m < x_{m+1} = d$ so that $u$ is constant on each open interval $I_j = (x_j, x_{j+1})$, $j = 0, \ldots, m$. Then $f \circ u$ is also constant on each $I_j$, hence $u' = (f \circ u)' = 0$ there. Noting that $\phi(x_0) = \phi(x_{m+1}) = 0$, we have

$$\begin{aligned}\text{LHS of (A.2)} &= \sum_{j=0}^{m} \int_{I_j} \phi (f \circ u)' + \sum_{j=0}^{m+1} \int_{\{x_j\}} \phi (f \circ u)' \\ &= \sum_{j=1}^{m} \phi (f \circ u)'(\{x_j\}) = \sum_{j=1}^{m} \phi(x_j) \left( (f \circ u)^+(x_j) - (f \circ u)^-(x_j) \right),\end{aligned}$$

$$\begin{aligned}\text{RHS of (A.2)} &= \sum_{j=0}^{m} \int_{I_j} \phi \left[ \int_0^1 f'(su^+(\cdot) + (1-s)u^-(\cdot)) ds \right] u' \\ &\quad + \sum_{j=0}^{m+1} \int_{\{x_j\}} \phi \left[ \int_0^1 f'(su^+(\cdot) + (1-s)u^-(\cdot)) ds \right] u' \\ &= \sum_{j=1}^{m} \phi(x_j) \left[ \int_0^1 f'(su^+(x_j) + (1-s)u^-(x_j)) ds \right] (u^+(x_j) - u^-(x_j)).\end{aligned}$$

Using the substitution $w = su^+(x_j) + (1-s)u^-(x_j)$ and $dw = (u^+(x_j) - u^-(x_j))ds$ we have

$$\text{RHS of (A.2)} = \sum_{j=1}^{m} \phi(x_j) \left( \int_{u^-(x_j)}^{u^+(x_j)} f'(w)dw \right)$$
$$= \sum_{j=1}^{m} \phi(x_j) \left( f(u^+(x_j) - f(u^-(x_j))dw \right)$$
$$= \sum_{j=1}^{m} \phi(x_j) \left( (f \circ u)^+(x_j) - ((f \circ u))^-(x_j))dw \right),$$

where the last equality is thanks to the continuity of $f$.

Suppose $u \in \mathrm{BV}_{\mathrm{loc}}((a,b))$, then $u \in \mathrm{BV}(c,d)$. By Lemma A.2, there is a sequence of step functions $j \mapsto u_j$ converging uniformly to $u$ and $\operatorname{Var}_{[c,d]} u_j \leq \operatorname{Var}_{[c,d]} u$, for all $j$.

Since each $u_j$ is a step function, we have

$$\int \phi (f \circ u_j)' = \int \phi \left[ \int_0^1 f'(su_j^+(\cdot) + (1-s)u_j^-(\cdot))ds \right] u_j' \tag{A.3}$$

Since $f \in C^1(J)$ and $J$ is bounded, $f$ is Lipschitz. Hence $f \circ u_j$ belongs to $\mathrm{BV}(c,d)$ and converges pointwise to $f \circ u$. Moreover, $\operatorname{Var}_I f \circ u_j \leq M_1 \operatorname{Var}_I u_j \leq M_1 \operatorname{Var}_I u$, where $M_1 > 0$ is the Lipschitz constant of $f$. Applying Theorem A.1 we have

$$\lim_{j\to\infty} \int \phi (f \circ u_j)' = \int \phi (f \circ u)'. \tag{A.4}$$

Define on $(a,b)$

$$v_j(\cdot) = \phi(\cdot) \int_0^1 f'(su_j^+(\cdot) + (1-s)u_j^-(\cdot))ds$$

$$v(\cdot) = \phi(\cdot) \int_0^1 f'(su^+(\cdot) + (1-s)u^-(\cdot))ds.$$

Then $v \in \mathrm{BV}((a,b))$ since it vanishes outside $[c,d]$ and for any partition $c = x_0 < x_1 < \ldots < x_{m+1} = d$ we have

$$\sum_{k=0}^{m} |v(x_{k+1}) - v(x_k)|$$
$$\leq \sum_{k=0}^{m} |\phi(x_{k+1})| M_2 \int_0^1 s|u^+(x_{k+1}) - u^+(x_k)| + (1-s)|u^-(x_{k+1}) - u^-(x_k)| ds$$
$$+ |\phi(x_{k+1}) - \phi(x_k)| \int_0^1 |f'(su^+(x_k) + (1-s)u^-(x_k))| ds$$
$$\leq M_2 \sup_{[c,d]} |\phi| \operatorname{Var}_{[c,d]} u + 2 \sup_{[c,d]} |\phi| \sup_{[c,d]} |f'|,$$

where $M_2 > 0$ is the Lipschitz constant for $f'$. Similarly, $v_j$'s are also of bounded variation.

Moreover, let $\varepsilon > 0$, there is $j_0 \in \mathbb{N}$ so that $|u_j^+(x) - u^+(x)| \leq \varepsilon$ and $|u_j^-(x) - u^-(x)| \leq \varepsilon$ for any $j \geq j_0$ and any $x \in (c,d)$. Hence $(v_j)$ converges uniformly in $x$ to $v$ since

$$|v_j(x) - v(x)|$$
$$\leq |\phi(x)| \int_0^1 |f'(su_j^+(x) + (1-s)u_j^-(x)) - f'(su^+(x) + (1-s)u^-(x))| ds$$
$$\leq |\phi(x)| \int_0^1 M_2 s|(u_j^+(x) - u^+(x))| + M_2(1-s)|u_j^-(x) - u^-(x)| ds$$
$$\leq \sup_{[c,d]} |\phi| M_2 \varepsilon, \qquad \text{for any } j \geq j_0 \text{ and any } x \in (c,d).$$

Since $v$ vanishes at the endpoints $c$ and $d$, applying the integration by parts formula $\int_{[c,d]} u^{\#}(v^{\#})' + v^{\#}(u^{\#})' = \int_{[c,d]} u_j^{\#}(v^{\#})' + v^{\#}(u_j^{\#})' = 0$. Let $S_v = \{x_k : k = 1, 2, \ldots\}$ be the set of jump discontinuities of $v$. We have

$$\begin{aligned}\left|\int v_j u_j' - \int v u'\right| &= \left|\int_{[c,d]} v_j (u_j^{\#})' - v(u^{\#})' - v(u_j^{\#})' + v(u_j^{\#})' \right.\\ &\qquad \left. -u_j^{\#}(v^{\#})' - v^{\#}(u_j^{\#})' + u^{\#}(v^{\#})' + v^{\#}(u^{\#})'\right|\\ &= \left|\int_{[c,d]} (v_j - v)(u_j^{\#})' + (u^{\#} - u_j^{\#})(v^{\#})' + (v - v^{\#})(u_j^{\#} - u^{\#})'\right|\\ &\leq \int_{[c,d]} |v_j - v||u_j'| + \int_{[c,d]} |u_j^{\#} - u^{\#}||v'| + \left|\sum_{k=1}^{\infty}(v - v^{\#})(x_k)(u_j - u)'(\{x_k\})\right|,\end{aligned}$$

where $|u_j'|$, $|v'|$ are the total variation measure of $u_j'$ and $v'$, respectively.

Let $\varepsilon > 0$. Since $(v_j)$ and $(u_j)$ converge uniformly in $x$ to $v$ and $u$, there is $j_0 \in \mathbb{N}$ such that $|v_j(x) - v(x)| < \varepsilon$ and $|u_j^{\#}(x) - u^{\#}(x)| < \varepsilon$ for any $j \geq j_0$ and any $x \in (c, d)$. Since $v - v^{\#} \in \mathrm{BV}((a, b))$, the series $\sum_{k=1}^{\infty} |(v - v^{\#})(x_k)|$ is convergent and there is $N \in \mathbb{N}$ such that $\sum_{k=N}^{\infty} |(v - v^{\#})(x_k)| < \varepsilon$. Let $C > 0$ be the bound of $|v - v^{\#}|$ over $(a, b)$, we have

$$\begin{aligned}\left|\int v_j u_j' - \int v u'\right| &\leq \varepsilon \operatorname{Var}_{[c,d]} u_j + \varepsilon \operatorname{Var}_{[c,d]} v\\ &\quad + \left|\sum_{k=1}^{N}(v - v^{\#})(x_k)(u_j - u)'(\{x_k\})\right| + \left|\sum_{k=N}^{\infty}(v - v^{\#})(x_k)(u_j - u)'(\{x_k\})\right|\\ &\leq \varepsilon \operatorname{Var}_{[c,d]} u + \varepsilon \operatorname{Var}_{[c,d]} v + 2C\varepsilon + \varepsilon \operatorname{Var}_{[c,d]}(u_j - u)\\ &\leq \varepsilon \operatorname{Var}_{[c,d]} u + \varepsilon \operatorname{Var}_{[c,d]} v + 2C\varepsilon + 2\varepsilon \operatorname{Var}_{[c,d]} u.\end{aligned}$$

Thus $\lim_j \int v_j u_j' = \int v u'$. This finishes the proof in case I.

Case II: $f'$ is not required to be Lipschitz.

Let $\psi(t) \in \mathcal{D}(-1, 1)$, $\psi_j(t) := j\psi(jt)$, and $f_j := f * \psi_j$ for $j = 1, 2, \ldots$. Since $\psi_j \in C_c^2(J)$, $f_j \in C^2$. Thus $f_j$ is $C^1$ and $f_j'$ is Lipschitz, and hence

$$\int \phi(f_j \circ u)' = \int \phi \left[\int_0^1 f_j'(su^+(\cdot) + (1 - s)u^-(\cdot))ds\right] u'. \tag{A.5}$$

All $f_j$'s share the same Lipschitz constant $M_1 > 0$ with $f$ since

$$\begin{aligned}|f_j(t_1) - f_j(t_2)| &= |f_j'(t_3)||t_1 - t_2| = |(f' * \psi_j)(t_3)||t_1 - t_2| \\ &= \left|\int f'(r - t_3)\psi_j(r)dr\right| |t_1 - t_2| \le \sup_J |f'||t_1 - t_2| \le M_1|t_1 - t_2|,\end{aligned}$$

for any $t_1, t_2 \in J$ and some $t_3 \in J$. Thus $\operatorname{Var} f_j \circ u \le M_1 \operatorname{Var} u$ for all $j$. Applying Theorem A.1 we get

$$\lim_j \int \phi(f_j \circ u)' = \int \phi(f \circ u)'. \tag{A.6}$$

The sequence of functions $I \ni x \mapsto \phi(x)\left[\int_0^1 f_j'(su^+(\cdot) + (1-s)u^-(\cdot))ds\right]$ is uniformly bounded since

$$|f_j'(t)| = |(f' * \psi_j)(t)| = \left|\int f'(r - t)\psi_j(r)dr\right| \le \sup_J |f'| \quad \forall t \in J.$$

Applying the dominated convergence theorem twice we have

$$\begin{aligned}\lim_j \int \phi &\left[\int_0^1 f_j'(su^+(\cdot) + (1-s)u^-(\cdot))ds\right] u' \\ &= \int \phi \left[\int_0^1 \lim_j f_j'(su^+(\cdot) + (1-s)u^-(\cdot))ds\right] u' \\ &= \int \phi \left[\int_0^1 f'(su^+(\cdot) + (1-s)u^-(\cdot))ds\right] u'.\end{aligned} \tag{A.7}$$

This finishes the proof of the theorem. □

If $u$ is continuous then the chain rule in (A.1) reduces to a more familiar formula.

Corollary A.4. *Given $u \in \mathrm{BV}_{\mathrm{loc}}((a,b))$ is continuous and $f \in C^1(\mathbb{R})$ then*

$$(f \circ u)' = (f' \circ u)u'. \tag{A.8}$$

The product rule is also an important consequence.

Corollary A.5. *If* $u, v \in \mathrm{BV}_{\mathrm{loc}}((a,b))$ *then* $uv \in \mathrm{BV}_{\mathrm{loc}}((a,b))$ *and*

$$(uv)' = u^{\#}v' + u'v^{\#}.$$

*If, in addition,* $u, v$ *are continuous then*

$$(uv)' = uv' + u'v. \tag{A.9}$$

Proof. Let $f(x) = x^2/4$. We have

$$uv = ((u+v)^2 - (u-v)^2)/4 = f(u+v) - f(u-v).$$

Differentiating both sides of this identity, by using Theorem A.3 we get

$$\begin{aligned}(uv)' = &\left[\int_0^1 f'(s(u+v)^+(\cdot) + (1-s)(u+v)^-(\cdot))ds\right](u+v)' \\ &- \left[\int_0^1 f'(s(u-v)^+(\cdot) + (1-s)(u-v)^-(\cdot))ds\right](u-v)' \\ &= \frac{1}{2}\left[\int_0^1 (s(u+v)^+(\cdot) + (1-s)(u+v)^-(\cdot))ds\right](u+v)' \\ &- \frac{1}{2}\left[\int_0^1 (s(u-v)^+(\cdot) + (1-s)(u-v)^-(\cdot))ds\right](u-v)' = u^{\#}v' + u'v^{\#}.\end{aligned}$$

The other claim of the theorem is trivial. □

## B. The Besicovitch Differentiation Theorem

The goal of this section is to present a proof of the Besicovitch Differentiation Theorem (Theorem B.4) by Abram S. Besicovitch for Radon measures in $\mathbb{R}$. This is a generalization of the Lebesgue Differentiation Theorem (see e.g., [6], Theorem 7.14, p. 143). The statements and proofs throughout this section are adapted from Ambrosio, Fusco, and Pallara [1] (Chapter 2).

Consider an open interval $I = (a,b)$, $-\infty \le a < b \le \infty$, and let $\mathcal{B}(I)$ be its Borel $\sigma$-algebra. If $x \in \mathbb{R}$ and $r \ge 0$ we denote $I_r(x) = (x-r, x+r)$ and $\overline{I}_r(x) = [x-r, x+r]$.

A set function $\mu$ defined on $\mathcal{B}(I)$ is said to be a *Radon measure* on $I$ if for any compact set $K \subset I$, $\mu$ is a measure on the measure space $(K, \mathcal{B}(K))$. If such a set function $\mu$ is indeed a measure on $(I, \mathcal{B}(I))$ then it is said to be a *finite Radon measure*.

For a positive measure $\mu$ on $I$ the *support* of a measure $\mu$ is the complement in $I$ of the largest open set $\tilde{U}$ such that $\mu(\tilde{U}) = 0$. (Remark: since $\mu$ is a positive measure, this condition implies that $\mu$ vanishes on $\tilde{U}$.) Another equivalent definition of the support is that $\operatorname{supp}\mu$ is the set of points $x \in I$ such that for any open neighborhood $U$ of $x$ we have $\mu(U) > 0$. Indeed, let $S_1$, $S_2$ be the support in the first and second definition, respectively. To prove $S_1 \subset S_2$, suppose $x \notin S_2$ and let $U$ be any open neighborhood of $x$, then $\mu(U) = 0$. Since $\tilde{U}$ is the largest (with respect to inclusion) of such sets with this property, $U \subset \tilde{U}$. Since $S_1 = I \setminus \tilde{U}$, $x \notin S_1$. To prove $S_2 \subset S_1$, observe that if $x \in \tilde{U}$ and $U$ is any open neighborhood of $x$ then $\mu(U) \leq \mu(\tilde{U}) = 0$. Thus $\mu(U) = 0$ and $x \notin S_2$.

Let $A$ be a subset of $\mathbb{R}$. A family $\mathcal{F}$ of closed intervals of $\mathbb{R}$ is said to be a *fine cover of* $A$ if for any $x \in A$ and for any $\varepsilon > 0$ there is a closed interval $\overline{I}_r(x)$ in $\mathcal{F}$ with radius $r \leq \varepsilon$.

For any family $\mathcal{F}$ of closed intervals let $C(\mathcal{F})$ be the set of the centers of the closed intervals belonging to $\mathcal{F}$. If $\mathcal{F}$ is a fine cover of $A$ then the *reduced* subfamily $\mathcal{F}_{\text{red}} = \{\overline{I}_r(x) \in \mathcal{F} : x \in A\}$ is also a fine cover of $A$ and we have $C(\mathcal{F}_{\text{red}}) = A$. We observe that $\mathcal{F}_{\text{red}}$ is also a fine cover of $A$. Indeed, to prove $A \subset C(\mathcal{F}_{\text{red}})$, let $x \in A$ then since $\mathcal{F}$ is a fine cover of $A$, there is a closed interval $\overline{I}_r(x) \in \mathcal{F}$. Also $\overline{I}_r(x) \in \mathcal{F}$ since $x \in A$. Thus $x \in C(\mathcal{F}_{\text{red}})$. To prove $C(\mathcal{F}_{\text{red}}) \subset A$, let $x \in C(\mathcal{F}_{\text{red}})$ then it is the center of some closed interval $\overline{I}_r(x) \in \mathcal{F}_{\text{red}}$. Thus $x \in A$ by the definition of $\mathcal{F}_{\text{red}}$.

Unless explicitly specified, most of the families $\mathcal{F}$ appear in this section are families of closed intervals. The notation $\overline{I}$ will be used to denote a generic element of $\mathcal{F}$.

We shall accept the following result by A. S. Besicovitch in which the character number $\xi \in \mathbb{N}_0$ is defined. We note here that $\xi \leq 42$. The proof of this result can be found in Ambrosio, Fusco, and Pallara [1] (Theorem 2.17, p. 49).

THEOREM B.1 (A. S. Besicovitch). *There exists a number* $\xi \in \mathbb{N}_0$ *with the following property: if* $\mathcal{F}$ *is a family of closed intervals in* $\mathbb{R}$ *such that the set* $C(\mathcal{F})$ *containing their centers is bounded, then there are* $\xi$ *disjoint countable subfamilies* $\mathcal{F}_1, \ldots, \mathcal{F}_\xi$ *of* $\mathcal{F}$ *such that*

$$C(\mathcal{F}) \subset \bigcup_{h=1}^{\xi} \bigcup_{\overline{I} \in \mathcal{F}_h} \overline{I}.$$

*Here, the second union is taken over all closed intervals* $\overline{I}$ *belonging to* $\mathcal{F}_h$.

THEOREM B.2 (Vitali-Besicovitch Covering). *Let* $A$ *be a bounded Borel set of* $\mathbb{R}$, *and let* $\mathcal{F}$ *be a fine cover of* $A$. *Then, for every positive Radon measure* $\mu$ *in* $\mathbb{R}$ *there is a disjoint countable subfamily* $\mathcal{F}'$ *of* $\mathcal{F}$ *such that*

$$\mu(A \setminus \bigcup_{\overline{I} \in \mathcal{F}'} \overline{I}) = 0.$$

PROOF. Let $\xi$ be the natural number defined in Theorem B.1.

Let $A_0 = A$. Without loss of generality, assuming that $\mathcal{F}$ is reduced so that it is still a fine cover of $A_0$ and $C(\mathcal{F}) = A_0$. Since $A_0$ is bounded, by Theorem B.1, there exist $\xi$ disjoint countable subfamilies $F_1, \ldots, \mathcal{F}_\xi$ such that $A_0 \subset \bigcup_{h=1}^{\xi} \bigcup_{\overline{I} \in \mathcal{F}_h} \overline{I}$. Thus

$$\mu(A_0) = \mu(A_0 \cap \bigcup_{h=1}^{\xi} \bigcup_{\overline{I} \in \mathcal{F}_h} \overline{I}) \leq \sum_{h=1}^{\xi} \mu(A_0 \cap \bigcup_{\overline{I} \in \mathcal{F}_h} \overline{I}).$$

This shows that there is some $h \in \{1, \ldots, \xi\}$ such that $\mu(A_0 \cap \bigcup_{\overline{I} \in \mathcal{F}_h} \overline{I}) \geq \frac{1}{\xi}\mu(A_0)$. Let $\mathcal{G}_1$ be this subfamily $\mathcal{F}_h \subset \mathcal{F}$.

Let $A_1 = A_0 \setminus \bigcup_{\overline{I} \in \mathcal{G}_1} \overline{I}$. Since $A_0$ is bounded we have $\mu(A_0) < \infty$ and

$$\mu(A_1) = \mu(A_0 \setminus \bigcup_{\overline{I} \in \mathcal{G}_1} \overline{I}) = \mu(A_0) - \mu(A_0 \cap \bigcup_{\overline{I} \in \mathcal{G}_1} \overline{I}) \leq \mu(A_0) - \frac{1}{\xi}\mu(A_0) < \delta\mu(A_0),$$

where $\delta = 1 - 1/2\xi \in (0, 1)$ (note that $\xi \geq 1$).

Consider $\mathcal{G} = \{\overline{I}_r(x) \in \mathcal{F} : \overline{I}_r(x) \cap \bigcup_{\overline{I} \in \mathcal{G}_1} \overline{I} = \emptyset\}$. $\mathcal{G}$ is again a fine cover of $A_1$ which is still bounded, so there exists a disjoint countable subfamily $\mathcal{G}_2 \subset \mathcal{G}$ such that $\mu(A_1 \cap \bigcup_{\overline{I} \in \mathcal{G}_2} \overline{I}) \geq \frac{1}{\xi}\mu(A_1)$. From the definition of $\mathcal{G}$, $\mathcal{G}_2$ is also a subfamily of $\mathcal{F}$ and none of the closed intervals in $\mathcal{G}_2$ intersects those belonging to $\mathcal{G}_1$. Thus $\mathcal{G}_1 \cup \mathcal{G}_2$ is still a disjoint countable subfamily of $\mathcal{F}$.

Repeating the procedure above we obtain a sequence of sets $n \mapsto A_n = A \setminus \bigcup_{n=1}^{k} \bigcup_{\overline{I} \in \mathcal{G}_n} \overline{I}$ such that $\mu(A_{n+1}) < \delta\mu(A_n)$, hence $\lim_n \mu(A_n) = 0$, and a sequence of subfamilies $n \mapsto \mathcal{G}_n$ such that $\mathcal{F}' := \bigcup_n \mathcal{G}_n$ is a disjoint countable subfamilies of $\mathcal{F}$.

Since $\mu(A) < \infty$ the proof is finished due to the continuity from above property of positive measures

$$\mu(A \setminus \bigcup_{\overline{I} \in \mathcal{F}'} \overline{I}) = \mu(A \setminus \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{k} \bigcup_{\overline{I} \in \mathcal{G}_n} \overline{I})$$

$$= \mu(\bigcap_{k=1}^{\infty} (A \setminus \bigcup_{n=1}^{k} \bigcup_{\overline{I} \in \mathcal{G}_n} \overline{I})) = \mu(\bigcap_{k=1}^{\infty} A_k) = \lim_n \mu(A_k) = 0.$$

This finishes the proof of the theorem. □

Let $\mu$, $\nu$ be two positive Radon measures and $x \in \operatorname{supp} \mu$, we define the *upper* and *lower derivatives of* $\nu$ *with respect to* $\mu$ by

$$D_\mu^+ \nu(x) = \limsup_{r \searrow 0} \frac{\nu(I_r(x))}{\mu(I_r(x))}$$

$$D_\mu^- \nu(x) = \liminf_{r \searrow 0} \frac{\nu(I_r(x))}{\mu(I_r(x))}.$$

If $D_\mu^+ \nu(x) = D_\mu^- \nu(x)$ then we define its common value $D_\mu \nu(x)$ to be the *symmetric derivative of* $\nu$ *with respect to* $\mu$.

It is easy to see that $D_\mu^\pm \nu$ are Borel measureable functions on $\operatorname{supp} \mu$. Moreover, $D_\mu^\pm \nu$ do not change if the open interval $I_r(x)$ is replaced by the closed interval $\overline{I}_r(x)$.

THEOREM B.3. *Let* $\mu$ *and* $\nu$ *be positive Radon measures in an open set* $I \subset \mathbb{R}$, $t \in [0, \infty)$, *and a Borel set* $E \subset \operatorname{supp} \mu$. *If* $D_\mu^- \nu(x) \leq t$ *for all* $x \in E$ *then* $\nu(E) \leq t\mu(E)$. *If* $D_\mu^- \nu(x) \geq t$ *for all* $x \in E$ *then* $\nu(E) \geq t\mu(E)$. *If, in addition,* $\nu$ *is finite then* $\mu(\{x \in \operatorname{supp} \mu : D_\mu^+(x) = \infty\}) = 0$.

PROOF.

Case I: If $E$ is bounded.

