2021

Cubic Symmetric Laminations

Sandeep chowdary Vejandla
University of Alabama at Birmingham

CUBIC SYMMETRIC LAMINATIONS

by

SANDEEP CHOWDARY VEJANDLA

NIKITA SELINGER, COMMITTEE CHAIR
ALEXANDER BLOKH
WU DONGSHENG
LEX OVERSTEEGEN
BULENT TOSUN
DA YAN

A DISSERTATION

Submitted to the faculty of the University of Alabama at Birmingham,
in partial fulfillment of the requirements of the degree of
Doctor of Philosophy

BIRMINGHAM, ALABAMA

2021

CUBIC SYMMETRIC LAMINATIONS

SANDEEP CHOWDARY VEJANDLA

APPLIED MATHEMATICS

ABSTRACT

To study the *parameter space of all polynomials of degree* d with connected Julia sets, Thurston proposed studying the space of all σ_d-*invariant laminations*, where $\sigma_d : \mathbb{S} \to \mathbb{S}$ is the degree d covering map of the unit circle defined by $\sigma_d(z) = z^d$. A lamination is a family of chords in the unit disk satisfying the following property. No two chords in a lamination intersect inside the disk. Thurston built a topological model for the space of quadratic polynomials $f(z) = z^2 + \lambda$ using a parametrization of the space of quadratic invariant laminations. He completed this approach for the space of quadratic polynomials but the case of higher degree has remained elusive.

Our goal is to gain a better understanding of the space of cubic polynomials $f(z) = z^3 + bz^2 + \lambda z$. We have studied a particular slice of the space of cubic polynomials. We call polynomials of the form $f(z) = z^3 + \lambda^2 z$ cubic symmetric polynomials. In the same spirit as Thurston's work, we will parametrize space of cubic symmetric laminations which will provide a model for the space of cubic symmetric polynomials.

ACKNOWLEDGEMENTS

First and foremost I am extremely grateful to my advisor, Dr. Nikita Selinger for introducing me to the subject of complex dynamics and patiently explaining complex mathematical concepts. I have greatly benefited from his immense knowledge on the subject. I would like to thank Dr. Oversteegen for guiding me at every step of the thesis preparation. I cannot thank him and Dr Selinger enough for all the discussions I had with them over the last five years.

I would also like to thank Dr Blokh for all his support. I have greatly benefited from his research papers.

Finally, I would like to express deep gratitude to UAB Mathematics Department in providing a helpful environment. I extend my deep gratitude to UAB Mathematics Department members for their support and encouragement.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

Studying the structure of polynomial families is one of the central problems of complex dynamics. The first non-trivial case here is that of the quadratic polynomial family $P_c(z) = z^2 + c$. The *Mandelbrot set* $\mathcal{M}_2$ is defined as the set of the parameters c such that the trajectory of the critical point 0 of P_c does not escape to infinity under iterations of P_c. Equivalently, this is the set of all parameters c such that the Julia set $J(P_c)$ of P_c is connected.

Thurston [**10**] constructed a combinatorial model for $\mathcal{M}_2$, which can be interpreted as follows. *Laminational equivalence relations* are closed equivalence relations $\sim$ on the unit circle $\mathbb{S}$ in the complex plane $\mathbb{C}$ such that all classes are finite and the convex hulls of all classes are pairwise disjoint. A laminational equivalence relation is said to be *(σ_d-) invariant* if it is preserved under the map $\sigma_d(z) = z^d : \mathbb{S} \to \mathbb{S}$ (precise definitions are given in the next section; if no ambiguity is possible, we will simply talk about *invariant* laminational equivalence relations). The map σ_d induces a *topological polynomial* $f_\sim : \mathbb{S}/\sim \to \mathbb{S}/\sim$ from the *topological Julia set* $J_\sim = \mathbb{S}/\sim$ to itself. If $J(P_c)$ is locally connected, then $P_c|_{J(P_c)}$ is conjugate to $f_\sim$ for a specific laminational equivalence relation $\sim$. If $d = 2$, then corresponding laminational equivalence relations, topological polynomials and Julia sets are said to be *quadratic*.

A q-lamination can be viewed as a geometric object consisting of all chords in the boundaries of the convex hulls (in the closed unit disk) of all equivalence classes of $\sim$. Thurston constructed a suitable topology on the set of all quadratic topological polynomials using q-laminations as follows. Call a chord a d-critical leaf if σ_d identifies its endpoints. Hence 2-critical leaves are diameters. The space of all the 2-critical leaves is called the space of critical portraits for the map σ_2. The set of σ_2 images of all diameters parameterizes the space of all critical portraits for the map σ_2. This space is clearly a circle and we denote it by another unit circle $\mathbb{T}$ in the complex plane $\mathbb{C}$. It is known in the quadratic case that for every point in the space of critical portraits $\mathbb{T}$, there exists a unique q-lamination and hence a unique quadratic topological polynomial. However there are multiple points in $\mathbb{T}$ which correspond to the same q-lamination. Thurston studies the family of q-laminations to construct a quotient space of the space of critical portraits so that the quotient space corresponds to the family of all quadratic topological polynomials as follows.

Let f be a topological polynomial defined through the equivalence relation $\sim_f$. There is one special topological polynomial f_0 (related to the unique complex polynomial which has a parabolic fixed point with derivative 1) for which the laminational equivalence relation $\sim_{f_0}$ has only degenerate classes. The topological polynomial f_0 corresponds to a single point $(1, 0)$ in the space of critical portraits $\mathbb{T}$. Now, let f be any other topological polynomial other than f_0. By considering all outer edges of the convex hulls of all equivalence classes of $\sim_f$ we get a geometric object, called the q-lamination, $\mathcal{L}_f$. Call the closure of a component of $\mathbb{D} \setminus \mathcal{L}_f$ a gap of $\mathcal{L}_f$.

Given the q-lamination $\mathcal{L}_f$, let C_f be either the unique gap or the unique critical leaf which contains the origin O of $\mathbb{D}$. We have the following possibilities:

(1) If C_f is a critical leaf, let $\sigma_2(C_f \cap \mathbb{S}) = m_f$. Then m_f is a single point and m_f is the only point in the space of critical portraits $\mathbb{T}$ which corresponds to the q-lamination $\mathcal{L}_f$. The point m_f is called the degenerate minor of the q-lamination $\mathcal{L}_f$.

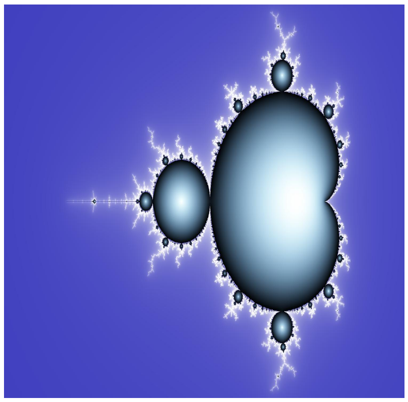


FIGURE 1.1. The Mandelbrot set $\mathcal{M}_2$

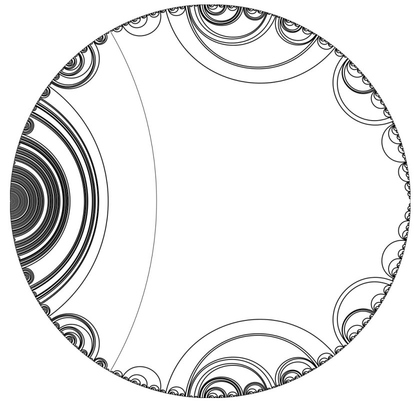


FIGURE 1.2. The quadratic minor lamination $\mathcal{L}_{\text{QML}}$

(2) If C_f is a gap and $C_f \cap \mathbb{S}$ is finite, let m_f be the convex hull of $\sigma_2(C_f \cap \mathbb{S})$. Then m_f is the minor gap of $\mathcal{L}_f$ and all the points of the set $m_f \cap \mathbb{S}$ in the space of critical portraits $\mathbb{T}$ correspond to the same q-lamination $\mathcal{L}_f$.

(3) If C_f is periodic and the first return map has degree 2 (then C_f is called a Fatou gap of $\mathcal{L}_f$). Let M_1 and M_2 be the two longest edges on the boundary of C_f (these are called majors). Then the convex hull of $\sigma_2(M_1 \cap \mathbb{S}) = \sigma_2(M_2 \cap \mathbb{S}) = m_f$ is a chord called the minor of $\mathcal{L}_f$. In this case both the points of the set $m_f \cap \mathbb{S}$ in $\mathbb{T}$ correspond to the same q-lamination $\mathcal{L}_f$.

The collection of all minors forms a q-lamination which Thurston calls QML (for quadratic minor lamination). If we collapse the space of critical portraits by identifying all points which are in the same leaf or a gap in QML, we get a locally connected continuum $\mathcal{M}_2^{Comb}$ which is called the combinatorial Mandelbrot set. It is conjecturally homeomorphic to the boundary $\partial(\mathcal{M}_2)$ of the Mandelbrot set $\mathcal{M}_2$. It is at least a good model for this space since it is known that there exist a continuous monotone function $\pi : \partial(\mathcal{M}_2) \to \mathcal{M}_2^{Comb}$ (a map π is monotone if $\pi^{-1}(y)$ is connected for all y).

The case of a similar model for $\mathcal{M}_d$ for $d > 2$ is much harder and, even after intense interest by many, is still open. For a σ_d map, $d > 2$, the choice of d-critical leaves has to be in such a way that they should not cross, i.e they should not intersect inside the open disk $\mathbb{D}$. It is easy to see that if $d = 3$, there are at most 2 non-crossing critical chords (except when they touch in which case we can add one more to obtain a triangle of 3-critical chords). The space of all non-crossing critical chords, called the space of 3-critical portraits, is already much more complicated. It can be described as follows.

For any pair of non-crossing critical chords $\{c, d\}$ that do not share an endpoint, let A be the shortest component of $\mathbb{S} \setminus \{c, d\}$. In general A is unique but it is possible that there are two such components (in that case the two critical chords are parallel). Let M_A be the midpoint of A and let L_A be the length of A. If A is unique, then the point $(M_A, L_A) \in \mathbb{S} \times (0, \frac{1}{6})$ uniquely determines the given critical portrait (here we normalize the total length of the circle to be 1). If A is not unique, there are two shortest components A and B of $\mathbb{S} \setminus \{c, d\}$ and the critical leaves c and d are parallel here. In this case, the lengths of A and B are $\frac{1}{6}$, $M_A = -M_B$ and $(M_A, L_A), (M_B, L_B) \in \mathbb{S} \times \{\frac{1}{6}\}$. Also each of the points $(M_A, \frac{1}{6})$ and $(M_B, \frac{1}{6})$ uniquely determines the given critical portrait of parallel critical leaves. Thus, the points $(x, \frac{1}{6})$ and $(-x, \frac{1}{6})$ in $\mathbb{S} \times \{\frac{1}{6}\}$ must be identified.

If the pair of non-crossing critical chords $\{c, d\}$ share an endpoint x, then let $M_A = x$ and $L_A = 0$. The point $(M_A, L_A) \in \mathbb{S} \times \{0\}$. Since two critical leaves c and d that share a common endpoint x are part of a critical triangle, the points $(x, 0), (x + \frac{1}{3}, 0)$ and $(x + \frac{2}{3}, 0)$ in $\mathbb{S} \times \{0\}$ must be identified.

Hence, the space of critical portraits for the σ_3 map is a quotient of the annulus $\mathbb{S} \times [0, \frac{1}{6}]$ where (i) the points $(x, \frac{1}{6})$ and $(-x, \frac{1}{6})$ are identified and (ii) the points $(x, 0), (x + \frac{1}{3}, 0)$ and $(x + \frac{2}{3}, 0)$ are identified.

The special role of symmetric (i.e. parallel) critical chords justifies the study of the corresponding cubic symmetric laminations (i.e., cubic invariant laminations which are invariant under 180^0 rotation).

In this paper we study cubic symmetric laminations. They correspond to affine conjugacy classes of complex polynomials of the form $P(z) = z^3 + \lambda^2 z$. We call such polynomials cubic symmetric polynomials since they correspond to cubic symmetric laminations. The results we obtain are similar to the quadratic case but the arguments are different. We show that all cubic symmetric laminations have a rotational central symmetric gap (or leaf). They do not admit wandering triangles but can have two orbits on the boundary of a finite periodic gap. The boundary of the connectedness locus in the parameter space of all cubic symmetric laminations is 1-dimensional. Rather than using minors to parametrize this space we will use the notion of comajors (siblings of the major with the same image). The collection of all comajors is a q-lamination called the cubic symmetric comajor lamination C_sCL. No comajor is periodic but we show that the collection of comajors of pre-period 1 is dense in C_sCL. Finally we produce an algorithm which produces all comajors of pre-period 1. This allows us to produce meaningful pictures of C_sCL.

In a subsequent joint paper we will show that the quotient space $\mathcal{M}_3^{Comb}$ of C_sCL, which collapses all chords of C_sCL to points is a good model for the boundary $\partial(\mathcal{M}_3^s)$ of the space of affine conjugacy classes of cubic symmetric polynomials with connected julia sets $\mathcal{M}_3^s$ since there exists a continuous and monotone function $\pi : \partial(\mathcal{M}_3^s) \to \mathcal{M}_3^{Comb}$.

CHAPTER 2

LAMINATIONS: CLASSICAL DEFINITIONS

2.1. Laminational equivalence relation

Let $\mathbb{C}$ be the complex plane and $\hat{\mathbb{C}}$ be the Riemann sphere. Let $\mathbb{D} \subset \mathbb{C}$ be the open unit disk.