Suppose $D^-_\mu \nu(x) \le t$ for all $x \in E$. Let $A$ be an open and bounded set containing $E$. Let $\varepsilon > 0$ and consider the family

$$\mathcal{F} = \{\overline{I}_r(x) \ : \ x \in E, \ \overline{I}_r(x) \subset A, \ \nu(\overline{I}_r(x)) < (t+\varepsilon)\mu(\overline{I}_r(x))\}.$$

For any $x \in E$, since $t + \varepsilon > t \ge \liminf_{r \searrow 0} \frac{\nu(\overline{I}_r(x))}{\mu(\overline{I}_r(x))}$ there are closed intervals $I_r(x)$ with arbitrarily small radius $r$ such that $\nu(\overline{I}_r(x)) < (t+\varepsilon)\mu(\overline{I}_r(x))$. Since $A$ is an open set, we can also fit these small intervals into $A$. So, the family $\mathcal{F}$ is a fine cover of $E$. Since $A$ is also bounded, by Theorem B.2, there exists a disjoint countable subfamily $\mathcal{F}'$ of $\mathcal{F}$ such that $\nu(E \setminus \bigcup_{\overline{I} \in \mathcal{F}'} \overline{I}) = 0$. Since $\nu(E) < \infty$ we get

$$\nu(E) = \sum_{\overline{I} \in \mathcal{F}'} \nu(E \cap \overline{I}) \le \sum_{\overline{I} \in \mathcal{F}'} \nu(\overline{I}).$$

Since $\mathcal{F}'$ is countable and any two different closed intervals $\overline{I}$ of $\mathcal{F}'$ are pairwise disjoint and contained in $A$ we have

$$\nu(E) \le \sum_{\overline{I} \in \mathcal{F}'} (t+\varepsilon)\mu(\overline{I}) = (t+\varepsilon)\nu(\bigcup_{\overline{I} \in \mathcal{F}'} \overline{I}) \le (t+\varepsilon)\mu(A).$$

Since this inequality is valid for any $\varepsilon > 0$ we have $\nu(E) \le t\mu(A)$.

$\mu$ is outer regular so $\mu(E) = \inf\{\mu(A') \ : \ A' \text{ is open and } E \subset A'\}$. Since $E$ is assumed to be bounded, say by an open interval $(-R, R)$ for some $R > 0$, for any open set $A'$ containing $E$ the set $(-R, R) \cap A'$ is open and bounded. Thus

$$\nu(E) \le t\mu((-R,R) \cap A') \le t\mu(A').$$

Thus $\nu(E) \le t\mu(E)$.

Suppose $D^+_\mu \nu(x) \ge t$ for all $x \in E$, we can prove in a similar fashion that $\nu(E) \ge t\mu(E)$.

Case II: If $E$ is not bounded then we consider the bounded Borel set $E_n = [-n, n] \cap E$. The statement now comes from the continuity from below property of positive measures. $\square$

Theorem B.4 (Besicovitch Differentiation Theorem). *Let $\mu$ be a positive Radon measure in an open set $I \subset \mathbb{R}$, and $\nu$ a Radon measure in $I$. Then the limit*

$$f(x) = \lim_{r \searrow 0} \frac{\nu(I_r(x))}{\mu(I_r(x))}$$

*exists for $\mu$-a.e. $x$ in the support of $\mu$ and the Lebesgue decomposition of $\nu$ with respect to $\mu$ is given by $\nu = f\mu + \nu_s$ where $\nu_s = \nu|_E$ and $E$ is the $\mu$-negligible set.*

$$E = (I \setminus \operatorname{supp} \mu) \cup \left\{ x \in \operatorname{supp} \mu : \lim_{r \searrow 0} \frac{|\nu|(I_r(x))}{\mu(I_r(x))} = \infty \right\}.$$

Proof.

Case I: If $\nu$ is positive and both $\mu$, $\nu$ are finite.

Let $\tilde{U} = I \setminus \operatorname{supp} \mu$, then $\mu(\tilde{U}) = 0$. Since $\nu$ is finite, by Theorem B.3 we have

$$\begin{aligned} \mu(E) &\le \mu(\tilde{U}) + \mu\left( \left\{ x : \lim_{r \searrow 0} \frac{\nu(I_r(x))}{\mu(I_r(x))} = \infty \right\} \right) \\ &\le \mu\left( \left\{ x : \limsup_{r \searrow 0} \frac{\nu(I_r(x))}{\mu(I_r(x))} = \infty \right\} \right) = \mu(\{x : D_\mu^+ \nu(x) = \infty\}) = 0. \end{aligned}$$

Hence, $E$ is a null set of $\mu$.

Let $F = I \setminus E$ and define the positive measures $\lambda^\pm$ on $\mathcal{B}(I)$ by

$$\lambda^\pm(B) = \int_B (D_\mu^\pm \nu)\, \mu, \ \forall B \in \mathcal{B}(I).$$

Note that $\lambda^\pm$ are well-defined since $\mu$ is a positive measure and $D_\mu^\pm \nu$ are Borel measureable functions.

If we can show that $\lambda^+ \le \nu|_F \le \lambda^-$ then $D_\mu^+ \nu \le D_\mu^- \nu$ for $\mu$-a.e. $x \in \operatorname{supp} \mu$, thence $f = D_\mu \nu = D_\mu^\pm \nu$ exists for $\mu$-a.e. $x \in \operatorname{supp} \mu$. The fact that $\nu|_F(B) = \int_B f\, \mu$

for all $B \in \mathcal{B}(I)$ shows that $f$ is the Radon-Nikodym derivative of $\nu$ with respect to $\mu$. The proof is finished since then we have $\nu = \nu|_F + \nu|_E = f\mu + \nu_s$.

To prove that $\lambda^+ \leq \nu|_F$, fix $B \in \mathcal{B}(I)$ and $t > 1$. Let $B^+_{-\infty} = B \cap \{x : D^+_\mu \nu(x) = 0\}$, $B^+_\infty = B \cap \{x : D^+_\mu \nu(x) = \infty\}$, and $B^+_n = B \cap \{x \in \operatorname{supp} \mu : t^n < D^+_\mu \nu(x) \leq t^{n+1}\}$ for any $n \in \mathbb{Z}$. Then by Theorem B.3, we have $\mu(B_\infty) = 0$ and $\nu(B^+_n) \geq t^n \mu(B^+_n)$.

Also, $B^+_n \subset F$ for any $n \in \mathbb{Z}$. Since if $x \in B^+_n$ then $\limsup_{r \searrow 0} \frac{\nu(I_r(x))}{\mu(I_r(x))} \leq t^{n+1} < \infty$. So, either $\lim_{r \searrow 0} \frac{\nu(I_r(x))}{\mu(I_r(x))}$ does not exist or it is a finite number. This means $x \in F = I \setminus E$.

Since $\lambda^+(B^+_{-\infty}) = 0$, $\mu(B^+_\infty) = 0$, and the sets $B_n$'s are pairwise disjoint we get

$$\lambda^+(B) = \sum_{n \in \mathbb{Z}} \int_{B^+_n} (D^+_\mu \nu)\, \mu \leq \sum_{n \in \mathbb{Z}} t^{n+1} \mu(B^+_n) \leq t \sum_{n \in \mathbb{Z}} \nu(B^+_n) \leq t\nu(B \cap F).$$

Since this inequality is valid for any $t > 1$ we have $\lambda^+(B) \leq \nu(B \cap F) = \nu|_F(B)$.

To prove that $\nu|_F \leq \lambda^-$, fix $B \in \mathcal{B}(I)$ and $t > 1$. Let $B^-_{-\infty} = B \cap \{x : D^-_\mu \nu(x) = 0\}$, $B^-_\infty = B \cap \{x : D^-_\mu \nu(x) = \infty\}$, and $B^-_n = B \cap \{x : t^n < D^-_\mu \nu(x) \leq t^{n+1}\}$ for any $n \in \mathbb{Z}$.

Since $D^-_\mu \nu(x) \leq 0$ for any $x \in B^-_{-\infty}$, applying Theorem B.3 for $t = 0$, we have $\nu(B^-_{-\infty}) \leq 0.\mu(B^-_{-\infty}) = 0$. Thus $\nu(B^-_{-\infty}) = 0$. We also have $\nu(B^-_n) \leq t^{n+1} \mu(B^-_n)$ for any $n \in \mathbb{Z}$.

Observe that $\liminf_{r \searrow 0} \frac{\nu(I_r(x))}{\mu(I_r(x))} = \infty$ if and only if $\lim_{r \searrow 0} \frac{\nu(I_r(x))}{\mu(I_r(x))} = \infty$, thus $B \cap F = B^-_{-\infty} \cup (\bigcup_{n \in \mathbb{Z}} B^-_n)$. Since $t^n < D^-_\mu \nu(x)$ for all $x \in B^-_n$ we have

$$\nu|_F(B) = \nu(B^-_{-\infty}) + \sum_{n \in \mathbb{Z}} \nu(B^-_n) \leq \sum_{n \in \mathbb{Z}} t^{n+1} \mu(B^-_n) < t \sum_{n \in \mathbb{Z}} \int_{B^-_n} (D^-_\mu \nu)\, \mu = t\lambda^-(B).$$

This inequality is valid for all $t > 1$, thus $\nu|_F(B) \leq \lambda^-(B)$.

<u>Case II:</u> If $\mu$ is a positive Radon measure and $\nu$ is a Radon measure.

Let $I_n = [-n, n] \cap I$ then $\mu$ and the positive/negative part $\nu^\pm$ of $\nu$ are positive measures on $I_n$. The proof is finished by applying the case I and let $I_n \to I$. $\square$

Theorem B.5 (Lebesgue points). *Let $\mu$ be a positive Radon measure in an open set $I \subset \mathbb{R}$ and $f \in L^1(I,\mu)$. Then*

$$\lim_{r \searrow 0} \frac{1}{\mu(I_r(x))} \int_{I_r(x)} |f(y) - f(x)| \, d\mu = 0 \tag{B.1}$$

*for $\mu$-a.e. $x \in I$.*

A point $x \in I$ is said to be a *Lebesgue point of $f$* if the equation (B.1) holds.

Proof. Let $q \in \mathbb{Q}$, applying Theorem B.4 for the Radon measures $|f - q|\mu$ and $\mu$ we have

$$|f(x) - q| = \lim_{r \searrow 0} \frac{1}{\mu(I_r(x))} \int_{I_r(x)} |f(y) - q| \, \mu(y)$$

for all $x \in F_q$ where $\mu(I \setminus F_q) = 0$. Hence

$$\begin{aligned} &\limsup_{r \searrow 0} \frac{1}{\mu(I_r(x))} \int_{I_r(x)} |f(y) - f(x)| \, \mu(y) \\ &\leq \limsup_{r \searrow 0} \frac{1}{\mu(I_r(x))} \int_{I_r(x)} |f(y) - q| \, \mu(y) + \limsup_{r \searrow 0} \frac{1}{\mu(I_r(x))} \int_{I_r(x)} |f(x) - q| \, \mu(y) \\ &\qquad = \limsup_{r \searrow 0} \frac{1}{\mu(I_r(x))} \int_{I_r(x)} |f(y) - q| \, \mu(y) + |f(x) - q| = 2|f(x) - q| \end{aligned}$$

for all $x \in F_q$.

Since $\mathbb{Q}$ is countable $\bigcap_{q \in \mathbb{Q}} F_q$ is also a set of full measure with respect to $\mu$. For a fixed $x \in \bigcap_{q \in \mathbb{Q}} F_q$, the last inequality is valid for every $q \in \mathbb{Q}$ and in particular for those $q$ very close to $f(x)$. Thus, $\limsup_{r \searrow 0} \frac{1}{\mu(I_r(x))} \int_{I_r(x)} |f(y) - f(x)| \, \mu(y) = 0$ for all $x \in \bigcap_{q \in \mathbb{Q}} F_q$. This finishes the proof of the theorem. $\square$

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# ON THE SPECTRAL THEORY FOR FIRST-ORDER SYSTEMS WITHOUT THE UNIQUE CONTINUATION PROPERTY

by

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# ON THE SPECTRAL THEORY FOR FIRST-ORDER SYSTEMS WITHOUT THE UNIQUE CONTINUATION PROPERTY

ABSTRACT. We consider the differential equation $Ju' + qu = wf$ on the real interval $(a, b)$ when $J$ is a constant, invertible skew-Hermitian matrix and $q$ and $w$ are matrices whose entries are distributions of order zero with $q$ Hermitian and $w$ nonnegative. In this situation it may happen that there is no existence and uniqueness theorem for balanced solutions of a given initial value problem. We describe the set of solutions the equation does have and establish that the adjoint of the minimal operator is still the maximal operator, even though unique continuation of balanced solutions fails.

## 1. Introduction

Ghatasheh and Weikard [4] investigated the spectral theory for the first-order system

$$Ju' + qu = wf$$

of differential equations on the real interval $(a, b)$ assuming that $J$ is a constant, invertible skew-Hermitian matrix and $q$ and $w$ are matrices whose entries are distributions of order zero[1] with $q$ Hermitian and $w$ nonnegative. Crucially, [4] requires that initial value problems for this equation have unique balanced[2] solutions. Indeed, unique continuation of a solution across a point where $Q$, the anti-derivative of $q$, has a discontinuity may fail. With the aid of linear algebra we were able to overcome this obstacle, describe the set of solutions of the differential equation, and establish

[1]Recall that distributions of order 0 are distributional derivatives of functions of locally bounded variation and hence may be thought of, on compact subintervals of $(a, b)$, as measures. For simplicity we might use the word measure instead of distribution of order 0 below.

[2]The concept of balanced solutions is defined below and the rationale of using it is explained in [4].

that the adjoint of the minimal operator is still the maximal operator, even if unique continuation of balanced solutions fails.

The relationship between minimal and maximal operators is a cornerstone for the spectral theory for differential equations. Of course, this topic is very well studied when the coefficients are locally integrable functions but in the case of measure coefficients much less is known. The first to consider an equation with a measure coefficient was Krein [5] in 1952 when he modeled a vibrating string. Also motivated by physical applications were Gesztesy and Holden [3] in 1987 who described Schrödinger equations with point interactions, specifically $\delta'$-interactions. In 1999 Savchuk and Shkalikov [6] treated Schrödinger equations with potentials in the Sobolev space $W^{-1,2}_{\mathrm{loc}}$, a paper which spurred many further developments. With the help of quasi-derivatives Eckhardt et al. [1] showed in 2013 that such equations can be cast as first-order $2 \times 2$-systems with locally integrable coefficients. Eckhardt and Teschl [2] considered a system where the coefficients are measures, viz. $Ju' + qu = wf$ where $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $q = \left(\begin{smallmatrix} \chi & 0 \\ 0 & -\varsigma \end{smallmatrix}\right)$, and $w = \left(\begin{smallmatrix} \rho & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Their approach covers both the Krein string ($\chi = 0$ and $\varsigma = 1$) as well as the $\delta'$-interaction ($\chi = 0$, $\varsigma = 1 + \beta\delta_0$, and $\rho = 1$ in the simplest case). Crucially, they require that the support of the discrete part of $\varsigma$ does not intersect the corresponding sets for $\chi$ or $\rho$, a condition which guarantees unique continuation. Both [1] and [2] are also excellent sources for a more thorough history of the subject.

Let us add a few words about notation. $\mathcal{D}'^0((a,b))$ is the space of distributions of order 0, i.e., the space of distributional derivatives of functions of locally bounded variation. Any function $u$ of locally bounded variation has left- and right-hand limits denoted by $u^-$ and $u^+$, respectively. Also, $u$ is called balanced if $u = u^\# = (u^+ + u^-)/2$. We use $\mathbb{1}$ to denote an identity matrix of appropriate size and superscripts $^\top$ and $^*$ indicate transposition and adjoint, respectively. The orthogonal complement of a subspace $S$ of a Hilbert space $H$ is denoted by $H \ominus S$ or by $S^\perp$. For $c_1, ..., c_N \in \mathbb{C}^n$ we abbreviate the vector $(c_1^\top, ..., c_N^\top)^\top \in \mathbb{C}^{nN}$ by $(c_1, ..., c_N)^\diamond$. Finally we note that, generally, our differential equations are represented by linear relations rather than linear operators. Consequently we work with graphs of such relations (even when they are operators).

## 2. Obtaining solutions

We begin by describing the set of solutions of the first-order system

$$Ju' + qu = wf$$

on the interval $(a, b)$ assuming that the coefficients satisfy the following hypothesis.

HYPOTHESIS 2.1. *$J$ is a constant, invertible and skew-Hermitian $n \times n$-matrix. Both $q$ and $w$ are in $\mathcal{D}'^0((a,b))^{n\times n}$, $w$ is nonnegative, and $q$ Hermitian.*

Associated with a nonnegative distribution $w \in \mathcal{D}'^0((a,b))^{n\times n}$ is a Hilbert space $L^2(w)$ with inner product $\langle u, v\rangle = \int u^* wv$ (recall that positive distributions are positive measures). Its elements are equivalence classes of functions $[f]$ satisfying $\|f\|^2 = \int f^* wf < \infty$ and, as usual, two functions $f$ and $g$ are equivalent, if $\|f - g\| = 0$.

We denote the left-continuous anti-derivatives of $q$ and $w$ by $Q$ and $W$, respectively. $\Delta_q(x) = Q^+(x) - Q^-(x)$ stands for the jump of $Q$ at a point $x$. Similarly, $\Delta_w(x) = W^+(x) - W^-(x)$.

Suppose $\xi_0 \in (\xi_1, \xi_2) \subset (a, b)$. When $f \in L^2(w)$ (as we shall henceforth assume) it was shown in [4] that the initial value problem $Ju' + qu = wf$, $u(\xi_0) = u_0 \in \mathbb{C}^n$ has a unique balanced solution of locally bounded variation in $(\xi_1, \xi_2)$ provided that the matrices $B_\pm(x) = J \pm \Delta_q(x)/2$ are invertible for all $x \in (\xi_1, \xi_2)$. At a point $x$ of discontinuity of $Q$ or $W$, the differential equation requires that

$$J(u^+(x) - u^-(x)) + \Delta_q(x)u(x) = \Delta_w(x)f(x)$$

where, $u$ being balanced, $u(x) = (u^+(x) + u^-(x))/2$. This is equivalent to

$$B_+(x)u^+(x) - B_-(x)u^-(x) = \Delta_w(x)f(x). \tag{2.1}$$

From this it is obvious that we may not be able to continue a solution across $x$ from left to right (or from right to left), if $B_+(x)$ (or $B_-(x)$) fails to be invertible. In our particular situation, where $J^* = -J$ and $\Delta_q(x)^* = \Delta_q(x)$, we have $B_-(x) = -B_+(x)^*$ and hence that $B_-(x)$ is invertible if and only if $B_+(x)$ is.

On account of the fact that $Q$ is locally of bounded variation, it is clear that the set of points $x$ where $B_\pm(x)$ are not invertible is discrete and finite on compact subintervals of $(a, b)$ even though the set of all jumps of $Q$ may be dense.

Let us now fix an interval $[\xi_1, \xi_2] \subset (a, b)$ assuming that the points in $(\xi_1, \xi_2)$ where $B_\pm$ are not invertible are among the points $x_1 < ... < x_N$. Normally one would choose these to be precisely the points where $B_\pm$ are not invertible but it is advantageous to avoid the case $N = 1$. For convenience let us also set $x_0 = \xi_1$ and $x_{N+1} = \xi_2$. As mentioned above we do have unique solutions of initial value problems and, indeed, a variation of constants formula in any of the intervals $(x_j, x_{j+1})$. These solutions have limits at the endpoints of the interval and we may even use, for instance, the left endpoint to pose an initial condition. Therefore the general solution of $Ju' + qu = wf$ in $(x_j, x_{j+1})$ is represented by

$$u^-(x) = U_j^-(x)(c_j + J^{-1} \int_{(x_j,x)} U_j^* wf) \tag{2.2}$$

where $c_j$ is an arbitrary element of $\mathbb{C}^n$ and $U_j$ a balanced fundamental matrix for $Ju' + qu = 0$ in $(x_j, x_{j+1})$ which we may choose so that $\lim_{x \downarrow x_j} U_j(x) = \mathbb{1}$. We then define $U_j(x_{j+1}) = \lim_{x \uparrow x_{j+1}} U_j^-(x)$. For $u$ to be a solution of $Ju' + qu = wf$ on $(x_0, x_{N+1})$ we need $u$ to be determined by (2.2) (for appropriate choices of the $c_j$) in the respective intervals. Moreover, according to equation (2.1), $u$ must satisfy

$$B_+(x_j)u^+(x_j) - B_-(x_j)u^-(x_j) = \Delta_w(x_j)f(x_j) \quad \text{for } j = 1, ..., N. \tag{2.3}$$

Note that $u^+(x_j) = c_j$ and $u^-(x_j) = U_{j-1}(x_j)(c_{j-1} + J^{-1}I_{j-1}(f))$ where $I_{j-1}(f) = \int_{(x_{j-1},x_j)} U_{j-1}^* wf$. Thus we may rewrite equation (2.3) as

$$(-B_-(x_j)U_{j-1}(x_j), B_+(x_j))\binom{c_{j-1}}{c_j} = \Delta_w(x_j)f(x_j) + B_-(x_j)U_{j-1}(x_j)J^{-1}I_{j-1}(f).$$

At this point it appears helpful to introduce the following notation. Let

$$\begin{aligned}
\mathcal{B} &= \operatorname{diag}(B_+(x_1), ..., B_+(x_N)), \\
\mathcal{U} &= \operatorname{diag}(U_0(x_1), ..., U_{N-1}(x_N)), \\
\mathcal{J} &= \operatorname{diag}(J, ..., J),
\end{aligned}$$

and $E_\top = (0, \mathbb{1})$ and $E_\perp = (\mathbb{1}, 0)$, two $nN \times n(N+1)$-matrices which, respectively, strip the first and the last $n$ coordinates off a vector. Then we have

$$\begin{aligned}
B &= \mathcal{B}^*\mathcal{U}E_\perp + \mathcal{B}E_\top \\
&= \begin{pmatrix} -B_-(x_1)U_0(x_1) & B_+(x_1) & 0 & \cdots & 0 \\ 0 & -B_-(x_2)U_1(x_2) & B_+(x_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & -B_-(x_N)U_{N-1}(x_N) & B_+(x_N) \end{pmatrix}.
\end{aligned}$$

If we now introduce the abbreviations

$$\begin{aligned}
\tilde{u} &= (c_0, ..., c_N)^\diamond, \\
\mathcal{R}(f) &= (\Delta_w(x_1)f(x_1), ..., \Delta_w(x_N)f(x_N))^\diamond, \\
\mathcal{I}(f) &= (I_0(f), ..., I_{N-1}(f))^\diamond,
\end{aligned}$$

and, for later purposes,

$$\tilde{\mathcal{I}}(f) = (0, ..., 0, I_N(f))^\diamond \in \mathbb{C}^{nN},$$

equations (2.3) may be written as

$$B\tilde{u} = \mathcal{R}(f) - \mathcal{B}^*\mathcal{U}\mathcal{J}^{-1}\mathcal{I}(f). \tag{2.4}$$

We have proved the following result.

Theorem 2.1. If $u$ is any solution of $Ju' + qu = wf$ on $(x_0, x_{N+1})$ then $\tilde{u} = (u^+(x_0), ..., u^+(x_N))^\diamond$ is a solution of equation (2.4). Conversely, a solution $\tilde{u} = (c_0, ..., c_N)^\diamond$ of equation (2.4) provides a solution of $Ju' + qu = wf$ on $(x_0, x_{N+1})$ given by (2.2) for $j = 0, ..., N$.

It is clear that the rank of $B$ is no larger than $nN$ and hence the kernel of $B$ has at least dimension $n$. Note, however, that it is possible for the dimension of the kernel of $B$ to be larger than $n$, i.e., to have more than $n$ independent solutions of the homogeneous differential equation $Ju' + qu = 0$.

When we consider balanced solutions of $Ju'+qu = 0$, the relationship between $\tilde{u}$ and the vector of values of $u$ at the points $x_1$, ..., $x_N$, i.e., the vector $\hat{u} = (u(x_1), ..., u(x_N))^\diamond$ is given by $\hat{u} = C\tilde{u}$ where

$$C = \frac{1}{2}(\mathcal{U}E_\perp + E_\top) = \frac{1}{2}\begin{pmatrix} U_0(x_1) & \mathbb{1} & 0 & \cdots & 0 \\ 0 & U_1(x_2) & \mathbb{1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & U_{N-1}(x_N) & \mathbb{1} \end{pmatrix}.$$

We also introduce the matrices $B_m$ and $C_m$ which are obtained from $B$ and $C$ respectively by removing the first and last $n$ columns. Earlier we chose, without loss of generality, $N \geq 2$ to avoid the case when $B_m$ and $C_m$ have no columns.

We have the following relationship between $B$ and $C$.

Lemma 2.2. $C^*B - B^*C = \operatorname{diag}(-J, 0, ..., 0, J)$. In particular, $C_m^*B - B_m^*C = 0$.