Let P be a complex polynomial of degree $d \geq 2$.

DEFINITION 2.1.1 (The basin of infinity and the Julia set). *The basin of infinity* is defined as the set of complex numbers 'z' that converge to infinity upon multiple iterations of complex polynomial P, i.e

$$\Omega_P = \{z \in \mathbb{C} \mid P^{\circ n}(z) \to \infty \ (n \to \infty)\}$$

Julia set is the boundary of the basin of infinity. $J(P) = \partial\Omega_P$.

REMARK 2.1.1. (1) Julia set is completely invariant under P, $P^{-1}(J(P)) = P(J(P)) = J(P)$.

(2) J(P) is the closure of a set of *repelling periodic points*.

(3) J(P) is the boundary of the *filled-in Julia set*; that is, those points whose orbits under iterations of P remain bounded.

From the Riemann mapping theorem, there exists a conformal map $\Psi : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \hat{\mathbb{C}} \setminus K$, where K is the *filled-in Julia set* (i.e., the complement of the unbounded component of $J(P)$ in $\mathbb{C}$). One can choose Ψ so that $\Psi'(\infty) > 0$ and $P \circ \Psi = \Psi \circ \theta_d$, where $\theta_d(z) = z^d$ and d is the degree of P.

$$\begin{array}{ccc}
\hat{\mathbb{C}} \setminus \overline{\mathbb{D}} & \xrightarrow{\theta_d} & \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \\
\Big\downarrow \Psi & & \Big\downarrow \Psi \\
\hat{\mathbb{C}} \setminus K & \xrightarrow{P} & \hat{\mathbb{C}} \setminus K
\end{array}$$

We choose polynomials P such that Julia set $J(P)$ is locally connected so that Ψ extends over the boundary $\mathbb{S}$ of $\mathbb{D}$.

Let $\psi = \overline{\Psi}|_{\mathbb{S}}$. Let $\mathbb{S} = \mathrm{Bd}(\mathbb{D})$ be the unit circle identified with $\mathbb{R}/\mathbb{Z}$ and define an equivalence relation $\sim_P$ on $\mathbb{S}$ by $x \sim_P y$ if and only if $\psi(x) = \psi(y)$. Define a map $\sigma_d : \mathbb{S} \to \mathbb{S}$ by $\sigma_d(z) = dz$ mod 1, $d \geq 2$.

Since Ψ defined above conjugates θ_d and P, the map ψ semi-conjugates σ_d and $P|_{J(P)}$, which implies that $\sim_P$ is invariant. Equivalence classes of $\sim_P$ have pairwise disjoint convex hulls. The *topological Julia set* $\mathbb{S}/\sim_P = J_{\sim_P}$ is homeomorphic to J_P, and the *topological polynomial* $f_{\sim_P} : J_{\sim_P} \to J_{\sim_P}$, induced by σ_d, is topologically conjugate to $P|_{J_P}$.

$$\begin{array}{ccc}
\mathbb{S}/\sim_P & \xrightarrow{f_{\sim_P}} & \mathbb{S}/\sim_P \\
\Big\downarrow & & \Big\downarrow \\
J(P) & \xrightarrow{P} & J(P)
\end{array}$$

An equivalence relation $\sim$ on the unit circle, with similar properties to those of $\sim_P$ above, can be introduced abstractly without any reference to the Julia set of a complex polynomial.

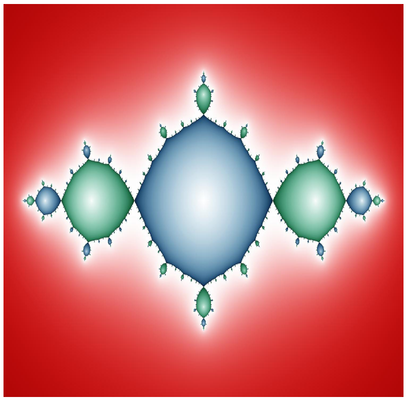


FIGURE 2.1. The Julia set of $f(z) = z^2 - 1$

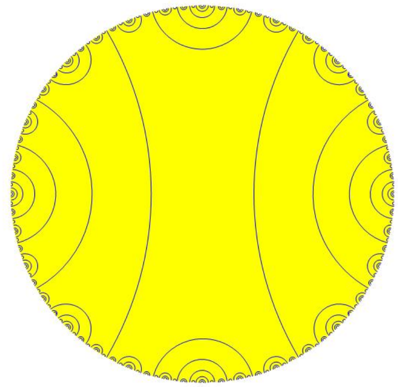


FIGURE 2.2. The topological Julia set of $z^2 - 1$ obtained by pinching the endpoints of chords.

DEFINITION 2.1.2 (Laminational equivalence relation). An equivalence relation $\sim$ on the unit circle $\mathbb{S}$ is called a *laminational equivalence relation* if it has the following properties:

(E1) the graph of $\sim$ is a closed subset in $\mathbb{S} \times \mathbb{S}$;

(E2) convex hulls of distinct equivalence classes are disjoint;

(E3) each equivalence class of $\sim$ is finite.

For a closed set $A \subset \mathbb{S}$ we denote its convex hull by $\mathrm{CH}(A)$. Then by an *edge* of $\mathrm{CH}(A)$ we mean a closed segment I of the straight line connecting two points of the unit circle such that I is contained in the boundary $\mathrm{Bd}(\mathrm{CH}(A))$ of $\mathrm{CH}(A)$. By an *edge* of a $\sim$-class we mean an edge of the convex hull of that class.

For the purpose of axiom D3 let an arc $\widehat{a\,b}$ of the circle $\mathbb{S}$ be represented as (a, b).

DEFINITION 2.1.3 (Invariance). A laminational equivalence relation $\sim$ is (σ_d-)*invariant* if:

(I1) $\sim$ is *forward invariant*: for a class $\mathbf{g}$, the set $\sigma_d(\mathbf{g})$ is a class too;

(I2) $\sim$ is *backward invariant*: for a class $\mathbf{g}$, its pre-image $\sigma_d^{-1}(\mathbf{g}) = \{x \in \mathbb{S} : \sigma_d(x) \in \mathbf{g}\}$ is a union of classes;

(I3) for any $\sim$-class $\mathbf{g}$ with more than two points, the map $\sigma_d|_{\mathbf{g}} : \mathbf{g} \to \sigma_d(\mathbf{g})$ is a *covering map with positive orientation*, i.e., for every connected component (s, t) of $\mathbb{S} \setminus \mathbf{g}$ the arc in the circle $(\sigma_d(s), \sigma_d(t))$ is a connected component of $\mathbb{S} \setminus \sigma_d(\mathbf{g})$;

2.2. Invariant Laminations

In the preceding section, we introduced the notion of a laminational equivalence relation on the unit circle $\mathbb{S}$ based on the identifications of a polynomial map on its locally connected and therefore connected Julia set. In this section ,we will define the notion of an invariant lamination on a more abstract level, which in turn gives rise to a laminational equivalence relation.

DEFINITION 2.2.1. A *lamination* $\mathcal{L}$ is a set of chords in the closed unit disk $\overline{\mathbb{D}}$, called *leaves* of $\mathcal{L}$, if it satisfies the following conditions:

(L1) leaves of $\mathcal{L}$ are disjoint, except possibly at their endpoints;

(L2) the union of $\mathcal{L}$ is closed.

We say two leaves/chords *cross each other* if they intersect inside the open disk $\mathbb{D}$. Thus, (L1) can be interpreted as the condition that no two leaves of a *lamination* can cross each other. A degenerate "leaf" is a point on $\mathbb{S}$. Given a leaf $\ell = \overline{ab} \in \mathcal{L}$, let $\sigma_d(\ell)$ be the chord with endpoints $\sigma_d(a)$ and $\sigma_d(b)$. If $\sigma_d(a) = \sigma_d(b)$, call ℓ a *critical leaf*. Let $\mathcal{L}^* = \cup_{\ell \in \mathcal{L}} \ell$ and $\sigma_d^* : \mathcal{L}^* \to \overline{\mathbb{D}}$ be the linear extension of σ_d over all the leaves in $\mathcal{L}$. It is not hard to check that σ_d^* is continuous. Also, σ_d is locally one-to-one on $\mathbb{S}$, and σ_d^* is one-to-one on any given non-critical leaf. Note that if $\mathcal{L}$ is a lamination, then $\mathcal{L}^*$ is a continuum.

Definition 2.2.2 (Gap). A *gap* G of a lamination $\mathcal{L}$ is the closure of a component of $\mathbb{D} \setminus \mathcal{L}^*$; its boundary leaves are called *edges (of a gap)*.

For each set $A \subset \overline{\mathbb{D}}$ we denote $A \cap \mathbb{S}$ by $\partial(A)$. If G is a leaf or a gap of $\mathcal{L}$, it follows that G coincides with the convex hull of $\partial(G)$. If G is a leaf or a gap of $\mathcal{L}$ we let $\sigma_d(G)$ be the convex hull of $\sigma_d(\partial(G))$. Also, by $\mathrm{Bd}(G)$ we denote the topological boundary of G. Notice that $\mathrm{Bd}(G) \cap \mathbb{S} = G \cap \mathbb{S} = \partial(G)$. We call $\partial(G)$ the vertices of G. A gap G is called *infinite* if and only if $\partial(G)$ is infinite. Similarly G is called *finite* if and only if $\partial(G)$ is finite, in particular the gap G is called *triangular gap* if $\partial(G)$ consists of three points.

Let $\mathcal{L}$ be a lamination. The equivalence relation $\sim_{\mathcal{L}}$ induced by $\mathcal{L}$ is understood by declaring that $x \sim_{\mathcal{L}} y$ if and only if there exists a finite concatenation of leaves of $\mathcal{L}$ joining x to y.

Definition 2.2.3 (q-lamination). A lamination $\mathcal{L}$ is called a *q-lamination* if the equivalence relation $\sim_{\mathcal{L}}$ is a laminational equivalence relation and $\mathcal{L}$ consists exactly of boundary edges of the convex hulls of $\sim_{\mathcal{L}}$ classes.

Remark 2.2.1. Since a *q-lamination* $\mathcal{L}$ consists of boundary edges of the convex hulls of $\sim_{\mathcal{L}}$ classes, if two leaves of $\mathcal{L}$ share an endpoint, they must form sides of a common gap. It follows that no more than two leaves of a q-lamination can share an endpoint.

Definition 2.2.4 (Invariant lamination). A lamination $\mathcal{L}$ is *(σ_d-)invariant* if,

(D1) $\mathcal{L}$ is *forward invariant.* For each $\ell \in \mathcal{L}$ either $\sigma_d(\ell) \in \mathcal{L}$ or $\sigma_d(\ell)$ is a point in $\mathbb{S}$ and

(D2) $\mathcal{L}$ is *backward invariant.*

(1) For each $\ell \in \mathcal{L}$ there exists a leaf $\ell' \in \mathcal{L}$ such that $\sigma_d(\ell') = \ell$.

(2) For each $\ell \in \mathcal{L}$ such that $\sigma_d(\ell)$ is a non-degenerate leaf, there exists **d disjoint** leaves $\ell_1, ...\ell_d$ in $\mathcal{L}$ such that $\ell = \ell_1$ and $\sigma_d(\ell_i) = \sigma_d(\ell)$ for all i.

DEFINITION 2.2.5 (Siblings). The d disjoint leaves from the above definition are part of what we call a *sibling* collection of leaves as they all map onto the same leaf under σ_d.

DEFINITION 2.2.6 (Monotone Map). Let X, Y be topological spaces and $f : X \to Y$ be continuous. Then f is said to be *monotone* if $f^{-1}(y)$ is connected for each $y \in Y$. It is known that if f is monotone and X is a continuum then $f^{-1}(Z)$ is connected for every connected $Z \subset f(X)$.

DEFINITION 2.2.7 (Gap-invariance). A lamination $\mathcal{L}$ is *gap invariant* if for each gap G , $\sigma_d(G)$ is either a gap or a leaf or a single point of $\mathcal{L}$ with the following rule:-

$\sigma_d^*|_{\mathrm{Bd}(G)} : \mathrm{Bd}(G) \to \mathrm{Bd}(\sigma_d(G))$ must map as the composition of a monotone map and a covering map to the boundary of the image gap, with positive orientation i.e, as you move through the vertices of G in clockwise direction around $\mathrm{Bd}(G)$, their corresponding images in $\sigma_d(G)$ must also be aligned clockwise in $\mathrm{Bd}(\sigma_d(G))$.

DEFINITION 2.2.8 (Degree). $\forall x \in \mathrm{Bd}(\sigma_d(G))$, number of components of $(\sigma_d^*)^{-1}(x)$ in $\mathrm{Bd}(G)$ is defined as the degree of the map $\sigma_d^*|_{\mathrm{Bd}(G)} : \mathrm{Bd}(G) \to \mathrm{Bd}(\sigma_d(G))$.
In other words, if every leaf of $\sigma_d(G)$ has k pre-image leaves in G, then the degree of the map σ_d^* is k. A gap G is called *critical* gap if $k > 1$.

The following important results about the invariant laminations are proved in [**2**].