Proof. Since $\mathcal{B} - \mathcal{B}^* = 2\mathcal{J}$ we get

$$C^*B - B^*C = (E_\top^*\mathcal{J}E_\top - E_\perp^*\mathcal{U}^*\mathcal{J}\mathcal{U}E_\perp).$$

It was shown in [4] that $u^{-*}Jv^{-}$ is constant on any interval on which $B_{\pm}$ are everywhere invertible when $u$ and $v$ are solutions of $Ju' + qu = 0$. In particular, $U_j(x_{j+1})^* J U_j(x_{j+1}) = \lim_{x \uparrow x_{j+1}} U_j^-(x)^* J U_j^-(x) = J$. This implies that $\mathcal{U}^* \mathcal{J} \mathcal{U} = \mathcal{J}$ and hence the claim. □

LEMMA 2.3. Suppose $\hat{u} \in \ker B_m^*$. Then there exists a unique vector $\tilde{u}$ such that $B\tilde{u} = 0$ and $C\tilde{u} = \hat{u}$. Moreover, if $\hat{u} \in \ker B^* \subset \ker B_m^*$, then the first and the last $n$ components of $\tilde{u}$ are equal to 0.

PROOF. If a solution $\tilde{u}$ indeed exists, it must satisfy $\mathcal{U}E_\perp \tilde{u} = 2\hat{u} - E_\top \tilde{u}$ and hence $0 = 2\mathcal{B}^* \hat{u} + (\mathcal{B} - \mathcal{B}^*) E_\top \tilde{u}$. This implies $E_\top \tilde{u} = -\mathcal{J}^{-1} \mathcal{B}^* \hat{u}$. Similarly, using $E_\top \tilde{u} = 2\hat{u} - \mathcal{U} E_\perp \tilde{u}$, we get $E_\perp \tilde{u} = \mathcal{J}^{-1} \mathcal{U}^* \mathcal{B} \hat{u}$. Thus a solution is unique.

To prove existence note that $\hat{u} = (\hat{u}_1, ..., \hat{u}_N)^\diamond \in \ker B_m^*$ implies

$$B_+(x_k)^* \hat{u}_k + U_k(x_{k+1})^* B_+(x_{k+1}) \hat{u}_{k+1} = 0$$

for $k = 1, ..., N-1$. Hence the assignments $E_\top \tilde{u} = -\mathcal{J}^{-1} \mathcal{B}^* \hat{u}$ and $E_\perp \tilde{u} = \mathcal{J}^{-1} \mathcal{U}^* \mathcal{B} \hat{u}$ define $\tilde{u}$ unambiguously. Also $\tilde{u}$ satisfies $B\tilde{u} = 0$ and $C\tilde{u} = \hat{u}$.

For the last claim notice that $(C^* B - B^* C)\tilde{u} = 0$. If $\tilde{u} = (c_0, ..., c_N)^\diamond$, Lemma 2.2 gives $-Jc_0 = Jc_N = 0$ and hence $c_0 = c_N = 0$ as claimed. □

THEOREM 2.4. If $\hat{u} \in \ker B_m^*$, then $Ju' + qu = 0$ has a unique solution $u$ on the interval $(x_0, x_{N+1})$ such that $\tilde{u} = (u^+(x_0), ..., u^+(x_N))^\diamond$ satisfies $B\tilde{u} = 0$ and $C\tilde{u} = (u(x_1), ..., u(x_N))^\diamond = \hat{u}$. If $\hat{u} \in \ker B^* \subset \ker B_m^*$ then, additionally, $u^+(x_0) = u^-(x_{N+1}) = 0$ so that $\operatorname{supp} u \in [x_1, x_N]$.

PROOF. This is an immediate consequence of Theorem 2.1 and Lemma 2.3. □

LEMMA 2.5. Suppose $f \in L^2(w)$ and $\hat{u} \in \ker B_m^*$. Then there is a function $u$ satisfying $Ju' + qu = 0$ on $(x_0, x_{N+1})$, $(u(x_1), ..., u(x_N))^\diamond = \hat{u}$, and $\hat{u}^* \mathcal{F}(f) = \int_{(x_0, x_{N+1})} u^* w f$, where

$$\mathcal{F}(f) = \mathcal{R}(f) - \mathcal{B}^* \mathcal{U} \mathcal{J}^{-1} \mathcal{I}(f) + \mathcal{B} \mathcal{J}^{-1} \tilde{\mathcal{I}}(f).$$

Proof. Let $u$ be the function furnished by Theorem 2.4 and let $\tilde{u}$ be the vector $(u^+(x_0), ..., u^+(x_N))^\diamond$. Then $\mathcal{B}E_\top\tilde{u} = -\mathcal{B}^*\mathcal{U}E_\perp\tilde{u}$ since $B\tilde{u} = 0$. We also have $2C\tilde{u} = \mathcal{U}E_\perp\tilde{u} + E_\top\tilde{u}$. Using $\mathcal{B} - \mathcal{B}^* = 2\mathcal{J}$ gives us that $\mathcal{B}C\tilde{u} = \mathcal{J}\mathcal{U}E_\perp\tilde{u}$ and $\mathcal{B}^*C\tilde{u} = -\mathcal{J}E_\top\tilde{u}$ or, taking adjoints,

$$\tilde{u}^*C^*\mathcal{B}^* = -\tilde{u}^*E_\perp^*\mathcal{U}^*\mathcal{J} \quad \text{and} \quad \tilde{u}^*C^*\mathcal{B} = \tilde{u}^*E_\top^*\mathcal{J}.$$

These and $\mathcal{U}^*\mathcal{J}\mathcal{U} = \mathcal{J}$ imply

$$\begin{aligned}
\hat{u}^*\mathcal{F}(f) &= \hat{u}^*\mathcal{R}(f) - \tilde{u}^*C^*\mathcal{B}^*\mathcal{U}\mathcal{J}^{-1}\mathcal{I}(f) + \tilde{u}^*C^*\mathcal{B}\mathcal{J}^{-1}\tilde{\mathcal{I}}(f) \\
&= \hat{u}^*\mathcal{R}(f) + \tilde{u}^*E_\perp^*\mathcal{I}(f) + \tilde{u}^*E_\top^*\tilde{\mathcal{I}}(f) \\
&= \hat{u}^*\mathcal{R}(f) + \tilde{u}^*(I_0(f), ..., I_N(f))^\diamond = \int_{(x_0,x_{N+1})} u^*wf
\end{aligned}$$

using that $u(x) = U_j(x)u^+(x_j)$ for $x \in (x_j, x_{j+1})$. □

## 3. Minimal and maximal relations

Our differential equation $Ju' + qu = wf$ gives rise to the following two linear relations. $T_{\max}$ is the set of all pairs $([u], [f]) \in L^2(w) \times L^2(w)$ for which there are representatives $u \in [u]$ and $f \in [f]$ such that $Ju' + qu = wf$ (in particular, $u$ is a balanced function of locally bounded variation). $T_{\min}$ is the set of those elements in $T_{\max}$ for which the solution $u \in [u]$ may be chosen with compact support.

Recall that the adjoint of a linear relation $S \subset L^2(w) \times L^2(w)$ is defined to be

$$S^* = \{(v, g) \in L^2(w) \times L^2(w) : \forall (u, f) \in S : \langle g, u\rangle = \langle v, f\rangle\}.$$

Our main result in this paper is the following theorem.

Theorem 3.1. $T_{\min}^* = T_{\max}$.

Our proof requires a little preparation with which we begin. Suppose $[\xi_1, \xi_2] \subset (a, b)$ and consider the relation $\breve{T}_{\max}$ associated with $J$, $q$, and $w$ but restricted to the interval

$(\xi_1, \xi_2)$. Of course, solutions of $Ju' + qu = wf$ have limits at $\xi_1$ and $\xi_2$. We denote the restriction of $w$ to $(\xi_1, \xi_2)$ by $\breve{w}$ and set $T_0 = \{([u],[f]) \in \breve{T}_{\max} : u^+(\xi_1) = u^-(\xi_2) = 0\}$ and $K_0 = \ker \breve{T}_{\max}$.

Lemma 3.2. $\operatorname{ran} T_0 = L^2(\breve{w}) \ominus K_0$.

Proof. Let $[f] \in \operatorname{ran} T_0$ and $[r] \in K_0$. Then $Ju' + qu = \breve{w}f$ for some $u$ which vanishes at $\xi_1$ and $\xi_2$ and $r$ (chosen appropriately in $[r]$) satisfies $Jr' + qr = 0$. Integration by parts shows then

$$\int f^* \breve{w} r = \int u^*(Jr' + qr) = 0.$$

Hence $\operatorname{ran} T_0 \subset L^2(\breve{w}) \ominus K_0$.

Conversely, suppose $[f] \in L^2(\breve{w}) \ominus K_0$. We want to show the existence of a balanced function $u$ of bounded variation defined on $(\xi_1, \xi_2)$, vanishing at the endpoints, and satisfying $Ju' + qu = \breve{w}f$. Using the notation established in Section 2 and, in particular, Theorem 2.1 we have to show the existence of a solution $\tilde{u} = (\gamma_0, ..., \gamma_N)^\diamond$ of equation (2.4) satisfying $\gamma_0 = 0$ and $\gamma_N = -J^{-1} I_N(f)$ (so that $u^+(\xi_1) = u^-(\xi_2) = 0$). Thus we need to find $\tilde{u}_0 = (\gamma_1, ..., \gamma_{N-1})^\diamond$ such that $B_m \tilde{u}_0 = \mathcal{F}(f)$ where, as in Lemma 2.5,

$$\mathcal{F}(f) = \mathcal{R}(f) - \mathcal{B}^* \mathcal{U} \mathcal{J}^{-1} \mathcal{I}(f) + \mathcal{B} \mathcal{J}^{-1} \tilde{\mathcal{I}}(f).$$

This system has a solution precisely when $\mathcal{F}(f)$ is in $\operatorname{ran} B_m = (\ker B_m^*)^\perp$.

Hence suppose $\hat{r} \in \ker B_m^*$. The function $r$ associated with $\hat{r}$ according to Theorem 2.4 is a representative of an element in $K_0$ so that $\int r^* \breve{w} f = 0$. But Lemma 2.5 shows that $\hat{r}^* \mathcal{F}(f) = \int r^* \breve{w} f$ guaranteeing the existence of $u$. □

Lemma 3.3. If $g \in \operatorname{ran} T_{\min}^*$, then the differential equation $Ju' + qu = wg$ has at least one solution on $(a, b)$.

Proof. Let $\tau_n$, $n \in \mathbb{Z}$, be an enumeration of points in $(a, b)$ which include all points where the matrices $J \pm \Delta_q(x)/2$ are not invertible. The labeling is such that

$\tau_n < \tau_{n+1}$ and we may arrange things so that $a$ and $b$ are accumulation points, and the only ones, of the sequence $\tau_n$.

According to Theorem 2.1 there is a balanced solution $v_j$ of $Ju' + qu = wg$ on $(\xi_1, \xi_2) = (\tau_{-j}, \tau_j)$ (at least when $j > 1$) provided $B\tilde{v}_j = G$ where

$$G = \mathcal{R}(g) - \mathcal{B}^*\mathcal{U}\mathcal{J}^{-1}\mathcal{I}(g) = \mathcal{F}(g) - \mathcal{B}\mathcal{J}^{-1}(0, ..., 0, I_N(g))^\diamond.$$

This, in turn, happens if and only if $G \in \operatorname{ran} B = (\ker B^*)^\perp$ which we show next.

For any $\hat{r} \in \ker B^*$ Theorem 2.4 and Lemma 2.5 show the existence of a solution $r$ of $Ju' + qu = 0$ on $(\xi_1, \xi_2)$ such that $\tilde{r} = (r^+(x_0), ..., r^+(x_N))^\diamond$ satisfies $B\tilde{r} = 0$, $r^+(x_0) = r^+(x_N) = 0$, and $\hat{r}^*\mathcal{F}(g) = \int_{(\xi_1,\xi_2)} r^* wg$ (here $N = 2j - 1$ and $x_\ell = \tau_{-j+\ell}$). In fact, since $r$ vanishes near $x_0$ and $x_{N+1}$, we may extend it by 0 to obtain a solution of $Ju' + qu = 0$ on all of $(a, b)$. Thus $\langle r, g\rangle = \hat{r}^*\mathcal{F}(g)$ and it follows, as in the proof of Lemma 2.5, that $\hat{r}^*G = \langle r, g\rangle - \tilde{r}^* E_\top^*(0, ..., 0, I_N(g))^\diamond$. Since $([r], 0) \in T_{\min}$ we have $\langle r, g\rangle = \langle 0, v\rangle = 0$ and since $r^+(x_N) = 0$ we also have $\tilde{r}^* E_\top^*(0, ..., 0, I_N(g))^\diamond = 0$. Thus $G \in \operatorname{ran} B$ and this guarantees the existence of $v_j$.

Now define, for any $j \geq k \geq 2$, the set $A_{k,j}$ to be the collection of restrictions to $(\tau_{-k}, \tau_k)$ of solutions of $Ju' + qu = wg$ on $(\tau_{-j}, \tau_j)$. According to the above the $A_{k,j}$ are nonempty and nested in the sense that $A_{k,j+1} \subset A_{k,j}$. Each $A_{k,j}$ is an affine subspace of, say, the space of all functions defined on $(\tau_{-k}, \tau_k)$ and their dimensions form, in $j$, a nonincreasing sequence of nonnegative integers which must eventually be constant (possibly zero). Hence, for a sufficiently large $m$, we have that $B_k = \bigcap_{j \geq k} A_{k,j} = A_{k,m}$. We now define inductively a sequence of functions $v_k$ whose pointwise limit is a solution of $Ju' + qu = wg$ on $(a, b)$. For $v_2$ we choose any element of $B_2$. Then suppose we had constructed a sequence $(v_2, ..., v_k)$ such that $v_j \in B_j$ and $v_{j-1} = v_j|_{(\tau_{-j+1}, \tau_{j-1})}$. Note that the elements of $B_k$ are restrictions of elements in $B_{k+1}$ to $(\tau_{-k}, \tau_k)$ (and vice versa). Thus we may choose for $v_{k+1}$ an element of $B_{k+1}$ which extends $v_k$ and this completes our definition of the sequence $v_k$, except that we extend each of its elements arbitrarily to $(a,b)$. Now $v$, the pointwise limit of the $v_k$, is the desired solution of $Ju'+qu=wg$ on $(a,b)$. $\square$

Proof of Theorem 3.1. If $([v],[g])$ and $([u],[f])$ are in $T_{\max}$ Lagrange's identity (cf. [4]) states that

$$\langle v,f\rangle-\langle g,u\rangle=(v^*Ju)^-(b)-(v^*Ju)^+(a). \tag{3.1}$$

Therefore, if $([u],[f])\in T_{\min}$, so that $u$ has compact support, we get $\langle v,f\rangle=\langle g,u\rangle$ which proves that $T_{\max}\subset T_{\min}^*$.

To prove $T_{\min}^*\subset T_{\max}$ assume that $([v],[g])\in T_{\min}^*$ and let $v_0$ be a solution of $Ju'+qu=wg$ on $(a,b)$ as constructed by Lemma 3.3. Next we will employ Lemma 3.2. We consider an interval $[\xi_1,\xi_2]\subset(a,b)$ and define $\breve{w}$, $T_0$ and $K_0$ as we did there. Given $([u],[f])\in T_0$ extend both $u$ and $f$ by 0 to all of $(a,b)$ (denoting the extensions also by $u$ and $f$). We then have $Ju'+qu=wf$ so that $([u],[f])\in T_{\min}$ and $\langle f,v\rangle=\langle u,g\rangle$. To establish a relationship between $v$ and $v_0$ we apply integration by parts to obtain

$$\begin{aligned}\int_{(\xi_1,\xi_2)} f^*\breve{w}v=\int_{(a,b)} u^*wg=\int_{(a,b)} u^*(Jv_0'+qv_0)\\ =\int_{(a,b)}(Ju'+qu)^*v_0=\int_{(\xi_1,\xi_2)} f^*\breve{w}v_0.\end{aligned}$$

Thus $\int f^*\breve{w}(v-v_0)=0$ so that, by Lemma 3.2, $[v-v_0]\in K_0$ showing that $[v]$ has a representative $v=v_0+k_0$ where $Jk_0'+qk_0=0$ and hence $Jv'+qv=wg$ on $(\xi_1,\xi_2)$. We obtain a solution on all of $(a,b)$ in a similar way as we did in the proof of Lemma 3.3. We only have to modify the definition of the sets $A_{k,j}$ to specify that the solutions $u$ considered are locally representatives of $v$, i.e, that $\int_{(\tau_{-j},\tau_j)}(u-v)^*w(u-v)=0$. $\square$

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**COMPARISON AND OSCILLATION THEOREMS FOR FIRST-ORDER SYSTEMS WITH DISTRIBUTIONAL COEFFICIENTS**

by

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# COMPARISON AND OSCILLATION THEOREMS FOR FIRST-ORDER SYSTEMS WITH DISTRIBUTIONAL COEFFICIENTS

Abstract. We establish a comparison theorem and an oscillation theorem for a $2 \times 2$-system of ordinary differential equations when the coefficients are (possibly singular) continuous, real, and finite measures.

## 1. Introduction

The goal of the present paper is to establish a comparison theorem and an oscillation theorem for a $2 \times 2$-system of ordinary differential equations when the coefficients are (possibly singular) continuous, real, and finite measures. This is a well studied subject for a Sturm-Liouville equation $-(py')' + vy = \lambda ry$ posed on a finite interval $(a, b)$ when $\lambda$ is a parameter and $1/p$, $v$, and $r$ are real and locally integrable functions. Hinton [12] gives an excellent survey on this subject. We also refer the reader to the books by Coddington and Levinson [6], Swanson [16], Teschl [17], and Zettl [21], but there are, of course, many others. Most of the extant results are for the definite case where $p, r > 0$. It appears that the so called indefinite case, where $p$ or $r$ may change sign, was first addressed by Richardson [15]. A (much more) recent contribution in this regard – with references to further literature – is by Binding and Volkmer [3].

Writing the Sturm-Liouville equation as a first order system yields

$$Ju' + qu = \lambda wu$$

where $u = (y, py')^\top$, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $q = \begin{pmatrix} v & 0 \\ 0 & -1/p \end{pmatrix}$, and $w = \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}$. Comparison and oscillation theorems for more general systems with integrable coefficients have also been studied. We refer, for instance, to Coppel [7], Reid [14], and Brown, Eastham, and Schmidt [4].

Our goal here, as we said above, is to allow the entries of the matrices $q$ and $w$ to be finite, continuous measures. We ask the measures to be finite as that leads to regular endpoints of the interval $(a, b)$. We are restricting ourselves to continuous measures for reasons which we shall explain in Section 3. Schrödinger equations with distributional potentials (worse than measures) have been studied by Shkalikov and Ben Amara [1] and by Homa and Hryniv [13]. However, these equations can be cast as a system with integrable coefficients as was pointed out by Eckhardt et al. [8]. A comparison theorem for such cases was obtained by Ghatasheh and Weikard [9].

We end this introduction by briefly describing the content of each section. In Section 2 we discuss distributions of order 0 and their connection to measures. In Section 3 we define Prüfer angles and prove two basic results about comparison of Prüfer angles for two equations $Ju' = p_j u$ where $p_2 \geq p_1$. Moreover, we establish a condition under which the zeros of the first component of a solution of $Ju' = pu$ are isolated since the solution can then cross the vertical axis only in a clockwise direction. These fundamental facts are then used in Section 4 to prove a comparison theorem, i.e., , a theorem comparing the number of zeros of the first components $u_{j;1}$ of solutions of $Ju' = p_j u$ when $p_2 \geq p_1$, and a separation theorem, i.e., , a theorem showing that between two consecutive zeros of $u_1$ there is always a zero of $v_1$ if $u$ and $v$ are linearly independent solution of $Ju' = pu$. Finally, in Section 5 we replace $p$ by $\lambda w - q$ with a spectral parameter $\lambda$ and appropriate measure coefficients $q$ and $w$. In this case the equation $Ju' + qu = \lambda wu$ gives rise to a self-adjoint relation and the object of the oscillation theorems proved there is to relate the eigenvalue count with the number of zeros of the first components of the eigenfunctions.

## 2. Some basic facts about distributions

Let $(a, b)$ be an open interval in the real line. The compactly supported infinitely often differentiable functions from $(a, b)$ to $\mathbb{C}$ are called *test functions* and a linear functional $p$ defined on the set of test functions is called a *distribution* on $(a, b)$, if it satisfies the following condition: for every compact subset $K$ of $(a,b)$ there are numbers $C \geq 0$ and $k \in \mathbb{N}_0$ such that

$$|p(\phi)| \leq C \sum_{j=0}^{k} \|\phi^{(j)}\|_\infty$$

whenever $\phi$ is a test function with support in $K$. The distribution $p$ is called a distribution of order 0, if one may choose $k = 0$ for any compact set $K$. The space of distributions of order 0 on $(a,b)$ is denoted $\mathcal{D}'^0((a,b))$.

If $U$ is an open set in $(a,b)$ and $p(\phi) = 0$ whenever the support of $\phi$ is contained in $U$ one says that the distribution $p$ vanishes on $U$. The *support* of $p$ is the complement of the largest open set in $(a,b)$ on which $p$ vanishes.

The most important fact about distributions is that they all have derivatives. If $p$ is a distribution its derivative $p'$ is defined by $p'(\phi) = -p(\phi')$ and is again a distribution. Distributions also have antiderivatives and we have the following important lemma.

LEMMA 2.1 (Du Bois-Reymond). *Suppose the derivative of the distribution $p$ is zero. Then $p$ is the constant distribution, i.e., , there is a complex number $C$ such that $p(\phi) = C \int \phi \, dx$ for every test function $\phi$.*

If $P$ is a function of locally bounded variation on $(a,b)$ and $dP$ is the associated local Lebesgue-Stieltjes measure[1] one may assign the number $p(\phi) = \int \phi \, dP$ to the test function $\phi$ to obtain a distribution of order 0. It is, in fact, a consequence of Riesz's representation theorem that every distribution of order 0 on $(a,b)$ is represented in this way by a (local) Lebesgue-Stieltjes measure. See, for instance, [10] for more details.

The most famous example of a distribution of order 0 is the Dirac distribution $\delta_0(\phi) = \phi(0)$ (assuming $0 \in (a,b)$). It may be represented by the integral of $\phi$ with respect to Dirac measure which is the Lebesgue-Stieltjes measure generated by the

[1]Only if $P$ is actually of bounded variation or if it is nondecreasing does it generate a measure on $(a,b)$. Otherwise it generates a measure on any compact subset of $(a,b)$.

Heaviside step function. The support of $\delta_0$ is $\{0\}$. Perhaps the most familiar example of a distribution is $\mathfrak{f}(\phi) = \int \phi f \, dx$ when $f$ is a locally integrable function.

Now suppose that $P \in \mathrm{BV}_{\mathrm{loc}}((a,b))$, the space of complex-valued functions of locally bounded variation on $(a,b)$. We then have the distributions $p(\phi) = \int \phi \, dP$. Since $P$ is also locally integrable we also have the distribution $P(\phi) = \int P\phi \, dx$ (we shall henceforth identify a function $P \in L^1_{\mathrm{loc}}((a,b))$ with the distribution $\phi \mapsto \int P\phi dx$; context will serve to distinguish the two meanings if necessary). Integration by parts shows that

$$P'(\phi) = -P(\phi') = -\int P\phi' dx = -\int P d\phi = \int \phi dP = p(\phi)$$

since the boundary terms vanish. Thus $P' = p$. This suggests to use the notation

$$p(\phi) = \int \phi dP = \int p\phi$$

even if $p$ is not a function.

It is perhaps useful to recall the product rule for functions of locally bounded variation underlying the integration by parts formula. If $F$ and $G$ are in $\mathrm{BV}_{\mathrm{loc}}((a,b))$, then so is their product and

$$d(FG) = (FG)' = F^+G' + F'G^- = F^+ \, dG + dF \, G^-,$$

see, e.g., , Hewitt and Stromberg [11], Theorem 21.67 and Remark 21.68. Note that, if one of $F$ and $G$ is continuous, one may ignore the superscripts $\pm$ on $F$ and $G$.

Not so simple is the chain rule which appears to have first been established by Vol'pert [18], see also Vol'pert and Hudjaev [19].

Theorem 2.2. *Suppose $u \in \mathrm{BV}_{\mathrm{loc}}((a,b))$, $V$ is an open set in $\mathbb{C}$ containing the range of $u$, and $f \in C^1(V)$. Then $f \circ u \in \mathrm{BV}_{\mathrm{loc}}((a,b))$ and*

$$(f \circ u)' = \left[ \int_0^1 f'(su^+(\cdot) + (1-s)u^-(\cdot)) ds \right] u'.$$

Corollary 2.3. *If $u \in \mathrm{BV}_{\mathrm{loc}}((a,b))$ is continuous and $f \in C^1(V)$, then $(f \circ u)' = (f' \circ u)u'$.*

The conjugate of a distribution $p$, denoted by $\overline{p}$, is defined by $\overline{p}(\phi) = \overline{p(\overline{\phi})}$. A distribution is called real if $p = \overline{p}$. Equivalently, $p$ is real if $p(\phi) \in \mathbb{R}$ whenever $\phi$ is real-valued. Similarly, $p$ is called nonnegative, written as $p \geq 0$, if $p(\phi) \geq 0$ for all $\phi \geq 0$. If $p$ is real or nonnegative, then so is the associated measure.