THEOREM 2.2.2 ([**2**]). *Every* (σ_d-)invariant *lamination is gap invariant.*

THEOREM 2.2.3 ([**2**]). *The space of all σ_d-invariant laminations is compact.*

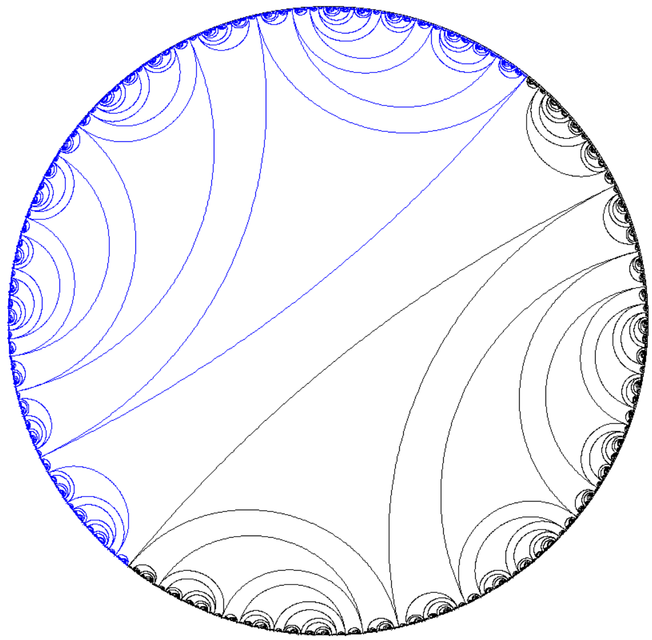


FIGURE 2.3. A σ_2 invariant lamination called Douady rabbit lamination.

CHAPTER 3

CUBIC SYMMETRIC LAMINATIONS

3.1. Cubic polynomials of the form $z \mapsto z^3 + \lambda^2 z$.

Consider a σ_3-invariant laminational equivalence relation $\sim$ corresponding to the map $z \mapsto z^3 + \lambda^2 z$. Apart from satisfying the axioms (I1)-(I3) of Definition 2.1.3, the laminational equivalence relation $\sim$ satisfies one additional property. Since the map $z \mapsto z^3 + \lambda^2 z$ is an odd function, $\sim$ has the property that for any equivalence class $\mathbf{g}$ in $\sim$, the equivalence class $-\mathbf{g}$ is also in it (note that the class $\mathbf{g}$ is understood as a subset of the unit circle $\mathbb{S} = \mathbb{R}/\mathbb{Z}$). In other words, $\sim$ has a symmetry of rotation by 180° with respect to the center of the circle $\mathbb{S}$. We will study σ_3-invariant laminations obtained from such laminational equivalence relations.

The unit circle $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ can be parametrized as [0,1) for the study of laminations. Under this parametrization, two endpoints of a diameter of $\mathbb{S}$ differ by $\frac{1}{2}$. It follows that if a leaf ℓ' is obtained by rotating a leaf ℓ by 180° with respect to the center of the circle $\mathbb{S}$, then their endpoints would differ by $\frac{1}{2}$.
Now, we define a specific kind of σ_3 invariant laminations called *cubic symmetric laminations* with the additional property mentioned above as follows.

Definition 3.1.1 (Cubic symmetric lamination). A σ_3-invariant lamination $\mathcal{L}$ is called a *cubic symmetric lamination* if:
(D3) for each $\ell \in \mathcal{L}$ there exists a leaf $\ell' \in \mathcal{L}$ such that ℓ' and ℓ are symmetric with respect to the center of the circle $\mathbb{S}$, i.e $\partial(\ell') = \partial(\ell) + \frac{1}{2}$ (note that $\partial(\ell) = \text{Bd}(\ell) \cap \mathbb{S}$).

We will show in the later sections that this property of cubic symmetric laminations leads to similar results that were obtained for quadratic (σ_2-invariant) laminations. The current and the following two chapters are dedicated to a detailed study of cubic symmetric laminations and their parameter space.

3.2. Cubic symmetric lamination: Basic properties

Below are a few basic definitions concerning periodic and (pre)periodic leaves/gaps.

DEFINITION 3.2.1 (Preperiodic and (pre)periodic points). A point $x \in \mathbb{S}$ is said to be *(pre)periodic* if $\sigma_3^{m+k}(x) = \sigma_3^m(x)$ for some $m \geq 0, k \geq 1$. Then, for the smallest m and k that satisfy the above equation, we call m the *preperiod* and k the *period* of x. The point x is said to be *preperiodic* if $m > 0$ or periodic (of *period* k) if $m = 0$.

(1) *Preperiodic and (pre)periodic leaves.* Let ℓ be a leaf of a *cubic symmetric lamination* $\mathcal{L}$. The leaf ℓ is said to be (pre)periodic of preperiod m and period k, if the endpoints a and b of ℓ are (pre)periodic of preperiod m and period k. The leaf ℓ is said to be preperiodic if $m > 0$ or periodic (of period k) if $m = 0$.
(2) *Preperiodic and (pre)periodic gaps.* Let G be a gap of a cubic symmetric lamination $\mathcal{L}$. The gap G is said to be (pre)periodic if $\sigma_3^{m+k}(G) = \sigma_3^m(G)$ for some $m \geq 0, k \geq 1$. Then, for the smallest m and k that satisfy the above equation, we call m the preperiod and k the period of G. The gap G is said to be preperiodic if $m > 0$ or periodic (of period k) if $m = 0$. If the period of G is 1, then G is said to be invariant.
(3) *(pre)critical gaps.* Similarly we can define a *(pre)critical* gap G if $\sigma_3^k(G)$ is critical gap for some $k \geq 0$.

Similar to an invariant gap, a leaf ℓ is said to be invariant under the map σ_3^k if $\sigma_3^k(\ell) = \ell$. It is not hard to see that invariant leaves (under the σ_3^k map) are periodic of period 1 or period 2. Period of an invariant leaf ℓ is 1 when its endpoints are fixed and the period is 2 when the endpoints map to each other. For the map σ_3, the leaves $\overline{0\frac{1}{2}}$ and $\overline{\frac{1}{4}\frac{3}{4}}$ are the only invariant leaves. The leaf $\overline{0\frac{1}{2}}$ is periodic of period 1 and the leaf $\overline{\frac{1}{4}\frac{3}{4}}$ is periodic of period 2.

The first two evident properties that are observed in cubic symmetric laminations are listed in the following proposition. Usually when we use the word 'symmetric', we mean symmetric with respect to the center of the circle $\mathbb{S}$.

PROPOSITION 3.2.1. *(1) The set of chords obtained by rotating the leaves of a cubic symmetric lamination $\mathcal{L}$ by $90°$ is a cubic symmetric lamination.*

(2) Every cubic symmetric lamination has an invariant gap or leaf which is symmetric with respect to the center of the circle $\mathbb{S}$.

PROOF. (1) Let $\mathcal{L}$ be a cubic symmetric lamination and $\mathcal{L}'$ be obtained by rotating the leaves of $\mathcal{L}$ by $90°$. Consider a leaf $\ell \in \mathcal{L}$ and its endpoints, denoted by $\partial(\ell)$ from the previous chapter. Let $\ell' \in \mathcal{L}'$ be obtained by rotating the leaf $\ell \in \mathcal{L}$ by $90°$. Rotating a leaf ℓ by $90°$ is equivalent to adding $\frac{1}{4}$ to its endpoints, i.e if $\partial(\ell) = \{a, b\}$, then $\partial(\ell') = \{a + \frac{1}{4}, b + \frac{1}{4}\} = \partial(\ell) + \frac{1}{4}$.

Claim1. (D1) $\mathcal{L}'$ is *forward invariant:*

Consider the image leaves of ℓ and ℓ' denoted by $\tilde{\ell} = \sigma_3(\ell)$ and $\sigma_3(\ell')$ respectively. The leaf $\sigma_3(\ell')$ has the endpoints $\sigma_3(\{a+\frac{1}{4}, b+\frac{1}{4}\}) = \sigma_3(\{a, b\}) + \frac{3}{4} = \partial(\tilde{\ell}) + \frac{3}{4} = \partial(\tilde{\ell}) + \frac{1}{2} + \frac{1}{4}$.

Using the property (D3) of cubic symmetric lamination $\mathcal{L}$, there exists a leaf $\hat{\ell} \in \mathcal{L}$ such that $\hat{\ell}$ and $\tilde{\ell}$ are symmetric with respect to the center of the circle, i.e $\partial(\hat{\ell}) = \partial(\tilde{\ell}) + \frac{1}{2}$. It implies that $\sigma_3(\ell')$ has the endpoints $\partial(\hat{\ell}) + \frac{1}{4}$.

Thus, $\ell' \in \mathcal{L}'$ maps to $\hat{\ell}' \in \mathcal{L}'$ making $\mathcal{L}'$ *forward invariant.*

Note that the angle between the leaves ℓ and ℓ' grows from 90° to 270° under the σ_3 map:

$$\begin{array}{ccc} \ell & \xrightarrow{\sigma_3} & \tilde{\ell} \\ {\scriptstyle 90^\circ \curvearrowleft}\downarrow & & {\scriptstyle 270^\circ \curvearrowleft}\downarrow \\ \ell' & \xrightarrow{\sigma_3} & \hat{\ell}' \end{array}$$

Claim2. (D2) $\mathcal{L}'$ is *backward invariant:*

We use the argument in claim1 to find pre-image leaves of $\ell' \in \mathcal{L}'$. First using the property (D2) of cubic symmetric lamination $\mathcal{L}$, we obtain three disjoint pre-image leaves, say, ℓ_1, ℓ_2, ℓ_3 of $\ell \in \mathcal{L}$, i.e $\sigma_3(\ell_1) = \sigma_3(\ell_2) = \sigma_3(\ell_3) = \ell$. Using (D3) now, let $\{\hat{\ell}_1, \hat{\ell}_2, \hat{\ell}_3\}$ be the leaves of $\mathcal{L}$ symmetric to $\{\ell_1, \ell_2, \ell_3\}$ respectively. There exists corresponding leaves $\{\hat{\ell}_1{}', \hat{\ell}_2{}', \hat{\ell}_3{}'\}$ in $\mathcal{L}'$.

Sub-claim. $\{\hat{\ell}_1{}', \hat{\ell}_2{}', \hat{\ell}_3{}'\}$ *are pre-image leaves of* ℓ' *in* $\mathcal{L}'$*:*

Leaves $\{\hat{\ell}_1, \hat{\ell}_2, \hat{\ell}_3\}$ map to a leaf $\hat{\ell}$ symmetric to leaf ℓ in $\mathcal{L}$. Correspondingly, leaves $\hat{\ell}'$ and ℓ' are symmetric pair of leaves in $\mathcal{L}'$.

In claim1, it is clearly seen that if ℓ maps to $\sigma_3(\ell)$, then ℓ' maps to the leaf symmetric to $\sigma_3(\ell)'$ in $\mathcal{L}'$. Thus, leaves $\{\hat{\ell}_1{}', \hat{\ell}_2{}', \hat{\ell}_3{}'\}$ map to the leaf symmetric to $\hat{\ell}'$ which is ℓ'.

$$\begin{array}{ccc} \{\hat{\ell}_1, \hat{\ell}_2, \hat{\ell}_3\} & \xrightarrow{\sigma_3} & \hat{\ell} \\ {\scriptstyle 90^\circ \curvearrowleft}\downarrow & & {\scriptstyle 270^\circ \curvearrowleft}\downarrow \\ \{\hat{\ell}_1{}', \hat{\ell}_2{}', \hat{\ell}_3{}'\} & \xrightarrow{\sigma_3} & \ell' \end{array}$$

Claim3: (D3) $\mathcal{L}'$ is *symmetric:*

For every pair of symmetric leaves ℓ and $\hat{\ell}$ in $\mathcal{L}$, corresponding leaves ℓ' and $\hat{\ell}'$ are also symmetric with respect to the center of the circle in $\mathcal{L}'$.

(2) For the cubic symmetric lamination $\mathcal{L}$, there exists either a gap G having center of the circle in its interior or a diameter of $\mathbb{S}$ as a leaf of $\mathcal{L}$. We will consider the case when it is a gap. The case of leaf is similar.

  (a) From Definition 3.1.1 (D3), there exists a gap $-G$ containing leaves that are symmetric to the leaves in the gap G. We claim that G coincides with $-G$. If gaps G and $-G$ are distinct gaps in the lamination $\mathcal{L}$, they would intersect resulting in leaves of the lamination $\mathcal{L}$ crossing each other. Thus, G is symmetric with respect to the center of the circle $\mathbb{S}$.

  (b) We claim that if a gap G is symmetric, then its image gap $\sigma_3(G)$ is also symmetric. Consider a pair of symmetric vertices p and $p + \frac{1}{2}$ of G. We observe that $\{p, p + \frac{1}{2}\} \in \partial(G)$ implies $\sigma_3(\{p, p + \frac{1}{2}\}) = \{\sigma_3(p), \sigma_3(p + \frac{1}{2})\} = \{\sigma_3(p), \sigma_3(p) + \frac{1}{2}\} \in \partial(\sigma_3(G))$. Hence $\sigma_3(G)$ is a symmetric gap, too. Now, since both G and $\sigma_3(G)$ are symmetric gaps, they share the center of the circle in its interior. It follows that both G and $\sigma_3(G)$ should coincide, for otherwise they would intersect resulting in the same contradiction that leaves of the lamination $\mathcal{L}$ cross each other. Thus, G is an invariant gap.