If $p = dP$ is a distribution of order 0 and $f \in L^1_{\mathrm{loc}}(|dP|)$, we may define the product $pf = fp$ as follows: $(pf)(\phi) = \int f\phi\, dP$, again a distribution of order 0.

Finally, a word about matrix-valued distributions. As usual the entry in row $j$ and column $k$ of a matrix $p$ is denoted by $p_{jk}$. Sometimes we will consider matrices $p_1$ and $p_2$. Then $p_{\ell;jk}$ denotes the entry in row $j$ and column $k$ of $p_\ell$. Similarly, $u_k$ and $u_{\ell;k}$ denote respectively the $k$-th component of the vectors $u$ and $u_\ell$. To avoid cumbersome notation we may use the symbol $u_1$ for the vector $u_1$ and for the first component of the vector $u$. Which meaning is appropriate will be clear from the context. We are interested in the case of $2 \times 2$-matrices whose entries are distributions of order 0 on $(a,b)$, a space which we denote by $\mathcal{D}'^0((a,b))^{2\times 2}$. A matrix $r \in \mathcal{D}'^0((a,b))^{2\times 2}$ is called hermitian if $\overline{r}_{jk} = r_{kj}$ for $j,k = 1,2$. This is equivalent to the requirement that $z^* r z$ is real for all $z \in \mathbb{C}^2$. One calls $r$ nonnegative, if $z^* r z$ is nonnegative for all $z \in \mathbb{C}^2$. If $r_1, r_2 \in \mathcal{D}'^0((a,b))^{2\times 2}$, we write $r_1 \geq r_2$ if $r_1 - r_2$ is nonnegative. We say that $r$ vanishes on an open set $U \subset (a,b)$ if this is true for each of its entries.

## 3. The Prüfer transform

In this section we consider matrix-valued distributions in $\mathcal{D}'^0((a,b))^{2\times 2}$ whose entries are real and correspond to finite and continuous measures. The set of such distributions will be denoted by $\mathcal{C}$.

For $p \in \mathcal{C}$ and $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ we consider the equation

$$Ju' = pu \tag{3.1}$$

on the interval $(a, b)$. Under our assumptions $\mathbb{C}^2$-valued solutions $u$ of (3.1) satisfying arbitrary initials conditions are guaranteed to exist, are continuous, and have limits at $a$ and $b$. Moreover, since $\overline{u}$ is a solution of (3.1) if $u$ is, it follows that the real and imaginary parts of $u$ of are also solutions. Consequently, the space of solutions is spanned by real (i.e., , $\mathbb{R}^2$-valued) solutions.

We now define the Prüfer variables $r$ and $\theta$ for a nontrivial real solution $u$ of (3.1) as the unique continuous functions on $[a, b]$ satisfying the following conditions

$$u(x) = r(x) \begin{pmatrix} \sin\theta(x) \\ \cos\theta(x) \end{pmatrix}, \quad r(x) > 0 \text{ and } \theta(a) \in [0, 2\pi).$$

Note that $r(x) = |u(x)|$ when $|\cdot|$ denotes the euclidean norm in $\mathbb{R}^2$. Using Theorem 2.2 and Corollary 2.3 gives that both $r$ and $\theta$ are functions of bounded variation and that

$$0 = Ju' - pu = r'J \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} + r\theta' J \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix} - rp \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}.$$

Multiplying this from the left by the row vectors $(\sin\theta, \cos\theta)$ and $(\cos\theta, -\sin\theta)$ gives, respectively,

$$\theta' = (\sin\theta, \cos\theta)p \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} \tag{3.2}$$

and

$$\frac{r'}{r} = -(\cos\theta, -\sin\theta)p \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}. \tag{3.3}$$

If we were to allow for the presence of discrete measures, solutions might have jump discontinuities. In such a case the tangent of the Prüfer angle would still be well defined but to define the angle itself would necessitate choosing a branch of arctangent. We are yet unclear how to do this in a consistent manner. In this context, it may be useful to point out that the variable $r$ has dropped out of equation (3.2). This would not be the case, if $r$ and $\theta$ had jump discontinuities, rendering, perhaps, Prüfer's approach less powerful.

We now return to the case of continuous measures. If the coefficients $p_1$ and $p_2$ of two different equations of the form (3.1) are comparable, then the Prüfer angles of their solutions are also comparable. The following results and their proofs are adopted from [2] and [20].

THEOREM 3.1 (Angles comparison I). *Let $p_1, p_2 \in \mathcal{C}$ and suppose that $p_2 \geq p_1$. If $u_j$, $\theta_j$ ($j = 1, 2$) are nontrivial real solutions of $Ju' = p_j u$ and their Prüfer angles, respectively, the following statements hold.*

*(a) If $\theta_2(c) \geq \theta_1(c)$ for some $c \in (a, b)$, then $\theta_2(x) \geq \theta_1(x)$ for all $x \in [c, b)$.*

*(b) If $\theta_2(c) \leq \theta_1(c)$ for some $c \in (a, b)$, then $\theta_2(x) \leq \theta_1(x)$ for all $x \in (a, c]$.*

PROOF. Define the continuous function $\delta = \theta_2 - \theta_1$ on $(a, b)$ and the distributions $h$ and $f$ in $\mathcal{D}'^0((a, b))$ by

$$h = (\sin\theta_1, \cos\theta_1)(p_2 - p_1)\begin{pmatrix}\sin\theta_1 \\ \cos\theta_1\end{pmatrix}$$
$$= (\sin\theta_1)^2(p_{2;11} - p_{1;11}) + (\cos\theta_1)^2(p_{2;22} - p_{1;22})$$
$$+ (\sin 2\theta_1)\big(\frac{p_{2;12} + p_{2;21}}{2} - \frac{p_{1;12} + p_{1,21}}{2}\big)$$

and

$$f = (p_{2;22} - p_{2;11})\frac{\sin\theta_2 - \sin\theta_1}{\theta_2 - \theta_1}(\sin\theta_2 + \sin\theta_1) - \frac{p_{2;12} + p_{2;21}}{2}\frac{\sin 2\theta_2 - \sin 2\theta_1}{\theta_2 - \theta_1}$$

where we replace the quotients by their limits when they become undefined. Then we have

$$\delta' = h - \delta f.$$

Let $F \in \mathrm{BV}((a, b))$ be an antiderivative of $f$ and define on $(a, b)$ the continuous function $k(x) = \exp(F(x))$. Since $F$ is continuous the chain rule shows that $k' = kf$.

Then the product rule gives

$$(k\delta)' = \delta k' + k\delta' = \delta k f + k\delta' = k(\delta f + \delta') = kh.$$

Since $k$ is a positive function and $h$ is a nonnegative distribution, $(k\delta)'$ is a nonnegative distribution on $(a, b)$.

Suppose $\theta_2(c) \geq \theta_1(c)$, i.e., , $\delta(c) \geq 0$, for some $c \in (a, b)$. Then we get, for any $x \in (c, b)$, that

$$(k\delta)(x) = (k\delta)(c) + \int_{(c,x)} (k\delta)' \geq 0.$$

Thus $\theta_2(x) \geq \theta_1(x)$ for all $x \in [c, b)$. This finishes the proof of (a); that of (b) is similar. □

THEOREM 3.2 (Angles comparison II). *Let $p_1, p_2 \in \mathcal{C}$ and suppose that $p_2 \geq p_1$. Let $u_j$, $\theta_j$ $(j = 1, 2)$ be nontrivial real solutions of $Ju' = p_j u$ and their Prüfer angles, respectively. If $(c, d)$ is a nonempty subinterval of $(a, b)$ such that $(p_2 - p_1)u_1$ does not vanish on $(c, d)$, the following statements hold.*

*(a) If $\theta_2(c) \geq \theta_1(c)$, then $\theta_2(x) > \theta_1(x)$ for all $x \in [d, b)$.*

*(b) If $\theta_2(d) \leq \theta_1(d)$, then $\theta_2(x) < \theta_1(x)$ for all $x \in (a, c]$.*

PROOF. By part (a) of Theorem 3.1, $\theta_2(x) \geq \theta_1(x)$ in $[c, b)$. Suppose there exists $x_0$ in $[d, b)$ such that $\theta_2(x_0) = \theta_1(x_0)$, then $\theta_2(x) \leq \theta_1(x)$ in $(a, x_0]$ by part (b) of Theorem 3.1. Thus $\theta_2(x) = \theta_1(x)$ on $[c, x_0]$. This means $\delta = \delta' = 0$, and hence $h = 0$ in $(c, x_0)$ using the notation of the proof of Theorem 3.1. Since $|u_1|^2 h = u_1^\top (p_2 - p_1)u_1$ this implies that $(p_2 - p_1)u_1$ is the zero distribution on $(c, x_0)$ and, in particular, on $(c, d)$. This finishes the proof of (a); that of (b) is similar. □

LEMMA 3.3. *Suppose $p \in \mathcal{C}$, $x_0 \in [a, b)$, and $\theta(x_0)$ is an integer multiple of $\pi$, i.e., , the first component of $u(x_0)$ is 0. If*

$$\int_{(x_0,x)} p_{22} > \frac{1}{2} \max \left\{ \Big| \int_{(x_0,t)} p_{22} \Big| : x_0 \leq t \leq x \right\} \tag{3.4}$$

*for all $x$ in some nonempty interval $(x_0, x_1)$, then there is a $\delta > 0$ such that $\theta(x) > \theta(x_0)$ for all $x \in (x_0, x_0 + \delta)$.*

Proof. If we define $\theta_0(t) = \int_{(x_0,t)} p_{22}$ equation (3.2) gives

$$\theta(t) - \theta(x_0) - \theta_0(t) = \int_{(x_0,t)} \big((p_{11} - p_{22})(\sin\theta)^2 + (p_{12} + p_{21})(\sin\theta)(\cos\theta)\big).$$

The mean value theorem implies that

$$\frac{\sin\theta(t)}{\theta(t) - \theta(x_0)} = \frac{\sin\theta(t) - \sin\theta(x_0)}{\theta(t) - \theta(x_0)} = \cos(t')$$

for some $t' \in (x_0, t)$. Hence $|\sin\theta(t)| \leq |\theta(t) - \theta(x_0)|$. Now we define $M(t) = \max\{|\theta(s) - \theta(x_0)| : x_0 \leq s \leq t\}$ to obtain

$$|\theta(t) - \theta(x_0) - \theta_0(t)| \leq M(t)\big(M(t)\int_{(x_0,t)} |p_{11} - p_{22}| + \int_{(x_0,t)} |p_{12} + p_{21}|\big).$$

The latter factor on the right of this inequality, which we call $F(t)$, is continuous as a function of $t$ and vanishes at $x_0$. Therefore there is a number $\delta \in (0, x_1 - x_0)$ such that $F(t)$ is less than $1/4$ as long as $x_0 \leq t \leq x < x_0 + \delta$. For such $t$ we have now

$$|\theta(t) - \theta(x_0)| - \theta_0(t) \leq |\theta(t) - \theta(x_0) - \theta_0(t)| \leq \frac{1}{4}M(t) \tag{3.5}$$

noting that $\theta_0(t) \geq 0$. Using our hypothesis we find $|\theta(t) - \theta(x_0)| \leq \theta_0(t) + \frac{1}{4}M(t) \leq 2\theta_0(x) + \frac{1}{4}M(x)$ so that $\frac{3}{4}M(x) \leq 2\theta_0(x)$. The latter inequality in (3.5) becomes then $|\theta(x) - \theta(x_0) - \theta_0(x)| \leq \frac{1}{4}M(x) \leq \frac{2}{3}\theta_0(x)$ which proves our claim. □

If $p_{22}$ is a positive measure with support $(a, b)$, one may choose $x_1 = b$ for every $x_0 \in (a, b)$ to satisfy condition (3.4). The positivity of $p_{22}$ is, however, not a necessary condition as the following example shows. The function $P_{22}$ defined by $P_{22}(x) = x + 2x^2 \sin(1/x)$ for $x > 0$ and $P_{22}(0) = 0$ is absolutely continuous on $[0, 1]$. Therefore its derivative gives rise to a measure $p_{22}$ which is absolutely continuous with respect to Lebesgue measure. It is easy to see that $p_{22}$ is not a positive measure. However, condition (3.4) holds for all $x \in (0, 1)$.

Considering instead $P_{22}(x) = 2x^2\sin(1/x)$, it turns out that condition (3.4) is violated. The system $Ju' = \left(\begin{smallmatrix} 0 & 0 \\ 0 & p_{22} \end{smallmatrix}\right) u$ is solved by the function $u = (-2x^2\sin(1/x), 1)^\top$. This shows that the conclusion of Lemma 3.3 is violated and hence that some condition on $p_{22}$ is necessary.

If condition (3.4) is replaced by

$$\int_{(x,x_0)} p_{22} > \frac{1}{2}\max\Big\{\Big|\int_{(t,x_0)} p_{22}\Big| : x_0 \le t \le x\Big\} \tag{3.6}$$

for all $x$ in some interval $(x_1, x_0)$ we may similarly conclude that $\theta(x) < \theta(x_0)$ for all $x$ in some interval $(x_0 - \delta, x_0)$. We have therefore the following theorem.

THEOREM 3.4. *Suppose $p \in \mathcal{C}$ and $u$ and $\theta$ are nontrivial real solution of $Ju' = pu$ and its Prüfer angle, respectively. If conditions (3.4) and (3.6) are satisfied near $x_0$ and if $u_1(x_0) = 0$, then $x_0$ is an isolated zero of $u_1$. Moreover, the Prüfer angle $\theta$ equals an integer multiple of $\pi$ at $x_0$ and is strictly increasing there. In other words, the point $u$ crosses the vertical axis clockwise.*

*If, instead, conditions (3.4) and (3.6) hold for $-p_{22}$ instead of $p_{22}$, then the Prüfer angle $\theta$ is strictly decreasing at $x_0$ and $u$ crosses the vertical axis counterclockwise.*

THEOREM 3.5 (Isolated zeros). *Suppose $p \in \mathcal{C}$ and $p_{22}$ is a positive measure with support $(a, b)$ and let $u$ be a nontrivial real solution of (3.1). Then the zeros of $u_1$, the first component of $u$, are isolated and the number of these zeros in $(a, b)$ is bounded above by*

$$1 + \frac{1}{\pi}\int_{(a,b)} \big(|p_{11}| + p_{22} + \frac{|p_{12} + p_{21}|}{2}\big),$$

*where $|p_{jk}|$ is the total variation measure of $p_{jk}$.*

PROOF. We mentioned earlier that condition (3.4) is satisfied for any $x_0 \in (a, b)$ if $p_{22}$ is a positive measure. Thus Theorem 3.5 shows that the zeros of $u_1$ are isolated.

Theorem 3.5 also shows that the trajectory of $u$ transverses the vertical axis in the clockwise direction at $x_0$. Thus $u_1$ picks up one more zero whenever $\theta$ increases an amount of $\pi$. The number of zeros of $u_1$ is therefore bounded above by

$$1+\frac{1}{\pi}|\theta(b)-\theta(a)| = 1+\frac{1}{\pi}\left|\int_{(a,b)}\big((\sin\theta)^2 p_{11}+(\cos\theta)^2 p_{22}+(\sin 2\theta)\frac{p_{12}+p_{21}}{2}\big)\right|$$
$$\leq 1+\frac{1}{\pi}\int_{(a,b)}\big(|p_{11}|+p_{22}+\frac{|p_{12}+p_{21}|}{2}\big).$$

□

Let $x_1, ..., x_N$ be finitely many points in $(a,b)=(x_0,x_{N+1})$. If $p_{22}$, restricted to $(x_{n-1},x_n)$, is either a positive or negative measure for $n=1,...,N+1$ and if $\operatorname{supp} p_{22}=(a,b)$, then the zeros of $u_1$ will still be isolated. However, the upper bound of these zeros mentioned in Theorem 3.5 may not hold anymore, since each interval $(x_{n-1},x_n)$ could contribute a zero without $\theta$, the Prüfer angle of $u$, changing very much.

## 4. Comparison and separation theorems

THEOREM 4.1 (Comparison theorem). *Let $p_1$ and $p_2$ be elements of $\mathcal{C}$ and suppose that $p_2 \geq p_1$. Furthermore, assume that $p_{1;22}$ and $p_{2;22}$ are positive measures with support equal to $(a,b)$. For $j=1,2$ let $u_j$ be a nontrivial real solution of $Ju'=p_ju$. If $x_0<x_1$ are two consecutive zeros of $u_{1;1}$ and if $u_1$ and $u_2$ are linearly independent, then $u_{2;1}$ has at least one zero in $(x_0,x_1)$.*

PROOF. First we note, in view of Theorem 3.5, that the zeros of $u_{1;1}$ and $u_{2;1}$ are isolated. Without loss of generality we can assume that $\theta_1(x_0)=0$ and $0\leq\theta_2(x_0)<\pi$. Since $x_0$ and $x_1$ are two consecutive zeros of $u_{1;1}$, we have $\theta_1(x_1)=\pi$. If $(p_2-p_1)u_1\neq 0$ we apply Theorem 3.2 with $c=x_0$ and $d=x_1$ and conclude that $\theta_2(x_1)>\theta_1(x_1)=\pi$. The continuity of $\theta_2$ shows that there is a point $y\in(x_0,x_1)$ such that $\theta_2(y)=\pi$ so that $u_{2;1}(y)=0$. If, on the other hand, $(p_2-p_1)u_1=0$, then both $u_1$ and $u_2$ satisfy the equation $Ju'=p_2u$. Since $u_1$ and $u_2$ are linearly independent, we must have $u_{2;1}(x_1)\neq 0$ and hence $\theta_2(x_1)\neq\pi$. Since, using Theorem 3.1, we also have

$\theta_2(x_1) \geq \theta_1(x_1) = \pi$ we conclude again that $\theta_2(x_1) > \pi$ and we may now argue as before for a zero of $u_{2;1}$ in $(x_0, x_1)$. $\square$

THEOREM 4.2 (Separation theorem). *Suppose $p \in \mathcal{C}$ and $p_{22}$ is a positive measure whose support equals $(a, b)$. Let $u_j$, $j = 1, 2$, be linearly independent (thus nontrivial) solutions of $Ju' = pu$. If $x_0 < x_1$ are two consecutive zeros of $u_{1;1}$, then $u_{2;1}$ has precisely one zero in $(x_0, x_1)$. In other words, the zeros of the first component of linearly independent solutions are interlacing.*

PROOF. Since $u_1$ and $u_2$ are linearly independent, it follows that $u_{2;1}(x_0) \neq 0$. Choosing $p_2 = p_1 = p$, Theorem 4.1 gives therefore that $u_{2;1}$ has at least one zero $y$ in $(x_0, x_1)$. By way of contradiction, let $y_0$ and $y_1$ be the first and second zero of $u_{2,1}$ in $(x_0, x_1)$. Interchanging the role of $u_1$ and $u_2$ in Theorem 4.1, $u_{1,1}$ has a zero in $(y_0, y_1)$. This contradicts the hypothesis on $x_0$ and $x_1$. $\square$

## 5. The oscillation theorem

In this section we let the coefficient $p$ in equation (3.1) depend on another real parameter $\lambda$. Specifically, we set

$$p = \lambda w - q$$

where $q$ is hermitian and $w$ is nonnegative. Since we still require that the entries of $q$ and $w$ are real, this entails, in fact, that they are symmetric. Thus the differential equation we consider reads now

$$Ju' + qu = \lambda wu. \tag{5.1}$$

Next we describe briefly how to define linear relations associated with the differential equation (5.1); more details may be found in [10]. Since $w \geq 0$ its trace is a positive scalar measure. Therefore we may introduce the vector space $\mathcal{L}^2(w)$ consisting of $\mathbb{C}^2$-valued functions $f$ where both components are measurable with respect to $\operatorname{tr} w$

and which satisfy $\int f^* w f < \infty$. The vector space $\mathcal{L}^2(w)$ is a semi-inner product space with semi-inner product $\langle f, g\rangle = \int f^* w g$ and semi-norm $\|f\| = \langle f, f\rangle^{1/2}$. The corresponding Hilbert space, i.e., , the quotient of $\mathcal{L}^2(w)$ by the kernel of $\|\cdot\|$, will be denoted by $L^2(w)$. We now introduce the linear relations

$$\mathcal{T}_{\max} = \{(u, f) \in \mathcal{L}^2(w) \times \mathcal{L}^2(w) : u \in \mathrm{BV}_{\mathrm{loc}}((a, b))^2, Ju' + qu = wf\}$$

and

$$\mathcal{T}_{\min} = \{(u, f) \in \mathcal{T}_{\max} : \operatorname{supp} u \text{ is compact in } (a, b)\}.$$

Then, in the Hilbert space setting, we represent our differential equation by the relations

$$T_{\max} = \{([u], [f]) \in L^2(w) \times L^2(w) : (u, f) \in \mathcal{T}_{\max}\}$$

and

$$T_{\min} = \{([u], [f]) \in T_{\max} : (u, f) \in \mathcal{T}_{\min}\}.$$

It was shown in [10] (see also [5] for a proof under more general circumstances) that $T_{\min}^* = T_{\max}$ which shows that $T_{\min}$ is a symmetric relation. Since $q$ and $w$ are finite measures $T_{\min}$ has equal deficiency indices so that we have self-adjoint extensions. The deficiency indices are equal to 1 if and only if there is a solution of $Ju' + qu = 0$ of norm 0. We will exclude this case, i.e., , we assume subsequently that the deficiency indices $n_\pm$ are equal to 2. For easy reference we collect our conditions in the following hypothesis.

Hypothesis 5.1. *Suppose $q, w \in \mathcal{D}'^0((a, b))^{2\times 2}$ have real entries whose associated measures are finite and continuous. Moreover, $q$ is symmetric and $w$ nonnegative. Finally, if $Ju' + qu = 0$ and $wu = 0$, then $u$ is identically equal to 0.*

Given numbers $\alpha \in [0, \pi)$ and $\beta \in (0, \pi]$ we consider now the relation

$$\mathcal{T}_{\alpha,\beta} = \{(u, f) \in \mathcal{T}_{\max} : (\cos\alpha, -\sin\alpha)u(a) = (\cos\beta, -\sin\beta)u(b) = 0\}$$

and the corresponding relation

$$T_{\alpha,\beta} = \{([u],[f]) \in T_{\max} : (u,f) \in \mathcal{T}_{\alpha,\beta}\}.$$

This is a self-adjoint relation, see Section 7.2 in [10].

Since $\overline{u}$ is an eigenfunction associated with $\lambda$ precisely when $u$ is, it follows that $\operatorname{Re} u$ and $\operatorname{Im} u$ are also eigenfunctions. In fact, $\operatorname{Re} u$ and $\operatorname{Im} u$ must be multiples of each other since this is true for their initial values at $a$. It follows that the eigenspace of $\lambda$ is one-dimensional and is spanned by a real solution of $Ju' + qu = \lambda wu$.

Theorem 5.2 (Oscillation theorem). *Suppose $q$ and $w$ satisfy Hypothesis 5.1, $-q_{22} \geq 0$, and $\operatorname{supp} q_{22} = (a,b)$. Then the eigenvalues of $T_{\alpha,\beta}$ are isolated and can accumulate only at $\infty$ and $-\infty$.*

*Let $\lambda_1 < \lambda_2 < \lambda_3 < \ldots$ be the positive eigenvalues and $u(\cdot,\lambda_n)$ be the corresponding real eigenfunctions. Let $\theta(\cdot,0)$ be the Prüfer angle for a real solution of $Ju' + qu = 0$ satisfying the boundary condition at $a$ and $k_r$ the unique nonnegative integer satisfying $\theta(b,0) = k_r\pi + \gamma$ where $\gamma \in (0,\pi]$. If $\gamma < \beta$, then $u_1(x,\lambda_n)$ has precisely $k_r + n - 1$ zeros in $(a,b)$. If $\gamma \geq \beta$, then $u_1(\cdot,\lambda_n)$ has precisely $k_r + n$ zeros in $(a,b)$.*

*If there is a number $\mu_0 < 0$ such that $q_{22} - \mu_0 w_{22} \geq 0$ let $\theta(\cdot,\mu_0)$ be the Prüfer angle for a real solution of $Ju' + (q - \mu_0 w)u = 0$ satisfying the boundary condition at $a$ and $k_\ell$ the unique nonnegative integer satisfying $\theta(b,\mu_0) = \delta - k_\ell\pi$ where $\delta \in [0,\pi)$. Let $\mu_1 > \mu_2 > \mu_3 > \ldots$ be the eigenvalues of $T_{\alpha,\beta}$ strictly below $\mu_0$ and $u(\cdot,\mu_n)$ the corresponding real eigenfunctions. If $\alpha > 0$ then $u_1(\cdot,\mu_n)$ has precisely $k_\ell + n$ or $k_\ell + n - 1$ zeros in $(a,b)$, depending on whether $\delta \leq \beta$ or $\delta > \beta$. If $\alpha = 0$ the count of zeros is smaller by 1.*

Proof. Let $u(\cdot,\lambda)$ be a real solution of $Ju' + qu = \lambda wu$ satisfying the initial condition $u(a,\lambda) = (\sin\alpha, \cos\alpha)^\top$. Then $u(\cdot,\lambda)$ satisfies the boundary condition at $a$. Consider the function $F$ defined by

$$F(\lambda) = (\cos\beta, -\sin\beta)u(b,\lambda).$$

It follows that $\lambda$ is an eigenvalue of $T_{\alpha,\beta}$ precisely if it is a zero of $F$. It was shown in [10] that the entries of $u(b,\cdot)$ are entire. Hence $F$ is also entire and this implies that the eigenvalues are isolated and that they cannot accumulate at any finite point.