□

The gap G obtained in Proposition 3.2.1 (2) is called the *central symmetric gap* of the lamination $\mathcal{L}$. Similarly, we call the leaf ℓ, a central symmetric leaf if it is invariant and symmetric. It is easy to see that, $\overline{0\frac{1}{2}}$ and $\overline{\frac{1}{4}\frac{3}{4}}$ are the only possible diameters of $\mathbb{S}$ which are invariant under the σ_3 map. Thus, the set of all possible central symmetric leaves of cubic symmetric laminations is $\{\overline{0\frac{1}{2}}, \overline{\frac{1}{4}\frac{3}{4}}\}$. We usually only talk about central symmetric gaps in the current and the next chapter, but all the statements that are true for central symmetric gaps also hold true for central symmetric leaves $\overline{0\frac{1}{2}}$ and $\overline{\frac{1}{4}\frac{3}{4}}$. The formal definition of a central symmetric gap is as follows.

Definition 3.2.2. If a gap G has center of the circle $\mathbb{S}$ in its interior and its vertices $\partial(G)$ consist of pairs of diametrically opposite points, then we call it a central symmetric gap.

From the above proposition, every cubic symmetric lamination contains an invariant central symmetric gap. In the next section, we will show that the degree of the map σ_3 acting on the central symmetric gap G has to be 1. Definition 2.2.6 part(2) tells us that order has to be preserved on the image of a gap. For a periodic/invariant gap G with the first return map g ($g(G) = G$) of degree 1, the order-preserving property translates to what we call as exhibiting to *rotational* (formal Definition 4.1.2) behavior. In particular all central symmetric gaps are rotational. Their rotational aspects are discussed in detail in the next chapter.

There are two properties of a cubic symmetric lamination namely *Short Strip Lemma* and *No Wandering Triangles* which resemble results that hold for quadratic laminations. These are quiet essential in understanding gaps of the cubic symmetric lamination and also help prove key results for the parameter space of the cubic symmetric laminations.

Let $\mathcal{L}$ be a cubic symmetric lamination. We define the length $\|\ell\|$ of a leaf ℓ in $\mathcal{L}$ to be the distance between its endpoints, measured along the circle in units of $\frac{radians}{2\pi}$. Thus, the maximum length of a leaf is $\frac{1}{2}$.

Let us characterize leaves of $\mathcal{L}$ into three categories by their lengths as follows-

Definition 3.2.3. A *short leaf* is a leaf ℓ such that $0 < \|\ell\| < \frac{1}{6}$, a *medium leaf* is a leaf ℓ such that $\frac{1}{6} \leq \|\ell\| < \frac{1}{3}$ and a *long leaf* is a leaf ℓ such that $\frac{1}{3} < \|\ell\| \leq \frac{1}{2}$.

A leaf of length $\frac{1}{3}$ is a *critical leaf.* Every leaf ℓ of $\mathcal{L}$ has siblings (Definition 2.2.5). It is easy to see that following are the only possible kinds of leaves in a sibling collection of ℓ.

(a) $\{\ell, \hat{\ell}, \tilde{\ell}\}$ are all *short* leaves.

(b) $\{\ell, \hat{\ell}, \tilde{\ell}\}$ are all *medium* leaves.

(c) $\{\ell, \hat{\ell}, \tilde{\ell}\}$ are *long,medium* and *short* leaves respectively.

Note that in a sibling collection, if two leaves are of the same kind, they have the same length. For a cubic symmetric lamination $\mathcal{L}$, we claim that the collection (b) is not possible. First thing we observe that the leaves in both the triplets, $\{\ell, \hat{\ell}, \tilde{\ell}\}$ and $\{-\ell, -\hat{\ell}, -\tilde{\ell}\}$ are all disjoint from each other and they have the same length. They have the same length because either they are siblings or they are symmetric to each other. Any two symmetric leaves in the two triplets map to disjoint symmetric images $\sigma_3(\ell)$ and $-\sigma_3(\ell)$, hence disjoint from each other. And any two sibling leaves are clearly disjoint from each other. It follows that in the collection (b), if all the leaves have length bigger than $\frac{1}{6}$, then the sum of lengths of the leaves in $\{\ell, \hat{\ell}, \tilde{\ell}\}$ and $\{-\ell, -\hat{\ell}, -\tilde{\ell}\}$ would add up to a number bigger than 1. Clearly, a contradiction.
If all of them have the length exactly $\frac{1}{6}$, then we would get a finite gap G with all of the above six leaves as sides. It implies that the image gap $\sigma_3(G)$ contains leaves $\sigma_3(\ell)$ and $-\sigma_3(\ell)$ which are of length $\frac{1}{2}$ each. Clearly, a contradiction since no gap of $\mathcal{L}$ can contain two diameters of the circle $\mathbb{S}$ on its boundary.

Thus, collections (a) and (c) are the only ones that are possible. Let us focus on collection (c). If a leaf ℓ is a medium or a long leaf, then there exists a long or a medium leaf $\hat{\ell}$ as its sibling. Consider the closed subset of the closed disk $\mathbb{D}$ bounded by the leaves ℓ and $\hat{\ell}$ that does not contain the center of $\mathbb{S}$. Let us call that region $C(\ell)$.

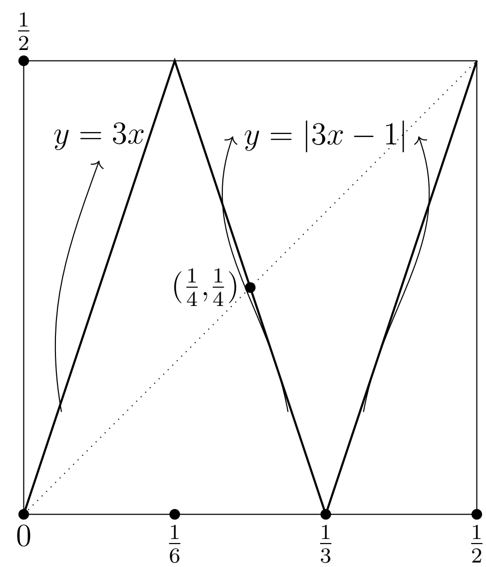


FIGURE 3.1. Graph of the length function. Length of the leaf ($\|\ell\|$) on x-axis and length of the image leaf ($\|\sigma_3(\ell)\|$) on the y-axis.

REMARK 3.2.2. (1) Length of the image leaves of the σ_3 map :-

$$\|\sigma_3(\ell)\| = \begin{cases} 3\|\ell\| & 0 \leq \|\ell\| \leq \frac{1}{6}, \\ |3\|\ell\| - 1| & \frac{1}{6} < \|\ell\| \leq \frac{1}{2} \end{cases}$$

(2) If $\frac{1}{4} < \|\ell\| < \frac{1}{2}$, we have $\|\sigma_3(\ell)\| < \|\ell\|$ and $\|\sigma_3(\ell)\| \geq \|\ell\|$ otherwise. Any leaf of length smaller than $\frac{1}{4}$ grows and leaves of length bigger than $\frac{1}{4}$ shrink and a leaf of length $\frac{1}{4}$ will have the image leaf of same length.
(3) Among leaves of length bigger than $\frac{1}{4}$, the closer the leaves get to a critical chord (of length $\frac{1}{3}$) of the circle, shorter their images get.

PROPOSITION 3.2.3. *Let $\mathcal{L}$ be a cubic symmetric lamination. Let ℓ be a medium or a long leaf. If k is minimal such that $\ell_k = \sigma_3^k(\ell)$ intersects the region $C(\ell) \cup -C(\ell)$, then ℓ_k is either a long or a medium leaf.*

Proof. Consider the sequence of lengths $L_i = \|\ell_i\|$ of the forward images $\ell_i = \sigma_3^i(\ell)$ of the leaf ℓ. Let $L = \|\ell\|$. We have two cases here.

(i) *If* $L < \frac{1}{4}$.

From Remark 3.2.2 (2), the leaf ℓ_1 is bigger than the leaf ℓ. We claim that the leaf ℓ_1 intersects the region $C(\ell) \cup -C(\ell)$. Consider the long sibling $\hat{\ell}$ of the leaf ℓ. We have $\sigma_3(\ell) = \sigma_3(\hat{\ell}) = \ell_1$. Using Remark 3.2.2 (2) again, we deduce that the leaf ℓ_1 is smaller than the leaf $\hat{\ell}$. It follows that the leaf ℓ_1 intersects the region $C(\ell) \cup -C(\ell)$ and ℓ_1 is clearly either a long or a medium leaf. In other words k in the lemma is equal to 1.

(ii) *If* $L \geq \frac{1}{4}$.

Claim. The leaf ℓ_1 is shortest among the forward images of the leaf ℓ until the leaf ℓ_{k+1}.

From Figure 3.1, we can deduce that if $L = \frac{1}{3}$, then ℓ_1 is a point, $L_1 = 0$. If $L = \frac{1}{4}$, then all the forward images ℓ_i have the same length $\frac{1}{4}$. It follows that in both the above sub-cases, claim is trivial.

Let $L \in (\frac{1}{4}, \frac{1}{3}) \cup (\frac{1}{3}, \frac{1}{2})$.

Using the length function in Remark 3.2.2 (1), length of the leaf ℓ_1 is $\|\sigma_3(\ell)\| = L_1 = |1 - 3L|$. Let us consider minimal j such that $j > 1$ and $L_j < L_1$. Since j is minimal, $L_{j-1} > L_j$. The leaf ℓ_{j-1} could not have been a short leaf as short leaves only grow (Remark 3.2.2 (2)). And using Remark 3.2.2 (3), we have that the long/medium leaf ℓ_{j-1} is more closer to a critical chord of $\mathbb{S}$ than ℓ, i.e $|L_{j-1} - \frac{1}{3}| < |L - \frac{1}{3}|$. It follows that the leaf ℓ_{j-1} intersects the region $C(\ell) \cup -C(\ell)$. As j is minimal, it we can conclude that $\ell_k = \ell_{j-1}$, i.e $j - 1 = k$.

Note that the width of each strip in the region $C(\ell) \cup -C(\ell)$ is $\frac{1}{3} * L_1$. So, for a short leaf to intersect the region $C(\ell) \cup -C(\ell)$, it has to have length smaller than $\frac{1}{3} * L_1$. And from the above claim, none of the forward images of the leaf ℓ till the leaf ℓ_{k+1} can have a length smaller than L_1. Thus, ℓ_k is either a long or a medium leaf.

□

DEFINITION 3.2.4. **Short Strips -** Given a medium or a long leaf ℓ in the cubic symmetric lamination $\mathcal{L}$, we call the region $C(\ell) \cup -C(\ell)$ as its Short Strips and each of the connected component, a Short Strip.

REMARK 3.2.4. Few things to be noted about Short Strips of ℓ are-

(a) Both the Short Strips are bounded by a pair of medium and a long leaf. Short Strip $C(\ell)$ is bounded by ℓ and its sibling $\hat{\ell}$ and Short Strip $-C(\ell)$ is bounded by $-\ell$ and its sibling $-\hat{\ell}$.

(b) If a critical chord of $\mathbb{S}$ has to be drawn without crossing any of the four leaves $\{\ell, \hat{\ell}, -\ell, -\hat{\ell}\}$ bounding the Short Strips, then it has to lie inside them.

(c) Any leaf or a gap of $\mathcal{L}$ in the complementary components of Short Strips map $1-1$ onto its image.

A useful corollary to Proposition 3.2.3 is as follows:

COROLLARY 3.2.5. ***(Short Strip Lemma).*** *If a leaf ℓ is such that it is closest to the critical chord (length of ℓ closest to $\frac{1}{3}$) among its forward images, then no forward image of ℓ can enter its Short Strips $C(\ell) \cup -C(\ell)$.*

PROOF. First,we observe that as every leaf on forward iterates of σ_3 grows to a leaf of length at least $\frac{1}{4}$ in the future, the leaf ℓ closest to a critical chord of the

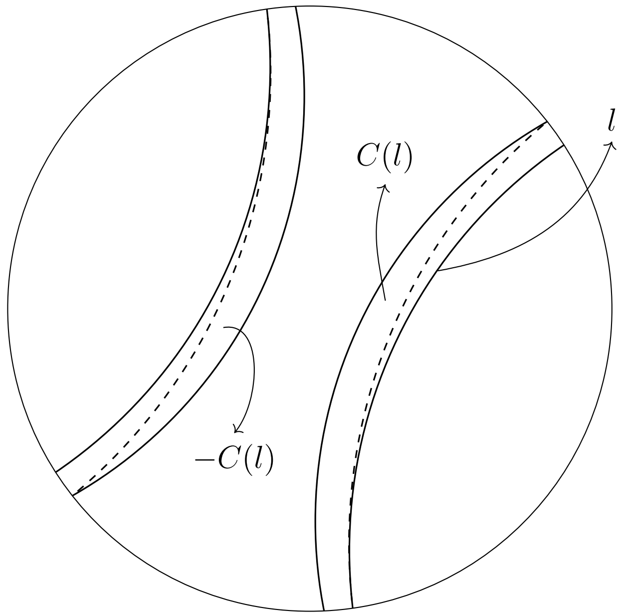


FIGURE 3.2. Short Strip of the leaf l : $C(l) \cup -C(l)$. Dashed line is a critical chord inside the strip.

circle has to have length at least $\frac{1}{4}$. There are no long or medium leaves ℓ_i among the forward images of leaf ℓ satisfying $|\|\ell_i\| - \frac{1}{3}| < |\|\ell\| - \frac{1}{3}|$. In other words, there are no long or medium leaves ℓ_i intersecting the Short Strips $C(\ell) \cup -C(\ell)$. This is a trivial condition in Proposition 3.2.3 and it follows that no forward image of ℓ can enter the Short Strips $C(\ell) \cup -C(\ell)$. □

THEOREM 3.2.6. *(**No Wandering Triangles**). Let $\mathcal{L}$ be a cubic symmetric lamination and G be a triangular gap of $\mathcal{L}$. If G does not eventually map to a leaf under the map σ_3, then G is (pre)periodic.*

PROOF. Let us assume by the way of contradiction that G is not (pre)periodic. Then there exists an infinite sequence of gaps $\{G_i\}_{i=0}^{\infty}$ where $G_i = \sigma_3^i(G)$ and $G_0 = G$. Let d_i be the length of the shortest leaf in G_i. As G does not eventually map to a leaf under the σ_3 map, $d_i \neq 0$ $\forall i$. We can show that the euclidean area in $\overline{\mathbb{D}}$ is bounded below in terms of d_i. And since the total area of $\overline{\mathbb{D}}$ is finite, $d_i \to 0$. It implies that there are infinitely many d_i that are smaller than all previous d_i and all of them get arbitrarily close to zero. Such short leaves cannot be the images of short leaves. It follows from Remark 3.2.2(3) that they must be the images of leaves of lengths converging to $\frac{1}{3}$. It follows that gaps G_i converge to a critical leaf $\ell \in \mathcal{L}$.

From property (D3), it follows that there exists gaps $-G_i$ that coexist with gaps G_i in the cubic symmetric lamination $\mathcal{L}$. Clearly, gaps $-G_i$ also should wander. Thus for the contradiction to work, the sequences of gaps G_i and $-G_i$ should converge to the pair of critical leaves ℓ and $-\ell$ respectively in $\mathcal{L}$. Let ℓ_i be the leaf of G_i closest to ℓ. In other words, length $L_i = \|\ell_i\|$ is closest to $\frac{1}{3}$ among all sides of G_i.

Claim1. First we claim that every gap G_i has a short leaf. In other words, $d_i < \frac{1}{6}\ \forall i$.

If all the leaves of G_i have length more than $\frac{1}{6}$, then there exists at least one leaf in G_i having length bigger than $\frac{1}{3}$. Similar thing happens with $-G_i$. It follows that the critical leaves ℓ and $-\ell$ run in the interior of G_i and $-G_i$ resulting in leaves of the lamination $\mathcal{L}$ crossing each other. If $d_i = \frac{1}{6}$ and other leaves of G_i have length bigger than $\frac{1}{6}$, then, too, we would have a leaf in G_i having length bigger than $\frac{1}{3}$ and the same argument works.

If two leaves of G_i have length $\frac{1}{6}$, the third side would be a critical leaf which means G_i collapses to a leaf contradicting the hypothesis.

Consider two gaps G_m and G_k in the sequence of gaps $\{G_i\}_{i=0}^{\infty}$ satisfying the following conditions.

(a) There are three conditions for G_m as follows. (i) G_m has a side ℓ_m closest to a critical leaf than any side of any G_i with $i < m$, i.e $|L_m - \frac{1}{3}| < |L_i - \frac{1}{3}|\ \forall i < m$, (ii) $|L_m - \frac{1}{3}| < d_0$ and (iii) $L_m \geq \frac{1}{4}$. We can make sure such a gap exists in the sequence because the gaps G_i converge to a critical leaf $\ell \in \mathcal{L}$.

(b) Let $k > m$ be minimal so that G_k is more closer to a critical leaf than G_m or in other words G_k has a side ℓ_k inside the short strips $C(\ell_m) \cup -C(\ell_m)$ such that $|L_k - \frac{1}{3}| < |L_m - \frac{1}{3}|$ (there exists such k since the gaps G_i have to get closer to ℓ).

Note that the whole triangle G_k has to be inside $C(\ell_m) \cup -C(\ell_m)$ which implies that $d_k \leq |\frac{1}{3} - L_m|$.

Claim2. $d_i > |3L_m - 1| \; \forall 1 \leq i \leq k$ and $i \neq m$.

Note that no gap in the sequence $\{G_i\}_{i=0}^{\infty}$ can map to a gap in the sequence $\{-G_i\}_{i=0}^{\infty}$ because it would result in the infinite sequences becoming finite. If G_i cannot map to $-G_j$, then G_i cannot map to its sibling gap $-\widehat{G_j}$, too. Combining these two arguments, we can now conclude that $\forall i < k$ and $i \neq m$, G_i is outside the short strips $C(\ell_m) \cup -C(\ell_m)$, i.e $|L_m - \frac{1}{3}| < |L_i - \frac{1}{3}| \; \forall i < k$ and $i \neq m$.

Claim1 says there will be at least one short leaf for every G_i. The shortest leaf of G_i is either obtained by image of either a leaf closer to a critical chord or image of a short leaf of G_{i-1}. It implies that $d_i = \min\{3d_{i-1}, |3L_{i-1} - 1|\}$.

We have $\forall i < k$ and $i \neq m$; $|L_m - \frac{1}{3}| < |L_i - \frac{1}{3}| \implies |3L_i - 1| > |3L_m - 1|$.

We will prove the claim2 by induction up to number k.

(a) $i = 1$ **:-** d_1 =min$\{3d_0, |3L_0 - 1|\}$. From the construction of G_m, we have $3d_0 > |3L_m - 1|$ and from the earlier inequality for $i = 0$, we have $|3L_0 - 1| > |3L_m - 1|$.

Thus, $d_1 > |3L_m - 1|$.

(b) *Induction step:* $i \to i+1$. Let us assume $d_i > |3L_m - 1|$. Since, $\forall i < k; |3L_i - 1| > |3L_m - 1|$, it implies that d_{i+1} =min$\{3d_i, |3L_i - 1|\} >$ min$\{3|3L_m -$

$1|, |3L_m - 1|\} > |3L_m - 1|$.

Note that the last induction step is $k-1 \to k$.

Now from the claim2, we have $|3L_m - 1| < d_k$ and preceding claim2, we have the inequality $d_k \leq \frac{|3L_m - 1|}{3}$. Combining both inequalities, we get $|3L_m - 1| < \frac{|3L_m - 1|}{3}$ giving us the contradiction. □

The above theorem tells us that every gap G of a cubic symmetric laminations is either *(pre)critical* or *(pre)periodic* (or both).

3.3. Classification of gaps

Using the results obtained in the previous section, we shall make a qualitative analysis of gaps for cubic symmetric laminations. Recall the notion of degree of the map $\sigma_d^*|_{\mathrm{Bd}(G)} : \mathrm{Bd}(G) \to \mathrm{Bd}(\sigma_d(G))$ from the Definition 2.2.8.

THEOREM 3.3.1. *Let $\mathcal{L}$ be a cubic symmetric lamination and G be a gap of $\mathcal{L}$. Then either*

(a) G eventually collapses to a leaf or

(b) G eventually maps to a periodic gap $\tilde{G}$ and the first return map g on $\tilde{G}$ ($g(\tilde{G}) = \tilde{G}$) is either of degree 1 ,2 or 4.

PROOF. From the *No Wandering Triangles* Theorem 3.2.6, it is clear that G eventually collapses to a leaf or maps to a periodic gap $\tilde{G}$.
Consider the periodic gap $\tilde{G}$ and the first return map g on it ($g(\tilde{G}) = \tilde{G}$). The possible degrees for the map g can be either 1,2,3 or 4. If a gap in the periodic orbit of $\tilde{G}$ has both the critical chords of the circle in their interior, then the degree of g is 3. If two of the gaps in the periodic orbit of $\tilde{G}$ has a critical chord of the circle each in their interior, then the degree of g is 4. Clearly, degree of g cannot be more than 4

as there cannot be more than two critical chords in a circle. If only one of the gaps in the periodic orbit of $\tilde{G}$ contain a critical chord in its interior, then degree of g is 2. If none of the gaps in the periodic orbit of $\tilde{G}$ have a critical chord in their interior, then degree of g is 1.

The thing left to prove is that there cannot be a periodic gap $\tilde{G}$ with the first return map g of degree 3. Let the gap $\tilde{G}$ contain two critical chords in its interior. From the property (D3) of a cubic symmetric lamination $\mathcal{L}$, there should be a gap $-\tilde{G}$ symmetric to $\tilde{G}$ in the lamination $\mathcal{L}$. It is clear that like $\tilde{G}$, -$\tilde{G}$ also contains two critical chords in its interior. It follows that the only way for the gaps $\tilde{G}$ and $-\tilde{G}$ to coexist in $\mathcal{L}$ is if both $\tilde{G}$ and $-\tilde{G}$ coincide, i.e for every leaf ℓ in $\mathrm{Bd}(\tilde{G})$ there exists a symmetric leaf $-\ell$ in $\mathrm{Bd}(\tilde{G})$.
Now consider the biggest leaf ℓ in $\mathrm{Bd}(\tilde{G})$. Clearly, ℓ has to be the biggest leaf in its orbit as any chord of $\mathbb{S}$ bigger than ℓ would intersect the interior of $\tilde{G}$. It follows that $\|\ell\| > \frac{1}{4}$ (see Remark 3.2.2 (2)). Also both the two siblings $\hat{\ell}$ and $\tilde{\ell}$ of leaf ℓ are in $\mathrm{Bd}(\tilde{G})$ and have the same length bigger than $\frac{1}{4}$. From the earlier discussion following Definition 3.2.3, it is clear that the leaves in the triplets $\{\ell, \hat{\ell}, \tilde{\ell}\}$ and $\{-\ell, -\hat{\ell}, -\tilde{\ell}\}$ are all disjoint from each other and have the same length bigger than $\frac{1}{4}$. It follows that the six disjoint leaves add up to a sum of lengths bigger than $\frac{6}{4} = \frac{3}{2} > 1$. Clearly, a contradiction. □

From the above theorem, it is clear that every gap is either a *(pre)periodic* gap or a *collapsing quadrilateral* or eventually maps to a *collapsing quadrilateral*. For the rest of the section, we will study periodic gaps. A lot of observations can be made about periodic gaps using the *Short Strip Lemma*, which will be used as the main criterion to classify gaps. Let G be a periodic gap in a cubic symmetric lamination $\mathcal{L}$.

From property (D3) of a cubic symmetric lamination it follows that for every gap G there is a gap $-G$ in $\mathcal{L}$ symmetric to G.

Periodic gaps G are of two kinds:

(1) *Gaps with symmetric orbits.* If a gap G eventually maps to the gap $-G$, i.e there exists k such that $\sigma_3^k(G) = -G$, then we say G is a gap with a symmetric orbit. For a gap G with symmetric orbit, the gaps G and $-G$ are in the same orbit of gaps because if $\sigma_3^k(G) = -G$, then $\sigma_3^k(-G) = G$, too.

(2) *Gaps without symmetric orbits:* If a gap G never maps to the gap $-G$ under any forward iterate, i.e $\sigma_3^k(G) \neq -G \ \forall k$, then we say G is a gap without a symmetric orbit. For a gap G without symmetric orbit, it follows that the gaps G and $-G$ have two disjoint orbits.

As we have seen in the proof of the Theorem 3.3.1, for the first return map g to have degree 4, the periodic gap G with a critical chord ℓ in its interior has to map to $-G$ having a critical chord $-\ell$ in its interior. In other words, G needs to be a gap with symmetric orbit. There are other kinds of periodic gaps which have symmetric orbits. By Proposition 3.2.1, central symmetric gap G is a gap with symmetric orbit since G is invariant and both G and $-G$ coincide.

Definition 3.3.1 (Primary major). Let G be a periodic gap of a cubic symmetric lamination $\mathcal{L}$. Consider the collection of all leaves in the boundaries of the gaps in the orbit of G. We call the closest leaf/leaves in the above collection to a critical chord of $\mathbb{S}$, primary majors of the orbit of G.

In the later section, we will show that there can be 1, 2 or 4 primary majors of an orbit of periodic gaps. Now, we need a few basic lemmas concerning periodic gaps before proceeding on to classify gaps of a cubic symmetric lamination. Let $\tilde{G}$ be the gap containing a primary major P of the orbit of G in $\mathcal{L}$. Clearly, P is a medium or a long leaf having length bigger than $\frac{1}{4}$ (see Remark 3.2.2 (2)).

Let ℓ and ℓ' be two disjoint chords of $\mathbb{S}$. There are 3 complementary components to $\ell \cup \ell'$ in $\overline{D}$. The component bounded by ℓ and ℓ' is called the *strip between the chords ℓ and ℓ'*. We denote it by $S_b(\ell, \ell')$.

Lemma 3.3.2. *If P is not a critical leaf, then there is exactly one other medium or a long leaf Q in $\tilde{G}$.*

Proof. We have two cases here.

(i) *If $\tilde{G}$ is a Central Symmetric gap.* Then there exists a leaf $-P$ in $\tilde{G}$ symmetric to the leaf P. The leaves P and $-P$ are of the same length. Thus, we have $Q = -P$.

(ii) *If $\tilde{G}$ is not a Central Symmetric gap.* We have two sub-cases here, when $\tilde{G}$ is critical and otherwise.

  (a) *$\tilde{G}$ is critical:* From the discussion following Definition 3.2.3, there exists a long or a medium sibling leaf $\hat{P}$ of the leaf P in the cubic symmetric lamination $\mathcal{L}$. And because $\tilde{G}$ is critical, a critical chord of $\mathbb{S}$ can be found in the interior of $\tilde{G}$ without crossing the leaves of $\tilde{G}$. It follows that $\hat{P}$ is in $\tilde{G}$. Thus, we have $Q = \hat{P}$.