Because of Theorem 3.4 and since $-q_{22} \geq 0$, the function $\theta(\cdot, 0)$ can pass (or reach) an integer multiple of $\pi$ only by increasing strictly. In particular, $\theta(b, 0) > 0$. When $\theta(\cdot, 0)$ passes an integer multiple of $\pi$, then $u_1$ has a zero and any zero of $u_1$ is of this kind. It follows that $u_1(\cdot, 0)$ has precisely $k_r$ zeros in the open interval $(a, b)$. For $\lambda > \lambda' > 0$ we have that $p_2 = \lambda w - q \geq \lambda' w - q = p_1$. Now Theorem 3.2 shows that $\theta(b, \lambda) > \theta(b, \lambda')$, i.e., , the function $\lambda \mapsto \theta(b, \lambda)$ is strictly increasing and continuous. If $\gamma < \beta$, then $\theta(b, \cdot)$ will not pass a multiple of $\pi$ before it reaches $k_r\pi + \beta$ at $\lambda_1$ the first positive eigenvalue. If $\gamma \geq \beta$, on the other hand, $\theta(b, \cdot)$ will pass a multiple of $\pi$ before it reaches $(k_r + 1)\pi + \beta$ at $\lambda_1$. The Prüfer angles at $b$ for the eigenfunctions of two consecutive eigenvalues will differ precisely by $\pi$ and will therefore cross a multiple of $\pi$ exactly once.

The proof for the last statement is almost the same, except that the Prüfer angle $\theta(\cdot, \mu)$ is now strictly decreasing as it passes a multiple of $\pi$. □

There may of course be no $\mu_0$ such that $\mu_0 w_{22} - q_{22} \leq 0$, for instance when $q_{22}(x) = -1$ and $w_{22}(x) = x$ on $(0, 1)$. In such a case the last part of Theorem 5.2 becomes void.

Theorem 5.3. *Suppose $q$ and $w$ satisfy Hypothesis 5.1, $w_{22} = 0$, $-q_{22} \geq 0$, and* $\operatorname{supp} q_{22} = (a, b)$. *Then the eigenvalues of $T_{\alpha,\beta}$ are bounded below. The eigenfunction associated with the smallest eigenvalue has no zeros in $(a, b)$ provided* $\operatorname{supp} w_{11} = (a, b)$.

Proof. A slight modification of the previous proof (whose notation and ideas we use freely) shows that the number of zeros of the eigenfunctions associated with consecutive nonpositive eigenvalues will differ by exactly 1 with the eigenfunction associated with the smaller eigenvalue having fewer zeros. It follows that there can be at most $k_r + 1$ nonpositive eigenvalues if $\theta(b, 0) = k_r\pi + \gamma$ with $\gamma \in (0, \pi]$. Hence there is a smallest eigenvalue. Denote the number of zeros in $(a, b)$ of the eigenfunction associated with the smallest eigenvalue by $k_0$.

Since $\theta(\cdot, \lambda)$ can reach an integer multiple of $\pi$ only from below, it follows that $\theta(x, \lambda) \geq 0$ for all $x \in (a, b)$ and all $\lambda \in \mathbb{R}$. We also know from Theorem 3.2 that $\theta(x, \cdot)$ is strictly increasing. Hence the function $\theta_0(x) = \lim_{\lambda \to -\infty} \theta(x, \lambda)$ exists and is nonnegative for every $x \in (a, b)$.

The Prüfer equation of $Ju' + qu = \lambda wu$ reads

$$\theta' = (\sin\theta, \cos\theta)(\lambda w - q) \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}.$$

Since $w_{22} = 0$ and $w \geq 0$ we have, in fact, $w = w_{11}\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Thus

$$(\sin\theta)^2 w_{11} = \frac{1}{\lambda}\Big[\theta' + (\sin\theta, \cos\theta) q \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}\Big] \tag{5.2}$$

and, integrating over $(a, b)$,

$$\begin{aligned} \int_{(a,b)} (\sin\theta)^2 w_{11} &= \frac{\theta(b,\lambda) - \theta(a,\lambda)}{\lambda} + \frac{1}{\lambda}\int_{(a,b)} (\sin\theta, \cos\theta) q \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} \\ &\leq \frac{\alpha}{|\lambda|} + \frac{1}{|\lambda|}\int_{(a,b)} \big(|q_{11}| + |q_{22}| + |q_{12}|\big). \end{aligned}$$

Applying the dominated convergence theorem to pass to the limit as $\lambda$ approaches $-\infty$ shows that $\int_{(a,b)} (\sin\theta_0)^2 w_{11} = 0$. Since $\operatorname{supp} w_{11} = (a, b)$, we get that $\sin\theta_0 = 0$ in a dense subset $S$ of $(a, b)$. Thus $\theta_0(x)$ is an integer multiple of $\pi$ if $x \in S$.

If $\lambda < 0$ and $a < s < t < b$, we get from (5.2)

$$\theta(t, \lambda) - \theta(s, \lambda) \leq \int_{(s,t)} \big( - q_{11}(\sin\theta)^2 - q_{22}(\cos\theta)^2 - q_{12}\sin 2\theta\big).$$

Using again the dominated convergence theorem gives $\theta_0(t) - \theta_0(s) \leq \mu((s, t))$ where $\mu = |q_{11}| + |q_{22}| + |q_{12}|$ is a finite, continuous measure on $(a, b)$. Hence there is a point $x_1 \in (a, b)$ such that $0 \leq \theta_0(x) \leq \theta_0(a) + \mu((a, x_1)) < \pi$ for $x \in (a, x_1)$. Therefore

$\theta_0 = 0$ on $S \cap (a, x_1)$. If $x_2$ satisfies $\mu((x_1, x_2)) < \pi$ we obtain similarly that $\theta_0 = 0$ on $S \cap (x_1, x_2)$. After finitely many steps we may conclude that $\theta_0 = 0$ on $S$. Now let $x$ be an any point in $(a, b]$ and $n \mapsto s_n$ a strictly increasing sequence in $S$ converging to $x$. Then $0 \leq \theta_0(x) \leq \theta_0(s_n) + \mu((s_n, x))$. But $\theta_0(s_n) = 0$ and $\mu((s_n, x))$ converges to 0 as $n$ tends to infinity showing that $\theta_0(x) = 0$. Therefore, finally, $\theta_0 = 0$ on $(a, b]$.

It follows now that $\theta(b, \lambda_0) = \beta \leq \pi$ if $\lambda_0$ is the smallest eigenvalue of $T_{\alpha,\beta}$ and this means that the corresponding eigenfunction $u(\cdot, \lambda_0)$ has no zero in $(a, b)$. $\square$

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# ASYMPTOTICS OF EIGENVALUES FOR FIRST-ORDER SYSTEMS WITH DISTRIBUTIONAL COEFFICIENTS

by

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# ASYMPTOTICS OF EIGENVALUES FOR FIRST-ORDER SYSTEMS WITH DISTRIBUTIONAL COEFFICIENTS

Abstract. We establish an asymptotic formula of the eigenvalues for a $2 \times 2$-system of ordinary equations provided with separated boundary conditions when the coefficients are (possibly singular) continuous, real, and finite measures.

## 1. Introduction

In this paper we prove an asymptotic formula for the eigenvalues of a two points boundary value problem corresponding to a system of ordinary differential equations when the coefficients are continuous (possibly singular), real, and finite measures.

A well-studied case of this eigenvalue problem is the regular Sturm-Liouville problem

$$-(py')' + vy = \lambda ry$$

on an interval $(a, b)$ when $\lambda$ is a spectral parameter and $1/p$, $v$, and $r$ are real and integrable functions, provided with two separated boundary conditions at $a$ and $b$.

For continuous and positive coefficients Hobson [10] showed in 1908 (see also Ince [12], Section 11.4) that the eigenvalues $\lambda_n$ may be labeled so that

$$\lambda_1 < \lambda_2 < \lambda_3 < \dots,$$

and that their asymptotic behavior is given by

$$\lim_{n\to\infty} \frac{\lambda_n^{1/2}}{n} = \pi \bigg/ \int_{(a,b)} \sqrt{r/p}\, dx,$$

where $dx$ is the Lebesgue measure on $(a, b)$.

If only $p > 0$ a.e. on $(a, b)$ and if $r_\pm$, the positive and negative parts of the function $r$, are positive on sets of positive measure, Atkinson and Mingarelli [2] (Theorem 2.3) showed that there exist infinitely many positive (or rather nonnegative) and negative eigenvalues $\lambda_n^\pm$ convergent to $\pm\infty$

$$\ldots < \lambda_3^- < \lambda_2^- < \lambda_1^- < \lambda_1^+ < \lambda_2^+ < \lambda_3^+ < \ldots$$

and that they follow the asymptotic formula

$$\lim_{n\to\infty} \frac{(\pm\lambda_n^\pm)^{1/2}}{n} = \pi \Big/ \int_{(a,b)} \sqrt{r_\pm/p}\, dx.$$

Omitting also the positiveness condition on $p$ and setting

$$\delta^\pm = \int_{(a,b)} \sqrt{r_\pm/p_+}\, dx - \int_{(a,b)} \sqrt{r_\mp/p_-}\, dx \neq 0,$$

Binding and Volkmer [4] (Corollary 4.3) showed that there is a (sub)sequence $\lambda_{n_k}^+$ of positive eigenvalues if $\delta^+ \neq 0$ and a (sub)sequence of negative eigenvalues $\lambda_{n_k}^-$ if $\delta^- \neq 0$ with the asymptotic behavior

$$\lim_{k\to\infty} \frac{(\pm\lambda_{n_k}^\pm)^{1/2}}{k} = \pi/|\delta^\pm|.$$

The first order system

$$Ju' + \left(\begin{smallmatrix} v & 0 \\ 0 & -1/p \end{smallmatrix}\right) u = \lambda \left(\begin{smallmatrix} r & 0 \\ 0 & 0 \end{smallmatrix}\right) u \tag{1.1}$$

naturally arises in the context of the Sturm-Liouville equation if we set $u = (y, py')^\top$, $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Or, more generally, we can consider the first order system

$$Ju' + qu = \lambda w u \tag{1.2}$$

where the off-diagonal entries of the matrices $q$ and $w$ can be nonzero.

Binding and Volkmer [4] (Theorem 4.2) considered the system (1.2) where the entries of $q$ and $w$ are locally integrable real-valued functions and

$$\delta(w) = \int_{\mathcal{P}} \sqrt{\det(w)}\, dx - \int_{\mathcal{N}} \sqrt{\det(w)}\, dx \neq 0,$$

where $\mathcal{P} = \{x \in (a,b) : (w(x) + w(x)^\top)/2 \geq 0\}$ and $\mathcal{N} = \{x \in (a,b) : (w(x) + w(x)^\top)/2 \leq 0\}$. In this case, they proved that there exists a (sub)sequence $\mathbb{Z} \ni k \mapsto \lambda_{n_k}$ of eigenvalues such that

$$\lim_{|k|\to\infty} \frac{\lambda_{n_k}}{k} = \pi/\delta(w).$$

The eigenvalue problem associated with first order systems can also be generalized to allow the coefficients to be measures instead of integrable functions.

In this direction, Bennewitz [3] (Theorem 7.3) considered the system (1.1) where $s = 1/p$, $v$, and $r$ are allowed to be finite measures and $r$ is a positive measure. It is shown that

$$\lim_{n\to\infty} \frac{|\lambda_n^\pm|^{1/2}}{n} = \pi \Big/ \int_{(a,b)} \sqrt{(s/r)_\pm}\, r,$$

where $(s/r)$ is the Radon-Nikodym derivative of the absolutely continuous part of $s$ with respect to the positive measure $r$.

Volkmer [15] (Theorem 5.2) also considered the system (1.1) where $r$ is not required to be a positive measure and

$$\delta^+ = \int_{(a,b)} \sqrt{(s^+/r^+)}\, r^+ - \int_{(a,b)} \sqrt{(s^-/r^-)}\, r^- > 0.$$

Here, $\mu^\pm$ denote the positive and negative part of a measure $\mu$ in its Jordan decomposition and $(s^\pm/r^\pm)$ are Radon-Nikodym derivatives. In this case, there exists an one-sided sequence $\mathbb{Z} \ni k \mapsto \lambda_{n_k}^+$ such that

$$\lim_{k\to\infty} \frac{(\lambda_{n_k}^+)^{1/2}}{k} = \pi/\delta^+.$$

A similar claim about the positive eigenvalue holds if $\delta^+ < 0$. There are also analogous results about the negative eigenvalues.

We note that, in addition to the conditions stated above, both Bennewitz and Volkmer required the same additional condition that $V$ and $R$ have no discontinuities in common with $S$, where $S$, $V$, and $R$ are the antiderivatives of $s = 1/p$, $v$, and

$r$, respectively. This is the Assumption 2.1 in [3] and the condition (5.4) in [15]. Under this condition, the Atkinson condition $(\Delta(x))^2 = 0$ is satisfied, where $\Delta(x) = \begin{pmatrix} 0 & \Delta_s(x) \\ \Delta_v(x)-\lambda\Delta_r(x) & 0 \end{pmatrix}$, while $\Delta_s(x)$, $\Delta_v(x)$, and $\Delta_r(x)$ are the jumps of $s$, $v$, and $r$ at a point $x \in (a, b)$. See also Eckhardt et al [5] (Corollary 7.2) and [6] (Theorem 3.4) for direct applications of Bennewitz's and Volkmer's results.

Our goal in this paper is to acquire an asymptotic formula for the first order systems (1.1) where the entries, including the off-diagonal ones, of the matrices $q$ and $w$ are now also allowed to be finite and continuous measures. For a technical difficulty explained in Section 3, we shall only deal with continuous measures, i.e., atomic parts of the coefficients are excluded.

We end this section by a brief summary for the content of each section. In Section 2 we discuss some basic facts about distributions of order zero and their connection to measures. In Section 3 we define Prüfer angles and their Kepler transformations. We shall also derive the differential equations of the Prüfer angles and their transformed versions, which are the main tool to study the asymptotics of the Prüfer angles in Section 4. In Section 5 we shall derive our main results (Theorem 5.1 & 5.2) about the asymptotic formula for the eigenvalues of the first order system (1.1) and its application.

## 2. Some Basic Facts About Distributions

Let $(a, b)$, $-\infty \le a < b \le \infty$, be an open interval on the real line. We call $\phi$ a *test function* if it is an infinitely often continuously differentiable function and its support is a subset of $(a, b)$. We denote the space of all test functions on $(a, b)$ by $C_c^\infty((a, b))$ or sometimes by $\mathcal{D}((a, b))$. A linear functional $p$ on $C_c^\infty((a, b))$ is called a *distribution* if for every compact subset $K$ of $(a, b)$ there are numbers $C \ge 0$ and $k \in \mathbb{N}_0$ such that

$$|p(\phi)| \le C \sum_{j=0}^{k} \|\phi^{(j)}\|_\infty$$

for all test functions $\phi$ such that $\operatorname{supp}\phi \subset K$. The distribution $p$ is called a *distribution of order* 0 if the number $k$ can be chosen as 0 independently from $K$. We denote $\mathcal{D}'((a,b))$ and $\mathcal{D}'^0((a,b))$ to be the space of distributions and distributions of order 0 on $(a,b)$, respectively.

The distribution $p$ is said to vanish on an open subset $U$ of $(a,b)$ if $p(\phi) = 0$ for all test function $\phi$ with $\operatorname{supp}\phi \subset U$. The *support* of $p$ is defined to be the complement of the largest open set in $(a,b)$ on which $p$ vanishes.

The *derivative*, of a distribution $p$ is defined via $p'(\phi) = -p(\phi')$. It is an important fact that antiderivatives of $p$ exist and the difference of any two of such antiderivatives is a constant distribution, i.e., the distribution $\phi \mapsto C\int \phi\, dx$, for some constant $C$. The antiderivative of a distribution of order 0 is also of order 0.

By $\mathrm{BV}_{\mathrm{loc}}((a,b))$ we denote the set of all functions of locally bounded variation on $(a,b)$. Let $f \in \mathrm{BV}_{\mathrm{loc}}((a,b))$ and $df$ be the Lebesgue-Stieltjes measure it generates, then two distributions of order 0 naturally arise: $\phi \mapsto \int f\phi\, dx$ and $\phi \mapsto \int \phi\, df$. We will oftentimes identify the function $f$ with the former distribution $\phi \mapsto \int f\phi\, dx$. That the latter distribution is the derivative of the former one is a consequence of the Integration by Parts formula.

Any function of locally bounded variation $f$ has at most countably many discontinuities $x \in (a,b)$ such that $f^+(x) \neq f^-(x)$, where $f^+$ and $f^-$ are the right continuous and the left continuous version of the function $f$, respectively. The function $f^\#$ defined as $f^\#(x) = (f^+(x) + f^-(x))/2$ is said to be the *balanced* version of $f$. $f$ is said to be a *balanced* function if $f = f^\#$.

We can multiple a function of locally bounded variation with a distribution of order zero. This is thanks to an important fact that antiderivatives of a distribution of order 0 are functions of locally bounded variation. Let $f \in \mathrm{BV}_{\mathrm{loc}}((a,b))$ and $p \in \mathcal{D}'^0((a,b))$ with an antiderivative $P \in \mathrm{BV}_{\mathrm{loc}}((a,b))$, then the product of $f$ and $p$ is the distribution of order 0 defined by $\phi \mapsto (fp)(\phi) = (pf)(\phi) = \int f\phi\, dP$.

For the distributional derivative of functions of locally bounded variation we also have a product rule, e.g., see Hewitt and Stromberg [9] (Theorem 21.67 and Remark 21.68) and a chain rule which is first established by Vol'pert [16], see also Vol'pert and Hudjaev [17] and Ambrosio, Fusco, and Pallara [1] (Theorem 3.96).

THEOREM 2.1 (Chain Rule I). *Suppose* $u \in \mathrm{BV}_{\mathrm{loc}}((a,b))$, $V$ *is an open set in* $\mathbb{C}$ *containing the range of* $u$, *and* $f \in C^1(V)$. *Then* $f \circ u \in \mathrm{BV}_{\mathrm{loc}}((a,b))$ *and*

$$(f \circ u)' = \left[ \int_0^1 f'(su^+(\cdot) + (1-s)u^-(\cdot))ds \right] u'.$$

COROLLARY 2.2 (Chain Rule II). *If* $u \in \mathrm{BV}_{\mathrm{loc}}((a,b))$ *is continuous and* $f \in C^1(V)$, *then* $(f \circ u)' = (f' \circ u)u'$.

COROLLARY 2.3. *If* $u, v \in \mathrm{BV}_{\mathrm{loc}}((a,b))$ *then* $uv \in \mathrm{BV}_{\mathrm{loc}}((a,b))$ *and*

$$(uv)' = u^{\#}v' + u'v^{\#}. \tag{2.1}$$

*If, in addition,* $u$ *and* $v$ *are continuous then*

$$(uv)' = uv' + u'v.$$

PROOF. Let $f(x) = x^2/4$. Differentiating both sides, by Theorem 2.1, of the identity $uv = f(u+v) - f(u-v)$ we get equation (2.1). □

The *conjugate* of a distribution $p$, denoted by $\overline{p}$, is defined by $\overline{p}(\phi) = \overline{p(\overline{\phi})}$. A distribution is called *real* if $p = \overline{p}$. Equivalently, $p$ is real if $p(\phi) \in \mathbb{R}$ whenever $\phi$ is real-valued. $p$ is called nonnegative, written as $p \geq 0$, if $p(\phi) \geq 0$ for all $\phi \geq 0$.

We are also interested in matrix-valued distributions. As usual the entry in row $j$ and column $k$ of a matrix $r$ is denoted by $r_{jk}$, and $u_k$ denotes respectively the $k$-th component of the vector $u$. We are interested in the case of $2 \times 2$-matrices whose entries are distributions of order 0 on $(a,b)$, a space which we denote by $\mathcal{D}'^0((a,b))^{2\times 2}$. A matrix $r \in \mathcal{D}'^0((a,b))^{2\times 2}$ is called *hermitian* (resp. *symmetric*) if $\overline{r_{jk}} = r_{kj}$ (resp. $r_{jk} = r_{kj}$) for $j,k = 1,2$. That $r$ is hermitian is equivalent to the requirement that

$z^*rz$ is real for all $z \in \mathbb{C}^2$. One calls $r$ *nonnegative*, if $z^*rz$ is nonnegative for all $z \in \mathbb{C}^2$. If $r_1, r_2 \in \mathcal{D}'^0((a,b))^{2\times 2}$, we write $r_1 \geq r_2$ if $r_1 - r_2$ is nonnegative. We say that $r$ *vanishes* on an open set $U \subset (a,b)$ if this is true for each of its entries.

## 3. Prüfer Angle and Its Kepler Transformation

In this paper we are interested in matrix-valued distributions in the subset $\mathcal{C}$ of $\mathcal{D}'^0((a,b))^{2\times 2}$ whose entries are real and correspond to finite and continuous measures.

Let $(a,b)$ be an interval on the real line. Consider the equation

$$Ju' + pu = 0, \tag{3.1}$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $p \in \mathcal{C}$.

Let $u$ be a nontrivial $\mathbb{R}^2$-valued balanced solution of the equation (3.1). Since $u^+(x) - u^-(x) = u'(\{x\}) = -J^{-1}\Delta_p(x)u(x) = 0$, $u$ is a continuous function on $(a,b)$. The Prüfer angle $\theta$ for the solution $u$ is defined as the unique continuous function on $(a,b)$ satisfying the following conditions

$$u(x) = r(x)\begin{pmatrix} \sin\theta(x) \\ \cos\theta(x) \end{pmatrix} \in \mathbb{R}^2, r(x) = ||u(x)||, \text{ and } \theta(a) \in [0, 2\pi).$$

Here, $||\cdot||$ is the Euclidean norm in $\mathbb{R}^2$. It is without loss of generality to assume that $\theta(a) \in [0,\pi)$ since if $\theta(a) \in [\pi, 2\pi)$ we can consider $-u$ instead.

Since $\theta$ and $r$ are continuous the chain rule for distributional derivatives, Corollary 2.2, shows that

$$0 = Ju' + pu = r'J\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} + r\theta' J\begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix} + rp\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}.$$

Multiplying from the left by the row vector $(\sin\theta, \cos\theta)$ we get

$$0 = r\theta' + r(\sin\theta, \cos\theta)p\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}$$

Since $u$ is a nontrivial solution of the equation (3.1), $r \neq 0$ on all $(a, b)$. Thus we obtain the *Prüfer equation* for $\theta$

$$\theta' = -(\sin\theta, \cos\theta) p \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}, \tag{3.2}$$

and, similarly, the second Prüfer equation for $r$

$$\frac{r'}{r} = (\sin\theta, \cos\theta) p \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}. \tag{3.3}$$

THEOREM 3.1 (Kepler transformation). *Let $k, \psi \in C^1([a, b])$, $k > 0$, and define a pair of real-valued functions $\phi, \rho \in C^1([a, b])$ via*

$$\rho \sin\phi = k \sin\psi, \quad \rho\cos\phi = \cos\psi, \quad |\phi(a) - \psi(a)| < 2\pi.$$

*Then $|\phi(x) - \psi(x)| < \pi/2$ for all $x \in [a, b]$, and $\phi$ satisfies the equation*

$$\phi' = \frac{k}{\rho^2}\psi' + \frac{k'}{k}\sin\phi\cos\phi.$$

PROOF. For any fixed $x \in [a, b]$ we have

$$\begin{pmatrix} \sin\phi(x) \\ \cos\phi(x) \end{pmatrix} = \begin{pmatrix} k(x)/\rho(x) & 0 \\ 0 & 1/\rho(x) \end{pmatrix} \begin{pmatrix} \sin\psi(x) \\ \cos\psi(x) \end{pmatrix}.$$

This means the unit vector $e_\psi(x) = (\sin\psi(x), \cos\psi(x))^\top$ is being mapped to the unit vector $e_\phi(x) = (\sin\phi(x), \cos\phi(x))^\top$ via the linear transformation

$$T(x) : \mathbb{R}^2 \to \mathbb{R}^2 \quad v \mapsto \begin{pmatrix} k(x)/\rho(x) & 0 \\ 0 & 1/\rho(x) \end{pmatrix} v.$$

Since $k(x)/\rho(x) > 0$ and $1/\rho(x) > 0$, $e_\phi(x)$ stays in the same quadrant as of $e_\psi(x)$. Hence $|\phi(x) - \psi(x)| < \pi/2$ for all $x \in [a, b]$.

Differentiating both sides of $\rho^2 = k^2(\sin\psi)^2 + (\cos\psi)^2$ we get

$$\rho' = \frac{1}{\rho}\left(kk'(\sin\psi)^2 + k^2\psi'\sin\psi\cos\psi - \psi'\sin\psi\cos\psi\right).$$

We also know that $\phi'\sin\phi = \frac{1}{\rho^2}\left(\psi'\rho\sin\psi + \rho'\cos\psi\right)$ by differentiating both sides of $\cos\phi = \cos\psi/\rho$. Hence, combining these two facts we get

$$\phi' = \frac{k}{\rho^2}\psi' + \frac{k'}{k}\sin\phi\cos\phi.$$

□

## 4. Asymptotic Behavior Of The Prüfer Angle

Let $\Omega$ be some domain in $\mathbb{C}$ containing the ray $(\lambda_0, \infty)$ for some given $\lambda_0 \in \mathbb{R}$. In this section we consider the first order system

$$Ju' + q(\lambda)u = \lambda w u, \tag{4.1}$$

on an interval $(a, b)$. We shall also adopt the following hypothesis throughout this section.

HYPOTHESIS 4.1. *$w \in \mathcal{C}$ be symmetric. $q$ is an analytic*[1] *function from $\Omega$ to $\mathcal{C} \subset \mathcal{D}'^0((a,b))$ such that $q(\lambda)$ is symmetric for any $\lambda \in \Omega$ and that*

$$\lim_{\lambda\to\infty} \frac{1}{\lambda}\int_{(a,b)} |q_{11}(\lambda) + q_{22}(\lambda) + q_{12}(\lambda)| = 0. \tag{4.2}$$

Let $u(\cdot, \lambda)$ be a solution of (4.1) satisfying the following separated boundary conditions at $a$ and $b$

$$(\cos\alpha, -\sin\alpha)u(a) = (\cos\beta, -\sin\beta)u(b) = 0, \tag{4.3}$$

for some given $\alpha \in [0, \pi)$ and $\beta \in (0, \pi]$. We also let $\theta(\cdot, \lambda)$ be the Prüfer angle of $u$ with $\theta(x, \lambda) = \alpha$.

[1] See [8] Section A.4.

The main result of this section is Theorem 4.7 which shows how fast the Prüfer angle $\theta$ increases as the spectral variable $\lambda$ approaches infinity along the ray $(\lambda_0, \infty)$. In order to do so we need some preparations.

In this paper, we adopt a slightly nonstandard definition of the Radon-Nikodym derivative. Suppose $\lambda$ and $\mu$ are two $\sigma$-finite measures, $\lambda$ is a signed measure, and $\mu$ is a positive measure. The Lebesgue decomposition of $\lambda$ with respect to $\mu$ is given by

$$\lambda = \lambda_{ac} + \lambda_{sc} + \lambda_d$$

where $\lambda_{ac}$ is absolutely continuous with respect to $\mu$, $\lambda_{sc}$ and $\lambda_d$ are singular with respect to $\mu$, the former being a continuous measure and the latter a discrete measure. The *Radon-Nikodym derivative* of $\lambda$ with respect to $\mu$, denoted by $(\lambda/\mu)$, is defined to be $(\lambda_{ac}/\mu)$, i.e., the standard Radon-Nikodym derivative of $\lambda_{ac}$ with respect to $\mu$ (see e.g., [**14**] Chapter 6).

If $\mu$ is instead a negative measure then we define the Radon-Nikodym derivative of $\lambda$ with respect to $\mu$ as $(\lambda/\mu) = -(\lambda/(-\mu)) = ((-\lambda)/(-\mu))$. If $\lambda \ll \mu$ then $(\lambda/\mu)$ is just the standard Radon-Nikodym derivative. If $\lambda_d = 0$, i.e., $\lambda$ is a continuous measure, then the Lebesgue decomposition of $\lambda$ with respect to $\mu$ is of the form

$$\lambda = (\lambda/\mu)\mu + \lambda_{sc}, \quad \text{where} \quad \lambda_{sc} \perp \mu.$$

LEMMA 4.2. *Let $w_{11}, w_{22}$ be two nonnegative distributions of order zero on $(a,b)$ associating with finite measures. For any $\varepsilon > 0$, there exists $k \in C^1((a,b))$, $k > 0$ such that*

$$\int_{(a,b)} \left| k w_{22} - \frac{1}{k} w_{11} \right| \leq \varepsilon \quad \textit{and} \quad \int_{(a,b)} \left| k w_{22} - \sqrt{(w_{11}/w_{22})}\, w_{22} \right| \leq \varepsilon. \tag{4.4}$$

Proof. Let $\varepsilon > 0$. Since $w_{11}$ and $w_{22}$ are nonnegative, $w_{11} \ll w_{22} + \varepsilon w_{11}$. Let $u = (w_{11}/(w_{22} + \varepsilon w_{11}))$, i.e., $u$ is the Radon-Nikodym derivative of $w_{11}$ with respect to $w_{22} + \varepsilon w_{11}$. We can also assume that $u$ is positive and defined on all $(a, b)$.

We shall show that $u \in L^1(w_{11} + w_{22})$. We have

$$\int_{(a,b)} w_{11} = \int_{(a,b)} (w_{11}/(w_{22} + \varepsilon w_{11}))(w_{22} + \varepsilon w_{11})$$
$$= \int_{(a,b)} u(w_{22} + \varepsilon w_{11}) = \int_{(a,b)} uw_{22} + \varepsilon \int_{(a,b)} uw_{11} \geq \varepsilon \int_{(a,b)} uw_{11}.$$

Hence $\int uw_{11} \leq \frac{1}{\varepsilon} \int w_{11}$. And thus

$$\int_{(a,b)} u(w_{11} + w_{22}) = \int_{(a,b)} u(w_{22} + \varepsilon w_{11}) + (1 - \varepsilon) \int_{(a,b)} uw_{11}$$
$$\leq \int_{(a,b)} u(w_{22} + \varepsilon w_{11}) + \frac{1 - \varepsilon}{\varepsilon} \int_{(a,b)} w_{11}$$
$$= \int_{(a,b)} w_{11} + \frac{1 - \varepsilon}{\varepsilon} \int_{(a,b)} w_{11} = \frac{1}{\varepsilon} \int_{(a,b)} w_{11} < \infty.$$

Since $C_c^1((a, b))$ is dense in $C_c^0((a, b))$ under the sup norm (see, e.g., Theorem 1.3.2 of [11]) and $C_c^0((a, b))$ is dense in $L^1(w_{11} + w_{22})$ (see, e.g., , Theorem 3.14 of [14]), $C_c^1((a, b))$ is dense in $L^1(w_{11} + w_{22})$. So, there exists $v \in C_c^1((a, b))$ such that

$$\int_{(a,b)} |u - v|(w_{11} + w_{22}) \leq \varepsilon^2.$$

Moreover, even though unstated in these two references, if $u > 0$ then one can choose $v$ so that $v \geq 0$.

Define on $(a, b)$ the function $k(x) = \sqrt{v(x) + \varepsilon^2}$. We know that $k \in C^1((a, b))$, $k > 0$, $1/k \leq 1/\varepsilon$, and $\varepsilon/k \leq 1$.

From the definition of $u$ we have $w_{11} - uw_{22} = \varepsilon u w_{11}$, and thus

$$\begin{aligned}
\int_{(a,b)} \left| k w_{22} - \frac{1}{k} w_{11} \right| &\\
\leq \int_{(a,b)} \frac{1}{k} |w_{11} - uw_{22}| + \int_{(a,b)} \frac{1}{k} |u - v| w_{22} + \int_{(a,b)} \frac{1}{k} |v - k^2| w_{22} &\\
\leq \int_{(a,b)} \frac{1}{k} \varepsilon u w_{11} + \int_{(a,b)} \frac{1}{k} |u - v| (w_{11} + w_{22}) + \int_{(a,b)} \frac{\varepsilon^2}{k} w_{22} &\\
\leq \int_{(a,b)} \frac{1}{k} \varepsilon u w_{11} + \frac{1}{\varepsilon} \int_{(a,b)} |u - v| (w_{11} + w_{22}) + \varepsilon \int_{(a,b)} \frac{\varepsilon}{k} w_{22} &\\
\leq \int_{(a,b)} \frac{1}{k} \varepsilon u w_{11} + \varepsilon \left( 1 + \int_{(a,b)} w_{22} \right). &
\end{aligned}$$

Let $I_1 = \{x \in (a,b) : u(x) \leq 2k(x)^2\}$ and $I_2 = (a,b) \setminus I_1$. If $x \in I_2$ then $u(x) > 2k(x)^2 = 2v(x) + 2\varepsilon^2 > 2v(x)$, thus

$$u(x) = 2u(x) - u(x) \leq 2u(x) - 2v(x) \leq 2|u(x) - v(x)|.$$

We have

$$\begin{aligned}
\int_{I_1} \frac{1}{k} \varepsilon u w_{11} &= \int_{I_1} \sqrt{\varepsilon} \sqrt{\frac{u}{k^2}} \sqrt{\varepsilon} \sqrt{(w_{11}/(w_{22} + \varepsilon w_{11}))}\, w_{11} \\
&\leq \sqrt{2\varepsilon} \int_{I_1} \sqrt{(\varepsilon w_{11}/(w_{22} + \varepsilon w_{11}))}\, w_{11} \leq \sqrt{2\varepsilon} \int_{I_1} w_{11}.
\end{aligned}$$

Note that in the second-to-last inequality we are multiplying a constant number with a Radon-Nikodym derivative, and that the last inequality is due to the fact that the Radon-Nikodym derivative $(\varepsilon w_{11}/(w_{22} + \varepsilon w_{11}))$ has absolute value no larger than 1.

We also have

$$\int_{I_2} \frac{1}{k} \varepsilon u w_{11} \leq \int_{I_2} \frac{\varepsilon}{k} 2|u - v| w_{11} \leq 2 \int_{(a,b)} |u - v|(w_{11} + w_{22}) \leq 2\varepsilon^2.$$

Thus

$$\int_{(a,b)} \left| k w_{22} - \frac{1}{k} w_{11} \right| \leq \sqrt{2\varepsilon} \int_{I_1} w_{11} + 2\varepsilon^2 + \varepsilon \left( 1 + \int_{(a,b)} w_{22} \right).$$

After recalibrating, the right hand side can be replaced by $\varepsilon$.

Let $(w_{11}/w_{22})$ be the Radon-Nikodym derivative of $w_{11}$ with respect to $w_{22}$, i.e., $w_{11} = (w_{11}/w_{22})w_{22} + s$ and $w_{22} \perp s$. There exists a Borel set $\Omega \subset (a, b)$ such that $w_{22}((a, b) \setminus \Omega) = 0$ and $s(\Omega) = 0$. Let $h = \chi_\Omega k$ and $g = \chi_\Omega \frac{1}{k}$, then $hw_{22} = kw_{22}$, $gs = 0$, and $hg = \chi_\Omega$. We also have

$$gw_{11} = g((w_{11}/w_{22})w_{22} + s) = g(w_{11}/w_{22})w_{22} + gs = g(w_{11}/w_{22})w_{22}.$$

We now get

$$\begin{aligned}
&\int_{(a,b)} \left|kw_{22} - \sqrt{(w_{11}/w_{22})}\, w_{22}\right| \\
&\qquad = \int_\Omega \left|hw_{22} - \sqrt{(w_{11}/w_{22})}\, w_{22}\right| = \int_\Omega \left|h - \sqrt{(w_{11}/w_{22})}\right| w_{22} \\
&\qquad \le \int_\Omega \left|h - \sqrt{(w_{11}/w_{22})}\right| \frac{h + \sqrt{(w_{11}/w_{22})}}{h} w_{22} \\
&\qquad = \int_\Omega |h - g(w_{11}/w_{22})|\, w_{22} = \int_\Omega |kw_{22} - gw_{11}| \\
&\qquad = \int_\Omega \left|kw_{22} - \frac{1}{k}w_{11}\right| \le \int_{(a,b)} \left|kw_{22} - \frac{1}{k}w_{11}\right| \le \varepsilon.
\end{aligned}$$

□

To illustrate Theorem 4.2, we consider an example with $w_{11} = \delta_0$, $w_{22} = dx$, and $(a, b) = \mathbb{R}$, where $\delta_0$ is the delta distribution concentrated at 0. Let $\varepsilon \in (0, 1)$. By the Besicovitch differentiation theorem (see, e.g., Theorem 2.22 of [1]), $u = (\delta_0/(dx + \varepsilon\delta_0)) = \frac{1}{\varepsilon}\chi_{\{0\}}$. Let $\phi \in C_c^\infty(\mathbb{R})$ be such that, $\phi \ge 0$, $\phi(x) = \phi(-x)$, $\phi(0) = 1$, $\operatorname{supp}\phi = [-1, 1]$, and $\int_\mathbb{R} \phi\, dx = 1$. For any $\eta \in (0, 1/\varepsilon)$, let $v_\eta(x) = \left(\frac{1}{\varepsilon} - \eta\right)\phi\left(\frac{x}{\eta}\right) \in C_c^1(\mathbb{R})$. We have

$$\begin{aligned}
||u - v_\eta||_{L^1(dx+\delta_0)} &= \int_\mathbb{R} \left|\frac{1}{\varepsilon}\chi_{\{0\}}(x) - \left(\frac{1}{\varepsilon} - \eta\right)\phi\left(\frac{x}{\eta}\right)\right| (dx + \delta_0(x)) \\
&= \left(\frac{1}{\varepsilon} - \eta\right)\int_\mathbb{R} \phi\left(\frac{x}{\eta}\right) dx + \left|\frac{1}{\varepsilon} - \left(\frac{1}{\varepsilon} - \eta\right)\right| = \left(\frac{1}{\varepsilon} - \eta\right)\eta + \eta \le \left(\frac{1}{\varepsilon} + 1\right)\eta.
\end{aligned}$$

Thus, $||u - v_\eta||_{L^1(dx+\delta_0)} \to 0$ as $\eta \to 0$. Let $\eta_0 > 0$ be such that $||u - v_{\eta_0}||_{L^1(dx+\delta_0)} \leq \varepsilon^2$. Choose $v = v_{\eta_0}$ and $k = \sqrt{v_{\eta_0} + \varepsilon^2}$ then the two inequalities in equation (4.4) will be satisfied.

For any real-valued Borel measure $\nu$ on $(a, b)$, by the Hahn decomposition theorem (see, e.g., , Theorem 3.3 of [7]), there exist disjoint Borel sets $\mathcal{P}(\nu)$ and $\mathcal{N}(\nu)$ such that $\mathcal{P} \cup \mathcal{N} = (a, b)$, $\nu(E) \geq 0$ for all $E \subset \mathcal{P}$, and $\nu(E) \leq 0$ for all $E \subset \mathcal{N}$. Such a pair $\mathcal{P}(\nu)$, $\mathcal{N}(\nu)$ is not unique. Also, by the Jordan decomposition theorem (See, e.g., , Theorem 3.4 of [7]), there exist unique positive measures $\nu^+$, $\nu^-$ such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$. Furthermore, $\nu^+$ and $\nu^-$ are concentrated on $\mathcal{P}(\nu)$ and $\mathcal{N}(\nu)$, respectively.

LEMMA 4.3. *Let $\mu$ be a regular Borel measure on $(a, b)$. If $E \subset (a, b)$ is a Borel set such that $\mu(E) < \infty$ then for any $\varepsilon > 0$, there exists a finite union $J$ of open disjoint subintervals of $(a, b)$ such that $\mu(E\Delta J) \leq \varepsilon$. Here, $E\Delta J$ is the symmetric difference between the two sets $E$ and $J$.*

PROOF. Let $\varepsilon > 0$, since $\mu(E) < \infty$ and $\mu$ is outer regular, there exists an open set $V$ which is the union of a sequence of open disjoint subintervals $n \mapsto J_n$ such that $E \subset V = \bigcup_{n=1}^{\infty} J_n$ and $\sum_{n=1}^{\infty} \mu(J_n) < \mu(E) + \epsilon/2$. Since $\mu(E) < \infty$, this series is convergent, so there exists $N \in \mathbb{N}$ such that $\sum_{n=N+1}^{\infty} \mu(J_n) < \varepsilon/2$. Let $J = \bigcup_{n=1}^{N} J_n$. We have

$$\mu(J \setminus E) \leq \mu(\bigcup_{n=1}^{\infty} J_n \setminus E) = \mu(\bigcup_{n=1}^{\infty} J_n) - \mu(E) \leq \sum_{n=1}^{\infty} \mu(J_n) - \mu(E) \leq \varepsilon/2.$$

Also

$$\mu(E \setminus J) \leq \mu(\bigcup_{n=1}^{\infty} J_n \setminus J) \leq \mu(\bigcup_{n=N+1}^{\infty} J_n) \leq \sum_{n=N+1}^{\infty} \mu(J_n) < \varepsilon/2.$$

Thus $\mu(E\Delta J) = \mu(E \setminus J) + \mu(J \setminus E) < \varepsilon$. □

LEMMA 4.4. *Let $w_{11}, w_{22}$ be two distributions of order zero on $(a, b)$ associating with finite measures. For any $\varepsilon > 0$, there exist a function $k \in C^1((a, b)), k > 0$ and a* *finite union of open intervals $J$ such that*

$$\int_J \left|kw_{22} - \frac{1}{k}w_{11}\right| \le \varepsilon \text{ and } \left|\int_J kw_{22} - \int_{(a,b)} \sqrt{(w_{11}^+/w_{22}^+)}\, w_{22}^+\right| \le \varepsilon.$$

Proof. By Lemma 4.2, there exists a $k \in C(a,b), k > 0$

$$\int_{(a,b)} \left|kw_{22}^+ - \frac{1}{k}w_{11}^+\right| \le \varepsilon \text{ and } \int_{(a,b)} \left|kw_{22}^+ - \sqrt{(w_{11}^+/w_{22}^+)}\, w_{22}^+\right| \le \varepsilon.$$

Let $J_1 = \mathcal{P}(w_{11}) \cap \mathcal{P}(w_{22}),\ J_2 = \mathcal{N}(w_{11}) \cap \mathcal{P}(w_{22})$, and note that $J_1 \cap J_2 = \emptyset$, $J_1 \cup J_2 = \mathcal{P}(w_{22})$.

Since Lebesgue-Stieltjes measures are Radon, they are regular (See, e.g., , Corollary 7.6 of [7]) and so are their total variation measures (See, e.g, Proposition 7.16 of [7]). Applying Lemma 4.3 for the regular measure $|kw_{22}| + |\frac{1}{k}w_{11} - kw_{22}|$, there exists a finite union $J$ of open subintervals of $(a,b)$ such that

$$\int_{J_1 \Delta J} |kw_{22}| \le \varepsilon \text{ and } \int_{J_1 \Delta J} \left|\frac{1}{k}w_{11} - kw_{22}\right| \le \varepsilon.$$

We have

$$\int_{J_1} \left|kw_{22} - \frac{1}{k}w_{11}\right| = \int_{J_1} \left|kw_{22}^+ - \frac{1}{k}w_{11}^+\right| \le \int_{(a,b)} \left|kw_{22}^+ - \frac{1}{k}w_{11}^+\right| \le \varepsilon.$$

Hence

$$\begin{aligned}\int_J \left|kw_{22} - \frac{1}{k}w_{11}\right| &= \int_{J \setminus J_1} \left|kw_{22} - \frac{1}{k}w_{11}\right| + \int_{J \cap J_1} \left|kw_{22} - \frac{1}{k}w_{11}\right| \\ &\le \int_{J \Delta J_1} \left|kw_{22} - \frac{1}{k}w_{11}\right| + \int_{J_1} \left|kw_{22} - \frac{1}{k}w_{11}\right| \le 2\varepsilon.\end{aligned}$$

Since $w_{11}^+(E) = 0$ for all $E \subset \mathcal{N}(w_{11})$ and thus for all $E \subset J_2$, We have

$$\int_{J_2} kw_{22}^+ = \int_{J_2} \left|kw_{22}^+ - \frac{1}{k}w_{11}^+\right| \le \int_{(a,b)} \left|kw_{22}^+ - \frac{1}{k}w_{11}^+\right| \le \varepsilon.$$

Since $J_1 \cup J_2 = \mathcal{P}(w_{22})$ and $J_1 \cap J_2 = \emptyset$, we have

$$\begin{aligned}
&\left| \int_{J_1} kw_{22} - \int_{(a,b)} \sqrt{(w_{11}^+/w_{22}^+)}\, w_{22}^+ \right| \\
&\qquad = \left| \int_{(a,b)} kw_{22}^+ - \int_{(a,b)} \sqrt{(w_{11}^+/w_{22}^+)}\, w_{22}^+ - \int_{J_2} kw_{22}^+ \right| \\
&\qquad\qquad \leq \int_{(a,b)} |kw_{22}^+ - \sqrt{(w_{11}^+/w_{22}^+)}\, w_{22}^+| + \int_{J_2} kw_{22}^+ \leq 2\varepsilon.
\end{aligned}$$

This inequality is equivalent to

$$-2\varepsilon + \int_{(a,b)} \sqrt{(w_{11}^+/w_{22}^+)}\, w_{22}^+ \leq \int_{J_1} kw_{22} \leq 2\varepsilon + \int_{(a,b)} \sqrt{(w_{11}^+/w_{22}^+)}\, w_{22}^+.$$

Since $k > 0$ and $w_{22}(E) \geq 0$ whenever $E \subset J_1$, we have

$$\begin{aligned}
\int_J kw_{22} &= \int_{J \setminus J_1} kw_{22} + \int_{J \cap J_1} kw_{22} \\
&\leq \int_{J \Delta J_1} |kw_{22}| + \int_{J_1} kw_{22} \leq 3\varepsilon + \int_{(a,b)} \sqrt{(w_{11}^+/w_{22}^+)}\, w_{22}^+.
\end{aligned} \tag{4.5}$$

Also

$$\begin{aligned}
\int_J kw_{22} &= \int_{J \setminus J_1} kw_{22} + \left( \int_{J \cap J_1} kw_{22} + \int_{J_1 \setminus J} kw_{22} \right) - \int_{J_1 \setminus J} kw_{22} \\
&\geq - \int_{J \setminus J_1} |kw_{22}| + \int_{J_1} kw_{22} - \int_{J_1 \setminus J} |kw_{22}| \\
&= - \int_{J \Delta J_1} |kw_{22}| + \int_{J_1} kw_{22} \geq -3\varepsilon + \int_{(a,b)} \sqrt{(w_{11}^+/w_{22}^+)}\, w_{22}^+.
\end{aligned} \tag{4.6}$$

Combining (4.5) and (4.6) we have

$$\left| \int_J kw_{22} - \int_{(a,b)} \sqrt{(w_{11}^+/w_{22}^+)}\, w_{22}^+ \right| \leq 3\varepsilon.$$

□

For a $(2 \times 2)$ matrix-valued measure $w$ on $(a,b)$, we denote $\sqrt{\det(w)}$ to be the distribution of order zero on a Borel subset $\Omega$ of $(a,b)$ defined via

$$\sqrt{\det(w)} = \sqrt{(w_{11}/w_{22}) - (w_{12}/w_{22})^2}\, |w_{22}|,$$

as long as the Radon-Nikodym derivatives are well-defined (i.e., $w_{22}$ is either a positive or a negative measure on $\Omega$ and that the function under the square-root is nonnegative).

For any $(2 \times 2)$ matrix-valued measure $w$ on $(a,b)$, let $\mathrm{n} = |w_{11}| + |w_{12}| + |w_{22}|$, and $(w/\mathrm{n})$ be the matrix of the Radon-Nikodym derivatives of entries of $w$ with respect to n. Define

$$\mathcal{P} = \{x \in (a,b) : (w/\mathrm{n})(x) \geq 0\}, \tag{4.7}$$

$$\mathcal{N} = \{x \in (a,b) : (w/\mathrm{n})(x) \leq 0\}, \tag{4.8}$$

i.e., the set of points $x \in (a,b)$ at which the matrix $(w/\mathrm{n})(x)$ is semi-positive definite or semi-negative definite.