  (b) *$\tilde{G}$ is non-critical:* By Proposition 3.2.1 (2), $\mathcal{L}$ has a central symmetric gap S or a central symmetric leaf s. We will first consider the case when $\mathcal{L}$ has a central symmetric gap S.

  Consider two longest leaves M and $-M$ in the gap S. Consider the Short Strips $C(M) \cup -C(M)$ bounded by the leaves $M, -M$ and their siblings.

Claim. The gap $\tilde{G}$ is inside the Short Strips $C(M) \cup -C(M)$.

Consider a symmetric pair of critical chords of $\mathbb{S}$ that do not cross the boundary leaves of the Short Strips $C(M) \cup -C(M)$. They have to lie inside the Short Strips $C(M) \cup -C(M)$ (see Remark 3.2.4 (c)). Clearly, a long leaf ℓ of $\mathcal{L}$ is in a strip between such critical chords. It follows that every long leaf ℓ of $\mathcal{L}$ is either in the Short Strips $C(M) \cup -C(M)$ or in the strip $S_b(M, -M)$ between the leaves M and $-M$. Since the central symmetric gap S is in the strip $S_b(M, -M)$, the long leaf ℓ cannot be in it too, without crossing the leaves of S. Thus, every long leaf ℓ of $\mathcal{L}$ is in the Short Strips $C(M) \cup -C(M)$. We can now argue that one among the leaves P and $\hat{P}$ in $\mathcal{L}$ is inside the Short Strips $C(M) \cup -C(M)$. It follows that the gap $\tilde{G}$ is inside the short strips $C(M) \cup -C(M)$.

As the gap $\tilde{G}$ is inside the Short Strips $C(M) \cup -C(M)$, there exists exactly one other medium or a long leaf Q in $\tilde{G}$. Either, both the leaves P and Q are long or both are medium depending on whether they are closer to the leaf M or its sibling $\hat{M}$.

□

If P is a critical leaf, there are examples of periodic gaps containing no other medium or a long leaf and there are periodic gaps which contain exactly one other medium or a long leaf Q.

Lemma 3.3.3. *If Q exists, each side $\ell \in \mathrm{Bd}(\tilde{G})$ eventually maps into the set $\{P, -P, Q\}$ and Q eventually maps to P or $-P$ or it is invariant under the first return map g. If Q does not exist, each side $\ell \in \mathrm{Bd}(\tilde{G})$ eventually maps to P or $-P$.*

Proof. Note that the first return map g of the periodic gap $\tilde{G}$ is an iterate of σ_3. We will consider two cases when $\tilde{G}$ is a gap with symmetric orbit and otherwise.

Consider a short leaf ℓ on $\tilde{G}$, it eventually lands on a leaf of length at least $\frac{1}{4}$ by some forward iterate of σ_3. Among all such leaves on the orbit of ℓ, consider the leaf $\hat{\ell}$ closest to a critical chord of the circle $\mathbb{S}$. We will first assume that the leaf Q exists.

(a) $\tilde{G}$ *does not map to* $-\tilde{G}$ under any iterate of σ_3. We claim that $\hat{\ell}$ has to be either P or Q. Let us assume that $\hat{\ell}$ is neither P nor Q. It implies that $\hat{\ell}$ is not a side of $\tilde{G}$. By the definition of primary major, either P is closer to a critical chord of $\mathbb{S}$ than the leaf $\hat{\ell}$ or both the leaves P and $\hat{\ell}$ are at the same distance away from a critical chord. The only leaves that can be at the same distance away from a critical chord as the primary major P are the leaves $-P$ and $-Q$ of the gap $-\tilde{G}$. As $\tilde{G}$ does not map to $-\tilde{G}$ under any iterate of σ_3, $\hat{\ell}$ can be neither $-P$ nor $-Q$. It follows that both P and Q are closer to a critical chord of $\mathbb{S}$ than the leaf $\hat{\ell}$. Thus, the Short Strips $C(\hat{\ell}) \cup -C(\hat{\ell})$ strictly cover the gap $\tilde{G}$. Recall that the leaf $\hat{\ell}$ is a forward image of the short leaf ℓ on $\tilde{G}$. It implies that $\hat{\ell}$ has to eventually map to a side of $\tilde{G} \subset C(\hat{\ell}) \cup -C(\hat{\ell})$ giving us the contradiction of Corollary 3.2.5 *Short Strip Lemma*. Thus, $\hat{\ell}$ is either P or Q and each side $\ell \in \mathrm{Bd}(\tilde{G})$ eventually maps to P or Q.

We claim that Q eventually maps to P or it is invariant under the first return map g. Consider the Short Strips $C(Q) \cup -C(Q)$. It is not hard to see that the gap $\tilde{G}$ lies in the Short Strip $C(Q)$. Either Q is invariant under the first return map g or Q maps to a side of $\tilde{G} \subset C(Q)$ under the map g. By Proposition 3.2.3, if Q is not invariant under g, $g(Q)$ is a long leaf in $C(Q)$. It follows that $g(Q) = P$.

(b) $\tilde{G}$ *maps to* $-\tilde{G}$ under some iterate of σ_3. In this case, $\hat{\ell}$ can be any of the leaves in the set $\{P, -P, Q, -Q\}$ as $\tilde{G}$ eventually maps to $-\tilde{G}$. Using the same arguments as before, if ℓ is neither of the leaves in the set $\{P, -P, Q, -Q\}$, we get a contradiction of Corollary 3.2.5 *Short Strip Lemma*. Thus, $\ell \in \mathrm{Bd}(\tilde{G})$ eventually maps into the set $\{P, -P, Q, -Q\}$.

Consider the leaf $-Q$ in $\mathrm{Bd}(-\tilde{G})$. We claim that the leaf $-Q$ eventually maps to P or Q. We already know from the arguments before that the gap $\tilde{G}$ lies in the Short Strip $C(Q) \cup -C(Q)$. As $\tilde{G}$ maps to $-\tilde{G}$ under some iterate of σ_3, the leaf $-Q$ has to a map to a side in $\mathrm{Bd}(\tilde{G})$ before getting back to itself. By Proposition 3.2.3, it can be only a long leaf. Thus, $-Q$ eventually maps to P or Q. Hence, each side $\ell \in \mathrm{Bd}(\tilde{G})$ eventually maps into the set $\{P, -P, Q\}$.

Using the same arguments in the last paragraph of case (a), we can conclude that Q eventually maps to P or $-P$ or it is invariant under the first return map g.

Finally, if P is a critical leaf, Q may not exist and using the same exact arguments as before, we can show that all the non-critical leaves of $\tilde{G}$ eventually map to P or $-P$. $\square$

The following few lemmas are about periodic gaps with the first return map g of degree 1.

LEMMA 3.3.4. *Let G be a periodic gap with the first return map g of degree 1. All the periodic points of the map g in the set $\partial(G)$ have the same period.*

PROOF. Consider two distinct periodic orbits $\{g^i(a)\}_{i=0}^{m-1}$ and $\{g^i(b)\}_{i=0}^{n-1}$ of periods m and n respectively. Arrange the m points in the periodic orbit of a around the circle $\mathbb{S}$ counterclockwise. They divide the circle into m arcs. There exists a point $b_k = g^k(b)$ in one of the m arcs. Let the two adjacent points of the periodic orbit of a on either side of b_k in the above arc are a and $a_s = g^s(a)$. By Definition 2.2.7, the circular order of the points a, b_k and a_s are preserved by the map g and its iterates. It follows that all the points $\{g^i(b_k)\}_{i=0}^{m-1}$ of the periodic orbit b are sandwiched in between two adjacent points of the periodic orbit of a. In other words, the points $\{g^i(b_k)\}_{i=0}^{m-1}$ occupy the m arcs mentioned above. The point $g^m(b_k)$ will be in the same arc $\widehat{a\, a_s}$ in which the point b_k lies.

Now we can repeat the same process for the point $g^m(b_k)$ until we exhaust all points of the periodic orbit of b. At the end of the process, we get the same number of points of the periodic orbit of b in each of the m arcs. Thus, we conclude that n should be a multiple of m.

We can do the same exact argument starting with the periodic orbit of b and a point a_j in the periodic orbit of a. It gives us a similar conclusion that m should be a multiple of n. Combining both the arguments, we get $m = n$.

□

Now, we will prove an important result about *fixed return triangles* using Short Strip Lemma. Consider a triangle T formed by three chords of $\mathbb{S}$. We say T is a fixed return triangle under the map $g = \sigma_3^k$ if it satisfies the following conditions.

(i) There exists a map $g = \sigma_3^k$ that fixes all the three vertices of the triangle T and no image of T under a smaller iterate $\sigma_3^i, 0 < i < k$ coincides with T,

(ii) the circular order of the vertices of T is preserved by all iterates of the σ_3 map and

(iii) two distinct forward images of T under the map σ_3 can only meet at a side or a vertex.

If both T and $-T$ are fixed return triangles under the map $g = \sigma_3^k$, then we call the pair $\{T, -T\}$ as symmetric fixed return triangles under the map $g = \sigma_3^k$. Note that the last two conditions mentioned above make sure that the fixed return triangles behave like gaps of a cubic symmetric lamination. So without the abuse of notation, we will refer to the fixed return triangles as gaps and its sides as leaves.

PROPOSITION 3.3.5 (No fixed return triangles). *There does not exist a pair of symmetric fixed return triangles under the map $g = \sigma_3^k$ for all k.*

PROOF. By the way of contradiction, let us assume that there exists a pair of symmetric fixed return triangles $\{T, -T\}$. Consider the gap $\tilde{T}$ that contains the primary major ℓ of the orbit of gaps of T. Similarly, we have the gap $-\tilde{T}$ that contains the primary major $-\ell$ of the orbit of gaps of $-T$.

Clearly, there exists at least one short leaf m on $\tilde{T}$. As every leaf under forward iterates of the map σ_3 grows to a leaf of length at least $\frac{1}{4}$ in the future, there exists a medium or a long leaf in the forward orbit of the leaf m having length at least $\frac{1}{4}$. Among all such leaves in the orbit of m, consider the leaf M closest to a critical chord of $\mathbb{S}$. Note that since the leaf m is periodic it maps to M and vice-versa. Also, leaf M cannot be an edge of $\tilde{T}$ as m under the first return map g maps to itself. We have two cases.

(a) $\tilde{T}$ *does not map to* $-\tilde{T}$ under any iterate of σ_3. Since $\tilde{T}$ does not eventually map to $-\tilde{T}$, leaf M cannot be an edge of $-\tilde{T}$, too. It implies that M has to lie outside the Short Strips $C(\ell) \cup -C(\ell)$. Now, consider the Short Strips of leaf M, $C(M) \cup -C(M)$. It is not hard to see that the gap $\tilde{T}$ is inside the region $C(M) \cup -C(M)$, particularly the short leaf m. As noted above, the leaf M has to eventually map to m which is inside its Short Strips $C(M) \cup -C(M)$ giving us the contradiction of Corollary 3.2.5 (Short Strip Lemma).

(b) $\tilde{T}$ *maps to* $-\tilde{T}$ under some iterate of σ_3. As $\sigma_3^k(\tilde{T}) = \tilde{T}$, using the basic notion of symmetry, we can deduce that $\sigma_3^{\frac{k}{2}}(\tilde{T}) = -\tilde{T}$. We claim that $\sigma_3^{\frac{k}{2}}(\ell) = -\ell$

and $\sigma_3^{\frac{k}{2}}(m) = -m$. If that is not the case, the implication is that $\tilde{T}$ rotates by one or two clicks when it maps to $-\tilde{T}$. It follows that $-\tilde{T}$ undergoes the same amount of clicks when it maps to $\tilde{T}$ and as a result, we have the first return map $g = \sigma_3^k$ that doesn't fix the vertices of $\tilde{T}$, a contradiction.

It follows that the leaf M cannot be an edge of $-\tilde{T}$ here, too. Now, using the same exact argument as in part(a) we get the required contradiction.

$\square$

COROLLARY 3.3.6. *Let G be a periodic gap with the first return map g of degree 1. Then $\partial(G)$ can contain at most two periodic orbits (under the first return map g of G).*

PROOF. By the way of contradiction, let us assume that there exists three distinct periodic orbits $\{g^i(a)\}_{i=0}^{m-1}$, $\{g^i(b)\}_{i=0}^{m-1}$ and $\{g^i(c)\}_{i=0}^{m-1}$ on $\partial(G)$. Note that by Lemma 3.3.5, all the three periodic orbits have the same period. Arrange the points from all the three periodic orbits around the circle $\mathbb{S}$ counterclockwise. Consider a triangle T formed by any three adjacent points in the set $\partial(G)$. We claim that T is a fixed return triangle under the map g^m.

(i) Let the vertices of T be denoted by x, y and z. Clearly all the points x, y and z are fixed by the map g^m as the points are chosen from the periodic orbits of the same period m.

We claim that any three adjacent vertices of the set $\partial(G)$ belong to one each of the three periodic orbits. This is because of the same argument used in the proof of Lemma 3.3.4. Two adjacent points from distinct periodic orbits stay adjacent all along their periodic orbits. It can be applied to three periodic orbits taken two at a time and the claim follows.

It follows that no images of T under the map g^i $0 < i < m$ coincide with T.

Also since G is a gap of cubic symmetric lamination, forward images of G meet only at a side or a vertex. We can combine the above two arguments and conclude that none of the forward images of T under a smaller iterate of the σ_3 when compared with the map g coincide with T.