LEMMA 4.5 (Chain Rule). *Let $w \in D'^0((a,b))^{2\times 2}$ be symmetric and $\tilde{w} = Q^\top w Q$, for some orthogonal-matrix-valued function $Q$ whose components are locally integrable on $(a,b)$. Then each entry of $\tilde{w}$ is absolutely continuous with respect to* n, *and there exist Borel sets $\tilde{C}^p_{11}, \tilde{C}^p_{12} \subset \mathcal{P}$ (resp. $\tilde{C}^n_{11}, \tilde{C}^n_{12} \subset \mathcal{N}$) such that*

$$(\tilde{w}_{11}/\tilde{w}_{22})(\tilde{w}_{22}/\mathrm{n}) = (\tilde{w}_{11}/\mathrm{n})$$

*almost everywhere with respect to* n *on $\tilde{C}^p_{11}$ (resp. $\tilde{C}^n_{11}$), and*

$$(\tilde{w}_{12}/\tilde{w}_{22})(\tilde{w}_{22}/\mathrm{n}) = (\tilde{w}_{12}/\mathrm{n})$$

*almost everywhere with respect to* n *on $\tilde{C}^p_{12}$ (resp. $\tilde{C}^n_{12}$). Furthermore, the measure $\chi_{\mathcal{P}}\tilde{w}_{22}$ (resp. $\chi_{\mathcal{N}}\tilde{w}_{22}$) is concentrated on $\tilde{C}^p_{11}, \tilde{C}^p_{12}$ (resp. $\tilde{C}^n_{11}, \tilde{C}^n_{12}$).*

Proof. Let $\mathrm{n} = |w_{11}| + |w_{12}| + |w_{22}|$ and $\mathcal{P}$, $\mathcal{N}$ defined as in (4.7), (4.8). Since each entry of $w$ is absolutely continuous with respect to n we have

$$\tilde{w} = Q^\top w Q = (Q^\top (w/\mathrm{n})Q)\mathrm{n}.$$

Thus, each entry of $\tilde{w}$ is also absolutely continuous with respect to n. Hence

$$(Q^\top (w/\mathrm{n})Q)\mathrm{n} = \tilde{w} = (\tilde{w}/\mathrm{n})\mathrm{n},$$

which means $Q^\top (w/\mathrm{n})Q = (\tilde{w}/\mathrm{n})$ on a Borel set $\mathcal{F} \subset (a,b)$ such that $\mathrm{n}((a,b) \setminus \mathcal{F}) = 0$. If $x \in \mathcal{F} \cap \mathcal{P}$ then the matrix $(\tilde{w}/\mathrm{n})(x) = (Q(x))^\top (w/\mathrm{n})(x)Q(x)$ is semi-positive definite since $Q(x)$ is an orthogonal matrix. Let $\tilde{\mathcal{P}}$, $\tilde{\mathcal{N}}$ be defined as

$$\tilde{\mathcal{P}} = \{x \in (a,b) : (\tilde{w}/\mathrm{n})(x) \geq 0\},$$

$$\tilde{\mathcal{N}} = \{x \in (a,b) : (\tilde{w}/\mathrm{n})(x) \leq 0\},$$

then $\mathcal{P}\Delta\tilde{\mathcal{P}}$ and $\mathcal{N}\Delta\tilde{\mathcal{N}}$ are null sets of n since then we have

$$(\mathcal{P}\Delta\tilde{\mathcal{P}}) \cap \mathcal{F} = ((\mathcal{P} \cap \mathcal{F}) \setminus \tilde{\mathcal{P}}) \cup ((\tilde{\mathcal{P}} \cap \mathcal{F}) \setminus \mathcal{P}) = \emptyset \cup \emptyset = \emptyset,$$

i.e., $\mathcal{P}\Delta\tilde{\mathcal{P}} \subset \mathcal{F}^c$, a null set of n. Similarly for $\mathcal{N}\Delta\tilde{\mathcal{N}}$.

For $j,k = 1,2$, since $\tilde{w}_{jk} \ll \mathrm{n}$, $\mathcal{P}\Delta\tilde{\mathcal{P}}$ is also a null set of $\tilde{w}_{jk}$; and hence $\chi_{\mathcal{P}}\tilde{w}_{jk}$ is identical to $\chi_{\tilde{\mathcal{P}}}\tilde{w}_{jk}$. Let $\mu_1$, $\mu_2$, $\tau$ be $\chi_{\mathcal{P}}\tilde{w}_{11}$, $\chi_{\mathcal{P}}\tilde{w}_{12}$, $\chi_{\mathcal{P}}\tilde{w}_{22}$, respectively. Then $\tau$ and n are nonnegative measures and the Radon-Nikodym derivatives $(\mu_1/\tau)$, $(\mu_2/\tau)$, $(\tau/\mathrm{n})$, $(\mu_1/\mathrm{n})$, $(\mu_2/\mathrm{n})$ are well-defined. Since $\mu_1$, $\mu_2$, $\tau$ are absolutely continuous with respect to n, we have the following Lebesgue decompositions

$$\mu_j = (\mu_j/\tau)\tau + s_j, \quad \tau = (\tau/\mathrm{n})\mathrm{n}, \quad \mu_j = (\mu_j/\mathrm{n})\mathrm{n},$$

where $s_j \perp \tau$. Let $C_1 = \tilde{C}^p_{11}$, $C_2 = \tilde{C}^p_{12}$ be Borel subsets of $\mathcal{P}$ on which $\tau$ is concentrated and such that $s_j$ is concentrated on $(a,b) \setminus C_j$.

We have

$$(\mu_j/\mathrm{n})\mathrm{n} - s_j = \mu_j - s_j = (\mu_j/\tau)\tau = (\mu_j/\tau)(\tau/\mathrm{n})\mathrm{n},$$

hence

$$[(\mu_j/\mathrm{n}) - (\mu_j/\tau)(\tau/\mathrm{n})]\mathrm{n} = s_j.$$

Since $s_j(C_j) = 0$ we get $(\mu_j/\mathrm{n}) = (\mu_j/\tau)(\tau/\mathrm{n})$ almost everywhere on $C_j$ with respect to n.

The existence of $\tilde{C}^n_{11}$, $\tilde{C}^n_{12}$ and their analogue properties can be proven in a similar fashion. □

As a consequence of Lemma 4.5, the measure $\sqrt{\det(w)}$ is well-defined on $\mathcal{P}$ and $\mathcal{N}$. Indeed, $(w_{22}/\mathrm{n})$ is a nonnegative function on $\mathcal{P}$ since $(w/\mathrm{n})(x)$ is semi-positive definite here. Thus, the restriction of $w_{22}$ on $\mathcal{P}$ is a positive measure. Also, the formula under the square root is nonnegative on $\mathcal{P}$ since

$$\left[(w_{11}/w_{22}) - (w_{12}/w_{22})^2\right](w_{22}/\mathrm{n})^2$$
$$= (w_{11}/\mathrm{n})(w_{22}/\mathrm{n}) - (w_{12}/\mathrm{n})^2 = \det((w/\mathrm{n})) \geq 0.$$

On $\mathcal{N}$, $w_{22}$ is a negative measure and the formula under the square root in $\sqrt{\det(w)}$ is also nonnegative since we still have $\det((w/\mathrm{n})) \geq 0$.

Lemma 4.6. *Let $w \in \mathcal{D}'^0((a,b))^{2\times 2}$ and $\gamma \in C^1((a,b),\mathbb{R})$. Define on $(a,b)$ the matrix-valued function $R_\gamma$ and the matrix-valued distribution $\tilde{w}$*

$$R_\gamma := \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix} \quad \textit{and} \quad \tilde{w} = R_\gamma^\top w R_\gamma.$$

*Then $\tilde{w} \in \mathcal{C}$.*

*Furthermore, let $\mathrm{n} = |w_{11}| + |w_{12}| + |w_{22}|$ and $\mathcal{P}$, $\mathcal{N}$ be the positive and negative set for $w$ as defined in (4.7) and (4.8), then $\sqrt{\det(w)} = \sqrt{\det(\tilde{w})}$ on $\mathcal{P}$ and $\mathcal{N}$ and* *that*

$$\int_{(a,b)} \sqrt{(\tilde{w}_{11}^{+}/\tilde{w}_{22}^{+})}\, \tilde{w}_{22}^{+} \geq \int_{\mathcal{P}} \sqrt{\det(\tilde{w})},$$

$$\int_{(a,b)} \sqrt{(\tilde{w}_{11}^{-}/\tilde{w}_{22}^{-})}\, \tilde{w}_{22}^{-} \leq \int_{\mathcal{N}} \sqrt{\det(\tilde{w})} + \int_{(a,b)} |\tilde{w}_{12}|.$$

Proof. Let $\mu_1$, $\mu_2$, $\tau$ be $\chi_{\mathcal{P}} w_{11}$, $\chi_{\mathcal{P}} w_{12}$, and $\chi_{\mathcal{P}} w_{22}$, respectively, where $\chi_{\mathcal{P}}$ is the characteristic function of $\mathcal{P}$. Choosing $Q$ to be the constant identity matrix in Lemma 4.5, then $(\mu_j/\tau)(\tau/\mathrm{n}) = (\mu_j/\mathrm{n})$, for $j = 1, 2$, almost everywhere with respect to n on a set $C^p \subset \mathcal{P}$ on which $\tau$ is concentrated. Thus, on $\mathcal{P}$ we have

$$\begin{aligned}\sqrt{\det(w)} &= \sqrt{(\mu_1/\tau) - (\mu_2/\tau)^2}\, \tau = \sqrt{(\mu_1/\tau) - (\mu_2/\tau)^2}\, (\tau/\mathrm{n})\mathrm{n} \\ &= \sqrt{(\mu_1/\tau)(\tau/\mathrm{n})^2 - (\mu_2/\tau)^2(\tau/\mathrm{n})^2}\, \mathrm{n} = \sqrt{(\mu_1/\mathrm{n})(\tau/\mathrm{n}) - (\mu_2/\mathrm{n})^2}\, \mathrm{n} \\ &= \sqrt{\det((w/\mathrm{n}))}\, \mathrm{n},\end{aligned}$$

where $(w/\mathrm{n}) = \begin{pmatrix} (\mu_1/\mathrm{n}) & (\mu_2/\mathrm{n}) \\ (\mu_2/\mathrm{n}) & (\tau/\mathrm{n}) \end{pmatrix}$. The fourth equality sign from the calculation above is from the fact that the measure $\sqrt{\det w}$ is concentrated on $C^p$.

Choosing $Q = R_\gamma$ in Lemma 4.5 and let $\tilde{\mu_1}$, $\tilde{\mu_2}$, $\tilde{\tau}$ be the restrictions of $\tilde{w}_{11}$, $\tilde{w}_{12}$, and $\tilde{w}_{22}$ on $\mathcal{P}$. Similarly, it can also be shown that $\sqrt{\det(\tilde{w})} = \sqrt{\det((\tilde{w}/\mathrm{n}))}\, \mathrm{n}$ on $\mathcal{P}$. Hence

$$\sqrt{\det(w)} = \sqrt{\det((w/\mathrm{n}))}\, \mathrm{n} = \sqrt{\det((\tilde{w}/\mathrm{n}))}\, \mathrm{n} = \sqrt{\det(\tilde{w})},$$

on $\mathcal{P}$ and similarly on $\mathcal{N}$.

For $j = 1, 2$, let $\tilde{\mathcal{P}}^{jj} = \{x : (\tilde{w}^{jj}/\mathrm{n})(x) \geq 0\}$, $\tilde{\mathcal{N}}^{jj} = \{x : (\tilde{w}^{jj}/\mathrm{n})(x) \leq 0\}$, and $\mathcal{D} = \{x : \det((\tilde{w}/\mathrm{n})(x)) \geq 0\}$. Then $\mathcal{P} = \tilde{\mathcal{P}}_{11} \cap \tilde{\mathcal{P}}_{22} \cap \mathcal{D}$, $\mathcal{N} = \tilde{\mathcal{N}}_{11} \cap \tilde{\mathcal{N}}_{22} \cap \mathcal{D}$, and we have

$$\begin{aligned}\int_{(a,b)} \sqrt{(\tilde{w}_{11}^{+}/\tilde{w}_{22}^{+})}\, \tilde{w}_{22}^{+} &= \int_{\tilde{\mathcal{P}}_{11}\cap\tilde{\mathcal{P}}_{22}} \sqrt{(\tilde{w}_{11}/\tilde{w}_{22})}\, |\tilde{w}_{22}| \geq \int_{\mathcal{P}} \sqrt{(\tilde{w}_{11}/\tilde{w}_{22})}\, |\tilde{w}_{22}| \\ &\geq \int_{\mathcal{P}} \sqrt{(\tilde{w}_{11}/\tilde{w}_{22}) - (\tilde{w}_{12}/\tilde{w}_{22})^2}\, |\tilde{w}_{22}| = \int_{\mathcal{P}} \sqrt{\det(\tilde{w})}.\end{aligned}$$

Assuming $\tilde{w}_{12} = (\tilde{w}_{12}/\tilde{w}_{22})\tilde{w}_{22} + s$ to be the Lebesgue decomposition of $\tilde{w}_{12}$ with respect to $\tilde{w}_{22}$ when restricting on $\tilde{\mathcal{N}}_{22}$, then

$$|\tilde{w}_{12}| = |(\tilde{w}_{12}/\tilde{w}_{22})||\tilde{w}_{22}| + |s| \geq |(\tilde{w}_{12}/\tilde{w}_{22})||\tilde{w}_{22}| = -|(\tilde{w}_{12}/\tilde{w}_{22})|\tilde{w}_{22}.$$

Let $\tilde{\mathcal{N}}_0 = \tilde{\mathcal{N}}_{11} \cap \tilde{\mathcal{N}}_{22}$, then $\mathcal{N} = \tilde{\mathcal{N}}_0 \cap \mathcal{D}$. For $x \in \tilde{\mathcal{N}}_0 \setminus \mathcal{N}$, the real symmetric matrix $(\tilde{w}/\mathrm{n})(x)$ has two eigenvalues of opposite signs, thus $\det((\tilde{w}/\mathrm{n})(x)) < 0$. Hence, on $\tilde{\mathcal{N}}_0 \setminus \mathcal{N}$, $\sqrt{(\tilde{w}_{11}/\mathrm{n})(\tilde{w}_{22}/\mathrm{n})} < |(\tilde{w}_{12}/\mathrm{n})|$ and that

$$\begin{aligned} -\sqrt{(\tilde{w}_{11}/\tilde{w}_{22})}\,\tilde{w}_{22} &= \sqrt{(\tilde{w}_{11}/\tilde{w}_{22})}\,|(\tilde{w}_{22}/\mathrm{n})|\mathrm{n} \\ &= \sqrt{(\tilde{w}_{11}/\tilde{w}_{22})(\tilde{w}_{22}/\mathrm{n})^2}\,\mathrm{n} = \sqrt{(\tilde{w}_{11}/\mathrm{n})(\tilde{w}_{22}/\mathrm{n})}\,\mathrm{n} \leq |(\tilde{w}_{12}/\mathrm{n})|\mathrm{n} \\ &= |(\tilde{w}_{12}/\tilde{w}_{22})(\tilde{w}_{22}/\mathrm{n})|\mathrm{n} = -|(\tilde{w}_{12}/\tilde{w}_{22})|\tilde{w}_{22} \leq |\tilde{w}_{12}|. \end{aligned}$$

For any two real numbers $A, B$ such that $A \geq 0$ and $A - B^2 \geq 0$ we have the inequality $0 \leq \sqrt{A} - \sqrt{A - B^2} \leq |B|$. (Applying the Cauchy-Schwarz inequality for the inner product in $\mathbb{R}^2$ here only gives a weaker result: $0 \leq \sqrt{A} - \sqrt{A - B^2} \leq \sqrt{2}|B|$.) Hence on $\mathcal{N}$ we have

$$\begin{aligned} &\left(\sqrt{(\tilde{w}_{11}/\tilde{w}_{22})} - \sqrt{(\tilde{w}_{11}/\tilde{w}_{22}) - (\tilde{w}_{12}/\tilde{w}_{22})^2}\right)(-\tilde{w}_{22}) \\ &\qquad \leq |(\tilde{w}_{12}/\tilde{w}_{22})|(-\tilde{w}_{22}) \leq |\tilde{w}_{12}|. \end{aligned}$$

We now have

$$\int_{(a,b)} \sqrt{(\tilde{w}_{11}^{-}/\tilde{w}_{22}^{-})}\,\tilde{w}_{22}^{-} - \int_{\mathcal{N}} \sqrt{\det(\tilde{w})}$$

$$= -\int_{\tilde{\mathcal{N}}_0} \sqrt{(\tilde{w}_{11}/\tilde{w}_{22})}\,\tilde{w}_{22} - \int_{\mathcal{N}} \sqrt{(\tilde{w}_{11}/\tilde{w}_{22}) - (\tilde{w}_{12}/\tilde{w}_{22})^2}\,|\tilde{w}_{22}|$$

$$= -\int_{\tilde{\mathcal{N}}_0\setminus\mathcal{N}} \sqrt{(\tilde{w}_{11}/\tilde{w}_{22})}\,\tilde{w}_{22}$$

$$+ \int_{\mathcal{N}} \left(\sqrt{(\tilde{w}_{11}/\tilde{w}_{22})} - \sqrt{(\tilde{w}_{11}/\tilde{w}_{22}) - (\tilde{w}_{12}/\tilde{w}_{22})^2}\right)(-\tilde{w}_{22})$$

$$\leq \int_{\tilde{\mathcal{N}}_0\setminus\mathcal{N}} |\tilde{w}_{12}| + \int_{\mathcal{N}} |\tilde{w}_{12}| \leq \int_{(a,b)} |\tilde{w}_{12}|.$$

□

We can now discuss about the main result of this section on the asymptotic behavior of the Prüfer angle.

Theorem 4.7. *Let $\Omega$ be some domain in $\mathbb{C}$ containing the ray $(\lambda_0, \infty)$ for some $\lambda_0 \in \mathbb{R}$. Suppose $w$ and $q$ satisfy Hypothesis 4.1. Let $u(\cdot, \lambda)$ and $\theta(\cdot, \lambda)$ be a solution and its Prüfer angle of the equation $Ju' + q(\lambda)u = \lambda w u$ provided with boundary conditions (4.3). Then*

$$\lim_{\lambda\to\infty} \frac{\theta(b,\lambda) - \theta(a,\lambda)}{\lambda} = \int_{\mathcal{P}} \sqrt{\det(w)} - \int_{\mathcal{N}} \sqrt{\det(w)},$$

*where $\mathcal{P}$ and $\mathcal{N}$ are the positive and negative part of $w$ as given in Equations (4.7) and (4.8).*

Proof. Let $\varepsilon > 0$ and $\nu = |w_{11} - w_{22}| + |w_{12}|$. Since $C_c^1((a,b))$ is dense in $L^1(\nu)$ and that $(w_{11} - w_{22})/2 \ll \nu$, $w_{12} \ll \nu$, there exist $\zeta_1, \zeta_2 \in C_c^1((a,b))$ such that

$$\int_{(a,b)} \left|\frac{w_{11} - w_{22}}{2} - \zeta_2 \nu\right| \leq \varepsilon \quad \text{and} \quad \int_{(a,b)} |\zeta_1 \nu - w_{12}| \leq \varepsilon.$$

The $C^1$-curve in $\mathbb{R}^2$ defined by $(a,b) \ni x \mapsto (\zeta_1(x), \zeta_2(x))^\top$ is a closed curve starting at the origin. Thus, fixing the angle at $a$, we can uniquely define $\gamma, r \in C^1((a,b))$ via the assignnments

$$\zeta_1(x) = r(x)\sin 2\gamma(x), \quad \zeta_2(x) = r(x)\cos 2\gamma(x), \quad \gamma(a) \in [0, \pi).$$

Let $\psi(\cdot, \lambda) = \theta(\cdot, \lambda) + \gamma(\cdot) \in \mathrm{BV}((a, b))$, we have

$$\begin{aligned}
\psi' &= \lambda(\sin(\psi - \gamma), \cos(\psi - \gamma))w \begin{pmatrix} \sin(\psi - \gamma) \\ \cos(\psi - \gamma) \end{pmatrix} - (\sin\theta, \cos\theta)q(\lambda) \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} + \gamma' \\
&= \lambda(\sin\psi, \cos\psi) \begin{pmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{pmatrix} w \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix} \begin{pmatrix} \sin\psi \\ \cos\psi \end{pmatrix} + d(\lambda) \\
&= \lambda((\sin\psi)^2 \tilde{w}_{11} + (\cos\psi)^2 \tilde{w}_{22} + (\sin 2\psi)\tilde{w}_{12}) + d(\lambda),
\end{aligned}$$

where $d(\lambda) = -(\sin\theta, \cos\theta)q(\lambda)\left(\begin{smallmatrix} \sin\theta \\ \cos\theta \end{smallmatrix}\right) + \gamma'$ and $\tilde{w} = \left(\begin{smallmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{smallmatrix}\right) w \left(\begin{smallmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{smallmatrix}\right)$.

Since $\tilde{w}_{11}$ and $\tilde{w}_{22}$ are distributions of order zero, there exist $k_1 \in C^1((a, b))$, $k_1 > 0$ and a finite union $J = \bigcup_{j=1}^N (a_j, b_j)$ of open disjoint subintervals of $(a, b)$ as in Lemma 4.4. Define $\phi_1$ to be the Kepler transformation of $\psi$ via the assignment

$$\rho_1 \sin\phi_1 = k_1 \sin\psi, \quad \rho_1 \cos\phi_1 = \cos\psi, \quad |\phi_1(a) - \psi(a)| < \pi/2.$$

By Theorem 3.1, we have

$$\begin{aligned}
\phi_1' &= \frac{k_1}{\rho_1^2}\psi' + \frac{k_1'}{k_1}\sin\phi_1 \cos\phi_1 \\
&= \lambda \left[ \frac{1}{k_1}(\sin\phi_1)^2 \tilde{w}_{11} + k_1(\cos\phi_1)^2 \tilde{w}_{22} + (\sin 2\phi_1)\tilde{w}_{12} \right] + c_1(\lambda) \\
&= \lambda \left[ \left( \frac{1}{k_1}\tilde{w}_{11} - k_1 \tilde{w}_{22} \right) (\sin\phi_1)^2 + k_1 \tilde{w}_{22} + (\sin 2\phi_1)\tilde{w}_{12} \right] + c_1(\lambda),
\end{aligned}$$

where $c_1(\lambda) = \frac{k_1}{\rho_1^2} d(\lambda) + \frac{k_1'}{k_1} \sin\phi_1 \cos\phi_1$.

Since $\rho_1^2 = k_1^2(\sin\psi)^2 + (\cos\psi)^2$ and $|k_1 + (\frac{1}{k_1} - k_1)(\cos\psi)^2| \geq \min\{k_1, \frac{1}{k_1}\}$ we have

$$\frac{1}{\lambda}\int_{(a,b)} |c_1(\lambda)| \leq \frac{1}{\lambda}\int_{(a,b)} \left| \frac{1}{\min\{k_1, k_1^{-1}\}} \right| |q_{11}(\lambda) + q_{22}(\lambda) + q_{12}(\lambda)| + \frac{1}{\lambda}\int_{(a,b)} \left| \frac{k_1'}{k_1} \right|.$$

Since $k_1'/k_1 \in C_c^0((a,b))$ and $k_1$ is bounded and bounded away from zero the right hand side of the above inequality tends to zero as $\lambda$ tends to infinity by equation (4.2).

For any large $\lambda > 0$ such that $\int_J |c_1(\lambda)| < \lambda\varepsilon$, we have

$$\begin{aligned}\int_J \phi_1' &\geq -\lambda \int_J \left|\frac{1}{k_1}\tilde{w}_{11} - k_1\tilde{w}_{22}\right| + \lambda\int_J k_1\tilde{w}_{22} - \lambda\int_J |\tilde{w}_{12}| - \lambda\varepsilon \\ &\geq \lambda\int_{(a,b)} \sqrt{(\tilde{w}_{11}^+/\tilde{w}_{22}^+)}\,\tilde{w}_{22}^+ - \lambda\int_J |\tilde{w}_{12}| - 3\lambda\varepsilon.\end{aligned}$$

Also, by Theorem 3.1, $|\psi(b_j) - \phi_1(b_j)|$ and $|\psi(a_j) - \phi_1(a_j)|$ are bounded above by $\pi/2$, thus $\psi(b_j) - \phi_1(b_j) + \frac{1}{N}\lambda\varepsilon \geq \psi(a_j) - \phi_1(a_j)$ for all $j = 1, 2, \ldots, N$ and for all large enough $\lambda > 0$. Hence

$$\begin{aligned}\int_J \psi' &= \sum_{j=1}^{N}\int_{(a_j,b_j)} \psi' = \sum_{j=1}^{N}(\psi(b_j) - \psi(a_j)) \geq \sum_{j=1}^{N}(\phi_1(b_j) - \phi_1(a_j)) - \lambda\varepsilon \\ &= \int_J \phi_1' - \lambda\varepsilon \geq \lambda\int_{(a,b)}\sqrt{(\tilde{w}_{11}^+/\tilde{w}_{22}^+)}\,\tilde{w}_{22}^+ - \lambda\int_J|\tilde{w}_{12}| - 4\lambda\varepsilon.\end{aligned} \tag{4.9}$$

Since $\tilde{w}_{11}^-$ and $\tilde{w}_{22}^-$ are nonnegative, by Theorem 4.2, there exists $k_2 \in C^1((a,b))$, $k_2 > 0$ such that

$$\int_{(a,b)}\left|k_2\tilde{w}_{22}^- - \frac{1}{k_2}\tilde{w}_{11}^-\right| \leq \varepsilon \quad \text{and} \quad \int_{(a,b)}\left|k_2\tilde{w}_{22}^- - \sqrt{(\tilde{w}_{11}^-/\tilde{w}_{22}^-)}\,\tilde{w}_{22}^-\right| \leq \varepsilon.$$

Define $\phi_2$ to be the Kepler transformation of $\psi$ via the assignment

$$\rho_2 \sin\phi_2 = k_2\sin\psi, \quad \rho_2\cos\phi_2 = \cos\psi, \quad |\phi_2(a) - \psi(a)| < \pi/2.$$

By Theorem 3.1, we have

$$\phi_2' = \lambda\left[\frac{1}{k_2}(\sin\phi_2)^2\tilde{w}_{11} + k_2(\cos\phi_2)^2\tilde{w}_{22} + (\sin 2\phi_2)\tilde{w}_{12}\right] + c_2(\lambda),$$

where $c_2(\lambda) = \frac{k_2}{\rho_2^2}d(\lambda) + \frac{k_2'}{k_2}\sin\phi_2\cos\phi_2$. We also have $\lim_{\lambda\to\infty}\frac{1}{\lambda}\int_{(a,b)}|c_2(\lambda)| = 0$.

Let $K = (a,b) \setminus J$. Since $\tilde{w}_{11} \geq -\tilde{w}_{11}^-$ and $\tilde{w}_{22} \geq -\tilde{w}_{22}^-$, for all large enough $\lambda > 0$ we have

$$\begin{aligned}\int_K \phi_2' &\geq -\lambda \int_K \left(\frac{1}{k_2}(\sin\phi_2)^2 \tilde{w}_{11}^- + k_2(\cos\phi_2)^2 \tilde{w}_{22}^-\right) - \lambda \int_K |\tilde{w}_{12}| - \lambda \int_K |c_2(\lambda)| \\ &\geq -\lambda \int_K \left(\frac{1}{k_2}\tilde{w}_{11}^- - k_2 \tilde{w}_{22}^-\right)(\sin\phi_2)^2 - \int_K k_2 \tilde{w}_{22}^- - \lambda \int_K |\tilde{w}_{12}| - \lambda \int_K |c_2(\lambda)| \\ &\geq -\lambda \int_{(a,b)} \left|\frac{1}{k_2}\tilde{w}_{11}^- - k_2 \tilde{w}_{22}^-\right| - \lambda \int_{(a,b)} k_2 \tilde{w}_{22}^- - \lambda \int_K |\tilde{w}_{12}| - \lambda \int_{(a,b)} |c_2(\lambda)| \\ &\geq -\lambda \int_{(a,b)} \sqrt{\tilde{w}_{11}^-/\tilde{w}_{22}^-}\, \tilde{w}_{22}^- - \lambda \int_K |\tilde{w}_{12}| - 3\lambda\varepsilon.\end{aligned}$$

$J$ is a finite union of disjoint subintervals of $(a,b)$ and so is $K$. Similarly as before we have

$$\int_K \psi' \geq \int_K \phi_2' - \lambda\varepsilon \geq -\lambda \int_{(a,b)} \sqrt{(\tilde{w}_{11}^-/\tilde{w}_{22}^-)}\, \tilde{w}_{22}^- - \lambda \int_K |\tilde{w}_{12}| - 4\lambda\varepsilon. \tag{4.10}$$

From (4.9), (4.10), and Lemma 4.6, for all large enough $\lambda > 0$ we have

$$\begin{aligned}\int_{(a,b)} \psi' &\geq \lambda \int_{(a,b)} \sqrt{(\tilde{w}_{11}^+/\tilde{w}_{22}^+)}\, \tilde{w}_{22}^+ - \lambda \int_{(a,b)} \sqrt{(\tilde{w}_{11}^-/\tilde{w}_{22}^-)}\, \tilde{w}_{22}^- - \lambda \int_{(a,b)} |\tilde{w}_{12}| - 8\lambda\varepsilon \\ &\geq \lambda \int_{\mathcal{P}} \sqrt{\det(\tilde{w})} - \lambda \int_{\mathcal{N}} \sqrt{\det(\tilde{w})} - 2\lambda \int_{(a,b)} |\tilde{w}_{12}| - 8\lambda\varepsilon.\end{aligned}$$

Since $\zeta_1 \cos 2\gamma = \zeta_2 \sin 2\gamma$ and $\tilde{w} = R_\gamma^\top w R_\gamma$, we have

$$\begin{aligned}\tilde{w}_{12} &= (\sin 2\gamma)\left(\frac{w_{22} - w_{11}}{2}\right) + (\cos 2\gamma) w_{12} \\ &= (\sin 2\gamma)\left(\frac{w_{22} - w_{11}}{2} + \zeta_2 \nu\right) + (\cos 2\gamma)(w_{12} - \zeta_1 \nu).\end{aligned}$$

Thus

$$\int_{(a,b)} |\tilde{w}_{12}| \leq \int_{(a,b)} \left|\frac{w_{11} - w_{22}}{2} - \zeta_2 \nu\right| + \int_{(a,b)} |\zeta_1 \nu - w_{12}| \leq 2\varepsilon.$$

And so for all large enough $\lambda > 0$ we have

$$\int_{(a,b)} \psi' \geq \lambda \int_{\mathcal{P}} \sqrt{\det(\tilde{w})} - \lambda \int_{\mathcal{N}} \sqrt{\det(\tilde{w})} - 12\lambda\varepsilon.$$

Since $\psi = \theta + \gamma$ and $\gamma$ are two finite distributions on $(a, b)$, we have for all large enough $\lambda > 0$

$$\int_{(a,b)} \theta' = \int_{(a,b)} \psi' - \int_{(a,b)} \gamma' \geq \lambda \int_{\mathcal{P}} \sqrt{\det(\tilde{w})} - \lambda \int_{\mathcal{N}} \sqrt{\det(\tilde{w})} - 13\lambda\varepsilon.$$

Thus for any $\lambda$ we have

$$\frac{\theta(b,\lambda) - \theta(a,\lambda)}{\lambda} \geq \int_{\mathcal{P}} \sqrt{\det(\tilde{w})} - \int_{\mathcal{N}} \sqrt{\det(\tilde{w})} - 13\varepsilon.$$

And so

$$\liminf_{\lambda\to\infty} \frac{\theta(b,\lambda) - \theta(a,\lambda)}{\lambda} \geq \int_{\mathcal{P}} \sqrt{\det(\tilde{w})} - \int_{\mathcal{N}} \sqrt{\det(\tilde{w})} - 13\varepsilon.$$

Since this inequality is true for any $\varepsilon > 0$ and by Lemma 4.6, we have

$$\begin{aligned}\liminf_{\lambda\to\infty} \frac{\theta(b,\lambda) - \theta(a,\lambda)}{\lambda} &\geq \int_{\mathcal{P}} \sqrt{\det(\tilde{w})} - \int_{\mathcal{N}} \sqrt{\det(\tilde{w})} \\ &= \int_{\mathcal{P}} \sqrt{\det(w)} - \int_{\mathcal{N}} \sqrt{\det(w)}.\end{aligned}$$

The proof is finished if we can show that

$$\limsup_{\lambda\to\infty} \frac{\theta(b,\lambda) - \theta(a,\lambda)}{\lambda} \leq \int_{\mathcal{P}} \sqrt{\det(w)} - \int_{\mathcal{N}} \sqrt{\det(w)}.$$

We have

$$\limsup_{\lambda\to\infty} \frac{\theta(b,\lambda) - \theta(a,\lambda)}{\lambda} = -\liminf_{\lambda\to\infty} \frac{(-\theta)(b,\lambda) - (-\theta)(a,\lambda)}{\lambda}.$$

Note that if $\theta(\cdot,\lambda)$ is the Prüfer angle of a solution $u(\cdot,\lambda)$ of $Ju' + qu = \lambda wu$, then $-\theta(\cdot,\lambda)$ is the Prüfer angle of a solution $v(\cdot,\lambda)$ of $Jv' + \breve{q}v = \lambda\breve{w}v$ where

$$\breve{q}_{11} = -q_{11}, \quad \breve{q}_{22} = -q_{22}, \quad \breve{q}_{12} = q_{12},$$

and

$$\breve{w}_{11} = -w_{11}, \quad \breve{w}_{22} = -w_{22}, \quad \breve{w}_{12} = w_{12}.$$

Repeating the proof above for the new equation $Jv' + \breve{q}v = \lambda\breve{w}v$ and note that $\mathcal{P}(\breve{w}) = \mathcal{N}$, $\mathcal{N}(\breve{w}) = \mathcal{P}$, and $\sqrt{\det(\breve{w})} = \sqrt{\det(w)}$. We now have

$$\liminf_{\lambda\to\infty} \frac{(-\theta)(b) - (-\theta)(a)}{\lambda} \geq \int_{\mathcal{P}(\breve{w})} \sqrt{\det(\breve{w})} - \int_{\mathcal{N}(\breve{w})} \sqrt{\det(\breve{w})}$$
$$= -\left(\int_{\mathcal{P}} \sqrt{\det(w)} - \int_{\mathcal{N}} \sqrt{\det(w)}\right).$$

□

## 5. Asymptotic Behavior Of Eigenvalues Of First Order Systems

Let $\Omega$ be some domain in $\mathbb{C}$ containing the ray $(0,\infty)$ (and possibly the ray $(-\infty,0)$). As mentioned in Section 4, we shall now consider the eigenvalue problem

$$Ju' + q(\lambda)u = \lambda wu, \tag{5.1}$$

provided with separated boundary conditions

$$(\cos\alpha, -\sin\alpha)u(a) = (\cos\beta, -\sin\beta)u(b) = 0, \tag{5.2}$$

where $q$ and $w$ satisfy the Hypothesis 4.1, $\alpha \in [0.\pi)$, $\beta \in (0,\pi]$

A real number $\lambda$ is said to be an *eigenvalue* of the problem (5.1) & (5.2) if there exists a nontrivial solution $u$ of equation (5.1) satisfying the boundary conditions (5.2). Such a solution $u$ is said to be an *eigenfunction* to $\lambda$.

For any $\lambda > 0$, let $u(.,\lambda)$ be the solution of $Ju' + q(\lambda)u = \lambda wu$ with the initial value $u(a) = (\sin\alpha, \cos\alpha)^\top$ and $\theta(.,\lambda)$ its Prüfer angle. Then $\theta(a,\lambda) = \alpha$, and a real number $\lambda$ is an eigenvalue of the problem (5.1) & (5.2) if and only if $\theta(b,\lambda) = \beta + n\pi$ for some integer $n$.

We can now discuss the main result of this paper.

THEOREM 5.1. *Suppose* $\Omega$ *is some domain in* $\mathbb{C}$ *containing the ray* $(0,\infty)$ *and that* $q$ *and* $w$ *satisfying Hypothesis 4.1 with* $\lambda_0 = 0$. *If*

$$\delta(w) = \int_{\mathcal{P}} \sqrt{\det(w)} - \int_{\mathcal{N}} \sqrt{\det(w)} > 0 \tag{5.3}$$

*then the positive eigenvalues of the eigenvalue problem* (5.1) & (5.2) *converge to* $\infty$ *and can be ordered as*

$$0 < \lambda_1^+ < \lambda_2^+ < \dots .$$

*Furthermore, there exist an integer* $k_0$ *and a sequence* $\{k_0, k_0+1, \dots\} \ni k \mapsto n_k \in \mathbb{N}_0$ *such that*

$$\lim_{k\to\infty} \frac{\lambda_{n_k}^+}{k} = \pi/\delta(w),$$

*thence*

$$0 \le \liminf_{n\to\infty} \frac{\lambda_n^+}{n} \le \limsup_{n\to\infty} \frac{\lambda_n^+}{n} \le \pi/\delta(w). \tag{5.4}$$

*If in addition* $\Omega$ *contains the whole real line* $\mathbb{R}$ *and* $q$ *also satisfy the condition* (4.2) *with* $\infty$ *substituted by* $-\infty$ *then the negative eigenvalues of the eigenvalue problem* (5.1) & (5.2) *converge to* $-\infty$ *and can be ordered as*

$$0 > \lambda_1^- > \lambda_2^- > \dots .$$

*Furthermore, there exist an integer* $k_0'$ *and a sequence* $\{\dots, k_0'-1, k_0'\} \ni k \mapsto n_k' \in \mathbb{N}_0$ *such that*

$$\lim_{k\to-\infty} \frac{\lambda_{n_k'}^-}{k} = \pi/\delta(w),$$

*and*

$$-\pi/\delta(w) \le \liminf_{n\to\infty} \frac{\lambda_n^-}{n} \le \limsup_{n\to\infty} \frac{\lambda_n^-}{n} \le 0.$$

PROOF. For any fixed $x \in (a,b]$, the function $\Omega \ni \lambda \mapsto u(x,\lambda)$ is analytic (see, e.g., Theorem 2.7 of [8]) and so is the function $\lambda \mapsto F(\lambda) = (\cos\beta, -\sin\beta)u(b,\lambda)$, whose zeros are exactly the eigenvalues of the problem (5.1) & (5.2).

By Theorem 4.7 we have

$$\lim_{\lambda\to\infty}\frac{1}{\lambda}\left(\theta(b,\lambda)-\beta\right)=\lim_{\lambda\to\infty}\frac{1}{\lambda}\left(\theta(b,\lambda)-\theta(a,\lambda)\right)+\lim_{\lambda\to\infty}\frac{1}{\lambda}\left(\alpha-\beta\right)=\delta(w)>0.$$

Thence each of the level set $E_k=\{\lambda>0:\theta(b,\lambda)-\beta=k\pi\}$, $k\in\mathbb{Z}$, is discrete since it is a subset of a zero set of a nonconstant analytic function $F$.

The function $\Omega\cap\mathbb{R}\ni\lambda\mapsto\theta(x,\lambda)-\beta$ is not bounded above since otherwise we would get $\delta(w)=\lim_{n\to\infty}(\theta(b,\lambda)-\beta)/\lambda=0$, a contradiction. Analogous arguments also show that the function $\lambda\mapsto\theta(x,\lambda)$ is bounded below and that each $E_k$ contains finitely many elements. Hence the set of indices $k\in\mathbb{Z}$ for which $E_k$ is nonempty is bounded below, says by some $k_0\in\mathbb{Z}$, and for any $k\geq k_0$ the set $E_k$ is nonempty.

Let $\lambda_n^+$, $n=1,2,\ldots$, be an enumeration, sorted in an increasing order, of the elements of $\cup_{k\geq k_0}E_k$. These $\lambda_n^+$'s are indeed all the positive eigenvalues of the problem (5.1) & (5.2).

For any $k\geq k_0$, let $n_k=\min\{j\in\mathbb{N}_0:\lambda_j^+\in E_k\}$. It is important to see that the sequence $k\mapsto\lambda_{n_k}^+$ has a strictly increasing tail. Let $k_1$ be the unique integer such that $k_1\pi<\theta(b,0)-\beta\leq(k_1+1)\pi$; hence $k_0\leq k_1$. Suppose there exist $k_1\leq k_2<k_3$ such that $\lambda_{n_{k_3}}^+\leq\lambda_{n_{k_2}}^+$; hence $n_{k_3}<n_{k_2}$. Since the function $\lambda\mapsto\theta(b,\lambda)-\beta$ is continuous and since $\theta(b,0)-\beta\leq(k_1+1)\pi\leq k_3\pi=\theta(b,\lambda_{n_{k_3}})-\beta$ there exists $j_0<n_{k_3}<n_{k_2}$ such that $\lambda_{j_0}\in E_{k_2}$, contradicting to the definition of $n_{k_2}$.

Fix an $m\in\mathbb{N}_0$, there exists $N_1(m)\in\mathbb{N}_0$ such that $|(\theta(b,\lambda)-\beta)/\lambda-\delta(w)|<2^{-m}$ whenever $\lambda\geq N_1(m)$. Equivalently,

$$\lambda(\delta(w)-2^{-m})<\theta(b,\lambda)-\beta<\lambda(\delta(w)+2^{-m})$$

whenever $\lambda\geq N_1(m)$.

Let $N_2(m)=\min\{n_k:\lambda_{n_k}^+\geq N_1(m)\}$ and $n_k\geq N_2(m)$. Since $k\mapsto\lambda_{n_k}^+$ is eventually strictly increasing, for sufficiently large $k$ we get $\lambda_{n_k}^+\geq\lambda_{N_2(m)}^+\geq N_1(m)$. Hence

$$\lambda_{n_k}^+(\delta(w)-2^{-m})<\theta(b,\lambda_{n_k}^+)-\beta=k\pi<\lambda_{n_k}^+(\delta(w)+2^{-m}),$$

and

$$\frac{\pi}{\delta(w)+2^{-m}} < \frac{\lambda^+_{n_k}}{k} < \frac{\pi}{\delta(w)-2^{-m}}.$$

Thus

$$\frac{\pi}{\delta(w)+2^{-m}} \le \liminf_{k\to\infty} \frac{\lambda^+_{n_k}}{k} \le \limsup_{k\to\infty} \frac{\lambda^+_{n_k}}{k} \le \frac{\pi}{\delta(w)-2^{-m}}.$$

This is true for any chosen $m \in \mathbb{N}_0$ so $\lim_k \lambda^+_{n_k}/k = \pi/\delta(w)$. The inequalities (5.4) come from the fact that $k \le n_k$.

It is a fact that if $u(x,\lambda)$ is a solution of the equation $J\psi'(x,\lambda) + q(\lambda)\psi(x,\lambda) = \lambda w\psi(x,\lambda)$ then $v(x,\lambda) = -u(x,-\lambda)$ is a solution of the equation $J\psi'(x,\lambda)+\tilde{q}(\lambda)\psi(x,\lambda) = \lambda\tilde{w}\psi(x,\lambda)$ where $\tilde{q}(\lambda) = q(-\lambda)$ and $\tilde{w} = -w$. Suppose $\Omega$ contains the whole real line and $q$ satisfies the condition (4.2) at $-\infty$ then $\tilde{q}$ satisfies the condition (4.2) at $\infty$. Applying Theorem 4.7 for the equation with $\tilde{q}$ and $\tilde{w}$ we get

$$\lim_{\lambda\to-\infty} \frac{\theta(b,\lambda)-\theta(a,\lambda)}{\lambda} = -\lim_{\lambda\to\infty} \frac{\theta(b,-\lambda)-\theta(a,-\lambda)}{\lambda} = -\delta(\tilde{w}) = \delta(w).$$

We note here that $\theta(x,-\lambda)$ is the Prüfer angle associating with $v(x,\lambda) = -u(x,-\lambda)$ and that $\sqrt{\det(\tilde{w})} = \sqrt{\det(w)}$, $\mathcal{P}(\tilde{w}) = \mathcal{N}(w)$, and $\mathcal{N}(\tilde{w}) = \mathcal{P}(w)$.

A modified version of the previous proof for the positive eigenvalues can now be repeated to get what being claimed about the negative eigenvalues. □

If $\delta(w) < 0$ then we also have the same claims about the asymptotic behavior for the eigenvalues $\lambda^\pm_n$.

If $q$ is in fact a constant function with respect to $\lambda$ then the subsequence $k \mapsto \lambda^\pm_{n_k}$ in Theorem 5.1 can be chosen to be the full sequence of eigenvalues. For, in this case, the function $\Omega \ni \lambda \mapsto \theta(x,\lambda) - \beta$ is not only analytic but also nondecreasing (see, e.g., Theorem 3.1 of [13]). Thus it is strictly increasing and each level set $E_n$ contains exactly one element. We have proven the following theorem.

THEOREM 5.2. *Suppose $q$ and $w$ satisfy Hypothesis 4.1 and that $q$ is a constant function in $\lambda$. If*

$$\delta(w) = \int_{\mathcal{P}} \sqrt{\det(w)} - \int_{\mathcal{N}} \sqrt{\det(w)} \neq 0$$

*then the eigenvalue problem $Ju' + qu = \lambda wu$ provided with boundary conditions (5.2) has countably many positive and negative eigenvalues $\lambda_n^{\pm}$ that can be sorted as*

$$-\infty \leftarrow \lambda_n^- < \ldots < \lambda_2^- < \lambda_1^- < 0 < \lambda_1^+ < \lambda_2^+ < \ldots < \lambda_n^+ < \ldots \rightarrow \infty.$$

*Furthermore, we have the following asymptotic formula*

$$\lim_n \frac{\lambda_n^{\pm}}{n} = \pm\pi \Big/ \left( \int_{\mathcal{P}} \sqrt{\det(w)} - \int_{\mathcal{N}} \sqrt{\det(w)} \right) = \pm\pi/\delta(w).$$

We can also pose the eigenvalue problem

$$Ju' + \left(\begin{smallmatrix} v & 0 \\ 0 & -s \end{smallmatrix}\right) u = \lambda \left(\begin{smallmatrix} r & 0 \\ 0 & 0 \end{smallmatrix}\right) u \tag{5.5}$$

provided with boundary condition (5.2) where $s$, $v$, and $r$ are distributions of order 0 in the class $\mathcal{C}$.

This eigenvalue problem was considered by Bennewitz [3] and Volkmer [15] under a more relaxed condition on the coefficients, in particular it is only required that $V$ and $R$ have no discontinuities in common with $S$, where $S$, $V$, and $R$ are the antiderivatives of $s$, $v$, and $r$, respectively (see the Assumption 2.1 of [3] and the condition (5.4) of [15]).

THEOREM 5.3. *Suppose $s$, $v$, and $r$ are distributions of order 0 in the class $\mathcal{C}$. If*

$$\delta^+ = \int_{\mathcal{P}(s)\cap\mathcal{P}(r)} \sqrt{(s/r)}\,|r| - \int_{\mathcal{N}(s)\cap\mathcal{N}(r)} \sqrt{(s/r)}\,|r| \neq 0$$

*and*

$$\delta^- = \int_{\mathcal{P}(s)\cap\mathcal{N}(r)} \sqrt{-(s/r)}\,|r| - \int_{\mathcal{N}(s)\cap\mathcal{P}(r)} \sqrt{-(s/r)}\,|r| \neq 0$$

*then the positive eigenvalues of the eigenvalue problem (5.5) & (5.2) converge to $\pm\infty$ and can be ordered as*

$$\ldots < \lambda_2^- < \lambda_1^- < 0 < \lambda_1^+ < \lambda_2^+ < \ldots.$$

*Furthermore, there exist integers $k_0$, $k_0'$ and two one-sided sequences $k \mapsto n_k \in \mathbb{N}_0$, $k \mapsto n_k' \in \mathbb{N}_0$ such that*

$$\lim_{|k|\to\infty} \frac{(\lambda_{n_k}^+)^{1/2}}{|k|} = \pi/|\delta^+|, \quad \lim_{|k|\to\infty} \frac{(-\lambda_{n_k'}^-)^{1/2}}{|k|} = \pi/|\delta^-|,$$

*thence*

$$0 \le \liminf_{n\to\infty} \frac{(\pm\lambda_n^\pm)^{1/2}}{n} \le \limsup_{n\to\infty} \frac{(\pm\lambda_n^\pm)^{1/2}}{n} \le \pi/|\delta^\pm|,$$

*If, in addition, $s$ is a positive measure and $r + \varepsilon v \ge 0$ for all $\varepsilon > 0$ then*

$$\lim_{n\to\infty} \frac{|\lambda_n^\pm|^{1/2}}{n} = \pi/|\delta^\pm|.$$

Proof. For $\lambda > 0$, the equation (5.5) can be rewritten as $Jz' + \tilde{q}(\mu)z = \mu\tilde{w}z$ where $\mu = \sqrt{\lambda}$, $z = \left(\begin{smallmatrix} 0 & -1 \\ \mu & 0 \end{smallmatrix}\right)u$, $\tilde{q}(\mu) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & \mu^{-1}v \end{smallmatrix}\right)$, and $w = \left(\begin{smallmatrix} s & 0 \\ 0 & r \end{smallmatrix}\right)$. Applying Theorem 5.1 for this first order system we get what is claimed about the positive eigenvalues. The negative eigenvalues can be dealt with by considering the equation with $r$ substituted by $-r$.

If $s$ is a positive measure and $r + \varepsilon v \ge 0$ for all $\varepsilon > 0$ then we have $\mu_2\tilde{w} - \tilde{q}(\mu_2) \ge \mu_1\tilde{w} - \tilde{q}(\mu_1)$ whenever $\mu_1 < \mu_2$. By Theorem 3.1 of [13] the Prüfer angle associating to the equation of $\tilde{q}(\mu)$ and $\tilde{w}$ is nondecreasing. The claim now follows by an analogous argument as in Theorem 5.2. □

We note that if the condition (5.3) fails then we may run into a situation where every complex number is an eigenvalue, for example with the system $Ju' = 0$, $\alpha = 0$, $\beta = \pi$. Or there are finitely many eigenvalues, for example with the system $Ju' = \lambda\left(\begin{smallmatrix} dx & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $\alpha = \pi/2$, $\beta = 3\pi/4$, where $dx$ is the Lebesgue measure.

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