(ii) Since G is a periodic gap with the first return map g of degree 1, circular order of any three points in the set $\partial(G)$ are preserved by the map σ_3 and its iterates. Thus, the circular order of the vertices of T is also preserved by the iterates of the σ_3 map.

(iii) As forward images of G meet only at a side or a vertex, forward images of T meet only at a side or a vertex, too.

Similarly we get a fixed return triangle $-T$ in the gap $-G$. Thus, there is a pair of symmetric fixed return triangles under the map g^m contradicting Proposition 3.3.5. □

3.3.1. Finite gaps. Consider a finite gap G of a cubic symmetric lamination $\mathcal{L}$. By Theorem 3.3.1,

(a) Either G eventually maps on to a *collapsing quadrilateral* and then collapse on to a leaf or

(b) G is *(pre)periodic.*

We call a finite periodic gap of cubic symmetric lamination, a *periodic polygon.*

LEMMA 3.3.7. *Let G be a periodic polygon, then the degree of the first return map g is one. And either,*

(a) g permutes the sides of G transitively as a rational rotation. We call such a gap 1-transitive rotational gap

(or)

(b) sides of G form two disjoint periodic orbits and on each orbit, g permutes the sides of G transitively. We call such a gap 2-transitive rotational gap. A 2-transitive rotational gap G eventually maps to the gap $-G$. If ℓ and ℓ' are two adjacent sides of G, then the leaf ℓ eventually maps to the side $-\ell'$ of $-G$.

PROOF. Let $\tilde{G}$ be the gap that contains the primary major P of the orbit of G. Clearly P is not a critical leaf, otherwise $\sigma_3(\tilde{G})$ would contain one point less than $\tilde{G}$ and $\tilde{G}$ can never come back to itself. Similarly, if the degree of the first return map g is bigger than 1, then images of G have less vertices than G and G cannot return to itself. Hence the degree of the first return map g is equal to one.

By Lemma 3.3.3, there exists exactly one other medium or a long leaf Q in $\tilde{G}$. First, we claim that no leaf $\tilde{\ell}$ of $\tilde{G}$ is invariant under the map g. If g fixes both the endpoints of $\tilde{\ell}$, it would have to fix all the vertices of $\tilde{G}$ because the circular order of any three vertices of $\tilde{G}$ is preserved by the σ_3 map and its iterates. It follows that we get a fixed return triangle T inside $\tilde{G}$ and a similar triangle $-T$ inside $-\tilde{G}$, a contradiction with Proposition 3.3.5. The map g cannot reverse the two endpoints of $\tilde{\ell}$, again because the circular points of the vertices of the gap $\tilde{G}$ will not be preserved by g. In particular, Q cannot map back to itself under the map g.

Thus, by Lemma 3.3.3, we conclude that the leaf Q eventually maps to P or $-P$. It follows that every leaf of $\tilde{G}$ eventually maps to P or $-P$.

By the Corollary 3.3.7, $\partial(\tilde{G})$ contains at most two periodic orbits (under the first return map g) which leads us to the following cases.

(i) $\partial(\tilde{G})$ contains one periodic orbit. It follows that there is one orbit of leaves of $\tilde{G}$ under the map g. Thus, from the above discussion, every side of $\tilde{G}$ (including Q) eventually maps to P. In other words, g permutes the sides of G transitively. Since the circular order of the vertices of $\tilde{G}$ has to be preserved by the map g, it follows that the action of g on leaves of G is *rotational* (Definition 4.1.2) in the sense that every leaf skips a certain fixed number of leaves while mapping on to its image leaf under the map g. We call such a gap *1-transitive rotational* gap.

(ii) $\partial(\tilde{G})$ contains two periodic orbits. It follows that there are two disjoint orbits of leaves of $\tilde{G}$ under the map g. By the earlier argument, we can conclude that half of the leaves of $\tilde{G}$ eventually map to P and the other half eventually map to $-P$ and the leaves P and $-P$ are in two disjoint orbits. On each of the two orbits separately, action of g is rotational and transitive similar to what we observed in case(i). We call such a gap *2-transitive rotational* gap. Since half of the leaves of $\tilde{G}$ eventually map to $-P$, it implies that at a certain point on its orbit, $\tilde{G}$ maps to $-\tilde{G}$ before returning back to itself. In other words, $\tilde{G}$ is a gap with symmetric orbit.

We have already seen that every two adjacent vertices of $\partial(\tilde{G})$ are a part of two disjoint periodic orbits of vertices. It follows that if ℓ and ℓ' are two adjacent sides of G, then they are in two disjoint orbits of leaves, too. One orbit of leaves contains P and the other orbit contains the leaf $-P$. Let us say the leaf ℓ is in the orbit that contains the leaf P, and the leaf ℓ' of is in the orbit that contains the leaf $-P$. It follows that the leaf $-\ell'$ of will be in the orbit that contains the leaf P. We can conclude that leaves ℓ and $-\ell'$ are in the same periodic orbit. Thus, the leaf ℓ eventually maps to the leaf $-\ell'$ of $-G$.

Note that the conclusions obtained for the gap $\tilde{G}$ hold for all the gaps on the orbit of G, since all of them are conjugate under the σ_3 map. $\square$

Below are the two important properties about (pre)periodic polygons resulting from the above lemma.

Corollary 3.3.8. *Let G be a finite gap of a cubic symmetric lamination $\mathcal{L}$ such that G is not (pre)critical. No diagonal of the polygon G can be a leaf of any cubic symmetric lamination.*

Proof. Let ℓ be a diagonal of G. Note that for ℓ to be part of a cubic symmetric lamination, $-\ell$ also has to exist in $\mathcal{L}$ along with it.
By Theorem 3.2.6, G is a (pre)periodic gap. Also ℓ does not map to a critical leaf and collapse to a point as G is not (pre)critical. Let $\tilde{G} = \sigma_3^i(G)$ be the periodic gap and $\tilde{\ell} = \sigma_3^i(\ell)$ be the corresponding diagonal leaf in $\tilde{G}$. By Lemma 3.3.7, $\tilde{G}$ is either a 1-transitive rotational gap or a 2-transitive rotational gap. In both the cases, we shall prove that a forward image of $\tilde{\ell}$ will either cross itself or cross the leaf $-\tilde{\ell}$.

(a) $\tilde{G}$ is a *1-transitive rotational gap.* The leaf $\tilde{\ell}$ splits the gap $\tilde{G}$ into two polygons. One polygon may contain more vertices of $\tilde{G}$ than the other one. If that is the case, consider the polygon G_0 having fewer vertices of $\tilde{G}$. Choose a point $p \in \partial(G_0)$ such that it is not an endpoint of the leaf $\tilde{\ell}$. Since, $\tilde{G}$ is a 1-transitive rotational gap, all the vertices of $\tilde{G}$ are in one periodic orbit under the first return map g ($g(\tilde{G}) = \tilde{G}$). It follows that there exists an iterate g^i such that the leaf $g^i(\tilde{\ell})$ has p as one of the endpoints. We claim that the leaf $g^i(\tilde{\ell})$ crosses the leaf $\tilde{\ell}$. Like the leaf $\tilde{\ell}$, the leaf $g^i(\tilde{\ell})$ also splits the gap $\tilde{G}$ into two polygons with unequal number of vertices. Consider the sub-gap G_1 with fewer vertices of $\tilde{G}$. Since, the circular order of the vertices of $\tilde{G}$ is preserved by the map g and its iterates, we can argue that both the sub-gaps G_0 and G_1 have the same number of vertices. Notice that p is the common vertex to both the sub-gaps G_0 and G_1. It follows that the polygons

G_0 and G_1 are not disjoint and their interiors intersect. Thus, leaves $g^i(\tilde{\ell})$ and $\tilde{\ell}$ cross each other.

If both the polygons have the same vertices of $\tilde{G}$, then choose a point $p \in \partial(\tilde{G})$. Any iterate $g^k(\tilde{\ell})$ splits the polygon into equal parts. It follows that the leaf $g^k(\tilde{\ell})$ crosses the leaf $\tilde{\ell}$ for any k.

(b) $\tilde{G}$ is a *2-transitive rotational gap*: For a 2-transitive rotational gap, things get slightly complicated because two adjacent vertices on such a gap $\tilde{G}$ are not part of 1 orbit, but two disjoint orbits. Like case (a), the leaf $\tilde{\ell}$ splits the gap $\tilde{G}$ into two polygons. If both the polygons have an unequal number of vertices, consider the polygon G_0 having fewer vertices of $\tilde{G}$. If the number of vertices of the polygon G_0 is more than 3, then we proceed in a similar way as before. Choose a point $p \in \partial(G_0)$ such that there is exactly one point in $\partial(G_0)$ between itself and one of the endpoints (say q) of the leaf $\tilde{\ell}$. Since, $\tilde{G}$ is a 2-transitive rotational gap, the points p and q are in one periodic orbit. It follows that there exists an iterate g^i such that the leaf $g^i(\tilde{\ell})$ has p as one of the endpoints. Using the similar arguments as in case (a), we can show that the leaves $g^i(\tilde{\ell})$ and $\tilde{\ell}$ cross each other.

If the number of vertices of the polygon G_0 is exactly 3, we do the following. Choose a point $p \in \partial(G_0)$ such that it is not an endpoint of the leaf $\tilde{\ell}$. Now, as discussed in the proof of Lemma 3.3.7, $\partial(\tilde{G})$ contains two disjoint periodic orbits. The periodic orbit of the point p is disjoint from the periodic orbit containing the endpoints of the leaf $\tilde{\ell}$. Also, the leaves $\tilde{\ell}$ and $-\tilde{\ell}$ are in two disjoint periodic orbits of leaves. It follows that there exists an iterate σ_3^i such that the leaf $g^i(-\tilde{\ell})$ has p as one of the endpoints. Clearly, the leaves $\sigma_3^i(-\tilde{\ell})$ and $\tilde{\ell}$ cross each other.

If both the polygons have the same vertices of $\tilde{G}$, then choose a point

$p \in \partial(\tilde{G})$. Any iterate $g^k(\tilde{\ell})$ splits the polygon into equal parts. It follows that the leaf $g^k(\tilde{\ell})$ crosses the leaf $\tilde{\ell}$ for any k.

Since there are crossings of leaves among the forward images of ℓ and $-\ell$, they cannot be leaves of a cubic symmetric lamination. □

COROLLARY 3.3.9. *Any two different (pre)periodic polygons have disjoint sets of vertices, unless both are preperiodic and share a common boundary leaf that eventually maps to a critical leaf.*

PROOF. By Corollary 3.3.8, it is clear that no two periodic polygons can share a side. So, it suffices to prove that two periodic polygons cannot share a vertex. If a periodic polygon G has a common vertex with another periodic polygon G', then by the transitive (1-transitive or 2-transitive) action each vertex of G must be a vertex of a polygon on the orbit of G', and each vertex of G' must be a vertex of a polygon on the orbit of G. This forces infinitely many further periodic polygons on the orbits of G and G', a contradiction.

Since we just proved that two periodic polygons cannot share a vertex, any two (pre)periodic polygons G and G' sharing a vertex have to eventually map to a single polygon $\hat{G}$ for otherwise when both G and G' become periodic they would be still be sharing a common vertex.

Preperiodic polygons whose image polygon is $\hat{G}$ share a critical leaf on their boundaries. Which means that both of the (pre)periodic polygons G and G' have to be preperiodic and share a common boundary leaf that eventually maps to the critical leaf. □

Corollaries 3.3.8 and 3.3.9 are used in a few important lemmas in chapter 5. Here is a useful lemma about gaps which eventually map onto *collapsing quadrilaterals*.

Lemma 3.3.10. *Let $\{G, -G\}$ be the collapsing quadrilaterals of a cubic symmetric lamination and s be the length of their shorter sides. Then any gap which eventually maps to G or $-G$ has a pair of opposite sides of length at most $s/3$.*

Proof. The other two sides of G (and $-G$) have length $\frac{1}{3} - s$, and all sides of G map to a single leaf of length $3s$.
Consider a leaf ℓ with length greater than $s/3$. During its forward orbit, ℓ can never get closer to a critical chord than the long leaves of G. It implies that no forward image of ℓ (barring ℓ itself) can get shorter than the leaf m of length $3s$.
Thus, we can conclude that no leaf with length greater $s/3$ can ever land on a short leaf of G (and $-G$). It follows that every eventually collapsing quadrilateral has a pair of opposite sides with length no more than $s/3$. □

3.3.2. Infinite gaps. Consider infinite periodic gaps of cubic symmetric laminations. We will be using the following standard topological result about *locally strictly expanding maps* for studying infinite gaps.

Theorem 3.3.11. *Any locally strictly expanding map on an infinite compact metric space is non-injective.*

Proof. **Locally strictly expanding-** A map $f : X \to X$ on a compact metric space X such that $\forall x \in X$ there is an open neighborhood U_x of x such that $d(f(u), f(v)) > d(u, v), \forall u, v \in U_x$ is said to be a locally strictly expanding map.

Let f be an injective locally strictly expanding continuous map , then we will prove that X is finite. First, since f is a continuous injection from a compact metric space, it is a homeomorphism onto its image $f(X)$. Hence, $\{f(U_x), x \in X\}$ is an open cover of $f(X)$. Since $f(X)$ is compact, $\exists$ Lebesgue number $\delta > 0$ such that for every $y \in f(X)$ there is $x \in X$ such that $B(y, \delta) \subset f(U_x)$. This implies, since f is locally strictly expanding map, if $d(y,z) < \delta$, then $d(f^{-1}(y), f^{-1}(z)) < d(y,z)$, where $y, z \in f(X)$.

Claim. $\exists N \in \mathbb{N} : f(X)$ can be covered by N sets of diameter at most $\frac{\delta}{2^n}$, $\forall\, n \in \mathbb{N}$. We will prove this inductively.

(1) *The case* $n = 1$. Since $f(X)$ is compact, there exists finite number N sets of diameter at most $\frac{\delta}{2}$ covering it. It is important to notice that all of these sets are inside the members of the cover $\{f(U_x), x \in X\}$.

(2) *Inductive assumption.* Assume that $f(X) = U_1 \cup ... \cup U_N$, where U_i are of diameter at most $\frac{\delta}{2^n}$. Again, all of the sets $\{U_i\}_{i=1}^{N}$ are inside the members of the cover $\{f(U_x), x \in X\}$.

(3) *Inductive step.* Consider U_i, we have $\forall\ (y,z) \in U_i,\ d(f^{-1}(y), f^{-1}(z)) < d(y,z)$.

For $\overline{U_i}$, we can have $\exists \alpha \in (0,1)\ : \forall\ (y,z) \in U_i,\ d(f^{-1}(y), f^{-1}(z)) \leq \alpha d(y,z)$.

Since clearly f is surjective onto $f(X)$, $\{f^{-1}(U_i)\}_{i=1}^{N}$ cover X , thereby $f(X)$ too.

And from above, $f^{-1}(U_i)$ are all of diameter at most $\alpha\frac{\delta}{2^n}$. Let $V_i^1 = f^{-1}(U_i) \cap f(X)$.

Proceeding like this, we can get sets $\{V_i^m\}_{i=1}^{N}$ in $f(X)$, all of diameter at most $\alpha^m \frac{\delta}{2^n} < \frac{\delta}{2^n+1}$. This completes the proof of claim.

Hence, it is clear from the above claim that $f(X)$ is finite. As f is a homeomorphism onto $f(X)$, X is injective too! □

Using the above theorem, we can deduce the following lemma about infinite gaps.

LEMMA 3.3.12. *Orbit of every infinite periodic gap has a critical gap or a gap with a critical leaf on its boundary.*

Proof. Every infinite periodic gap G is clearly a compact subset of $\overline{\mathbb{D}}$. And the map σ_3 and its iterates are locally strictly expanding. Consider the first return map $g = \sigma_3^k$ of G. By Theorem 3.3.11, the map g has to be non-injective. Thus, there exists a gap $\hat{G}$ in the orbit of G such that $\sigma_3|_{\partial(\hat{G})}$ is not one-to-one onto its image. It follows that the gap $\hat{G}$ is either a critical gap or a gap with a critical leaf on its boundary. □

Suppose that G is an infinite gap. By Theorem 3.3.1, G is *(pre)periodic* and from the above lemma, G eventually maps to a periodic gap $\tilde{G}$ which is either a critical gap or a gap with a critical leaf on its boundary.

Definition 3.3.2 (Fatou gaps). We call an infinite periodic gap G, a *Fatou* gap if $\partial(G)$ is uncountable and G is critical.

Definition 3.3.3 (Siegel gaps). We call an infinite periodic gap G, a *Siegel* gap if $\partial(G)$ is uncountable and the degree of the first return map g ($g(G) = G$) is 1. Note that G must have a critical leaf on its boundary.

Definition 3.3.4. We call an infinite periodic gap G, a *caterpillar* gap if $\partial(G)$ is countable.

Lemma 3.3.13 (Fatou gaps). *Let G be a periodic gap such that the primary major P of the orbit of G is not a critical leaf.*

(i) If the first return map g is of degree two, then G is a Fatou gap without symmetric orbit that intersects the circle in a Cantor set.

(ii) If the map g is of degree four, then G is a Fatou gap with symmetric orbit that intersects the circle in a Cantor set.

PROOF. Consider the gap $\tilde{G}$ that contains the primary major P of the orbit of G. The conclusions obtained for $\tilde{G}$ hold for all the gaps on the orbit of G, too, since all of them are conjugate under the map σ_3.

(i) The first return map $\tilde{g} : \tilde{G} \to \tilde{G}$ has degree two.

Clearly $\tilde{G}$ is a critical gap with a critical chord in its interior. As the degree of the map $\tilde{g}$ is two, $\tilde{G}$ does not map to $-\tilde{G}$ under any iterate of the σ_3 map. By Lemma 3.3.3, each side of $\tilde{G}$ eventually maps to P where P is chosen such that it is invariant under the map $\tilde{g}$. Let $\hat{P}$ be the long or a medium sibling of P. We have $\tilde{g}(\hat{P}) = P$. Note that the first return map $\tilde{g}$ acts on $\mathrm{Bd}(\tilde{G})$ as a double covering. Since every side of $\tilde{G}$ eventually lands on P, all sides of $\tilde{G}$ can be obtained by taking pre-images of P under iterates of $\tilde{g}$. As leaves of any invariant lamination are dense on the circle, it follows that $\tilde{G}$ touches the circle in a cantor set obtained by removing a countable dense sequence of open intervals determined by the pre-images of P. Thus, $\partial(\tilde{G})$ is an uncountable set and $\tilde{G}$ is a Fatou gap.

Finally, $\tilde{G}$ is a gap without symmetric orbit because $\tilde{G}$ does not map to $-\tilde{G}$ under any iterate of the σ_3 map.

(ii) $\tilde{g} : \tilde{G} \to \tilde{G}$ has degree 4:

As the degree of the map $\tilde{g}$ is 4, $\tilde{G}$ eventually maps to $-\tilde{G}$ under the σ_3 map. Thus, $\tilde{G}$ is a gap with symmetric orbit.

The first return map $\tilde{g}$ acts on $\mathrm{Bd}(\tilde{G})$ as a four fold covering. By Lemma 3.3.3, each side of $\tilde{G}$ eventually maps to P or $-P$, where P is chosen such that it is invariant under the map $\tilde{g}$. Also because $\tilde{G}$ is a gap with symmetric orbit, $-P$ has to eventually map to a side of $\tilde{G}$. By Short Strip Lemma, it has to be a long leaf and we conclude that $-P$ eventually maps to P. In other words, leaves P and $-P$ are in the same periodic orbit of leaves. It follows that each side of $\tilde{G}$ eventually maps to P and sides of $\tilde{G}$ can be obtained by taking pre-images of P under iterates of $\tilde{g}$ as in case (i).

The difference here is that every leaf in $\tilde{G}$ will have four pre-images under the map g. Similar to the case (i), we can argue that intersection of $\tilde{G}$ with the circle is a Cantor set. □

LEMMA 3.3.14 (Siegel and caterpillar gaps). *Let G be a periodic gap such that the primary major P of the orbit of G is a critical leaf. Then, the degree of the first return map g is one and*

(i) if $\partial(G)$ does not have periodic points, then G is a Siegel gap that intersects the circle in a cantor set. The map g acts on $\mathrm{Bd}(G)$ as an irrational rotation.

(ii) if $\partial(G)$ has periodic points, then G is a Caterpillar gap that intersects the circle in a countable set of points. Furthermore, at most one side S of G is fixed by g. Except for S, all sides of G eventually collapse to points, and they are connected to each other by one-sided infinite chains.

PROOF. Consider the gap $\tilde{G}$, that contains the primary major P of the orbit of G. If the primary major P is a critical leaf, then the first return map $\tilde{g}$ maps the sides of $\tilde{G}$ other than P bijectively to the sides of $\tilde{G}$. Therefore the degree of the map $\tilde{g}$ is one and $\tilde{G}$ has infinite sides.

(i) **(Siegel case)** Vertices of $\tilde{G}$, $\partial(\tilde{G})$ does not have periodic points. In the absence of periodic points, no side can ever return to itself. By Lemma 3.3.3 that every side of $\tilde{G}$ including Q (if it exists) eventually maps to P or $-P$. We claim that no two sides of $\tilde{G}$ can touch. If two sides of $\tilde{G}$ meet at a common endpoint p, there exists two distinct forward iterates of p that are same as the image of the critical leaf P (or $-P$). It follows a forward image of the point p is periodic, a contradiction with the fact that $\partial(\tilde{G})$ does not contain periodic points.

There are two kinds of Siegel gaps as follows.

If $\tilde{G}$ is a gap without a symmetric orbit, then every side of $\tilde{G}$ eventually maps to P under the map $\tilde{g}$. Sides of $\tilde{G}$ are obtained by taking the pre-images of critical leaf P under iterates of $\tilde{g}$. Since leaves of an invariant lamination are dense on the circle, we can conclude that $\tilde{G}$ touches the circle in a cantor set obtained by removing a countable dense sequence of open intervals determined by the pre-image leaves.

Consider the quotient space of $\text{Bd}(\tilde{G})$ obtained by the equivalence relation which collapses each leaf to a point. It is a circle on which $\tilde{g}$ induces a homeomorphism h. As discussed above, pre-images of critical leaf P under $\tilde{g}$ are dense in $\text{Bd}(\tilde{G})$, and hence h has a dense backward orbit. A standard topological result is that any homeomorphism of a circle that has a dense orbit is conjugate to an irrational rotation.

We conclude that $\tilde{g}$ acts on $\text{Bd}(\tilde{G})$ as an irrational rotation along the backward orbit of critical leaf P.

If $\tilde{G}$ is a gap with a symmetric orbit, then every side of $\tilde{G}$ eventually maps to P or $-P$. Note that both the critical leaves P and $-P$ cannot map to each other because it would result in one of the endpoints of the critical leaf being a periodic point. Thus, $\text{Bd}(\tilde{G})$ contains two disjoint infinite orbits of leaves, symmetric with respect to the center of the circle $\mathbb{S}$. Sides of $\tilde{G}$ are obtained by taking the pre-images of critical leaves P under the iterates of $\tilde{g}$ and the pre-images of the critical leaves $-P$ under the iterates of $\tilde{g}^{\frac{1}{2}}$. Since leaves of an invariant lamination are dense on the circle, we can conclude that $\tilde{G}$ touches the circle in a cantor set obtained by removing a countable dense sequence of open intervals determined by the pre-image leaves.

Similar to the above case, $\tilde{g}$ acts on $\text{Bd}(\tilde{G})$ as an irrational rotation along each of the two disjoint backward orbits of critical leaves P and $-P$.

(ii) **(Caterpillar case)** $\mathrm{Bd}(\tilde{G})$ has periodic points. By Lemma 3.3.4, all the periodic points have the same period. We have two cases.

(a) *Periodic points of $\tilde{G}$ are fixed points under the map $\tilde{g}$.* We claim that there is either exactly one fixed point or one fixed leaf on $\mathrm{Bd}(\tilde{G})$. There cannot be three or more fixed points in $\mathrm{Bd}(\tilde{G})$, because it would result in a fixed return triangle T. We get a similar fixed return triangle $-T$ in $\mathrm{Bd}(-\tilde{G})$, thereby contradicting Proposition 3.3.5. Two fixed points in $\mathrm{Bd}(\tilde{G})$ can always be joined to form a new fixed leaf and construct a new cubic symmetric lamination. We will use that approach in the last chapter.

By Lemma 3.3.3, all the non-fixed leaves eventually map to the critical leaf P or $-P$. Thus, there is at least one fixed point of the map $\tilde{g}$ which is an endpoint of the critical leaf P. There is a possibility of one more fixed point which makes Q a fixed leaf. One interesting observation about the action of $\tilde{g}$ on $\tilde{G}$ is that the leaves form a one-sided infinite collapsing chain. Consider the endpoint x of a non-fixed leaf in $\mathrm{Bd}(\tilde{G})$ and its image point $\tilde{g}(x)$. Now, consider the arc of $\mathbb{S}$ containing points x and $\tilde{g}(x)$ not containing the fixed point of $\tilde{g}$. If $\tilde{G}$ is a gap without symmetric orbits, this arc has to eventually map to an arc under the critical leaf P in a 1-1 fashion. It follows that the arc cannot contain any other point of $\partial(\tilde{G})$. Thus, all the non-fixed leaves of $\tilde{G}$ form a one-sided infinite collapsing chain, eventually collapsing on to the critical leaf P under the map $\tilde{G}$. We will show in the last chapter that there are two fixed points of $\tilde{g}$ in $\partial(\tilde{G})$. It follows that the backward orbit of P under the map $\tilde{G}$ converge to a second fixed endpoint in $\partial(\tilde{G})$ and the fixed leaf Q is formed by joining the two fixed points.

For a *gap with symmetric orbit*, there can be one other vertex of $\mathrm{Bd}(\tilde{G})$ in the arc of $\mathbb{S}$ containing points x and $\tilde{g}(x)$. It is because of the fact that there are exactly two disjoint symmetric orbits of leaves in $\mathrm{Bd}(\tilde{G})$ determined by the critical leaves P and $-P$. The two non-fixed adjacent leaves of $\tilde{G}$ belong to the two disjoint orbits. When a non-fixed leaf maps to P, its adjacent one maps to $-P$. This does not change the fact that the leaves of $\tilde{G}$ form a one-sided infinite collapsing chain. It is also interesting to note that both the endpoints of the critical leaves P and $-P$ are in the same orbit because when $\tilde{G}$ eventually maps to $-\tilde{G}$, the only place P can map in $\mathrm{Bd}(-\tilde{G})$ is the fixed endpoint of $-P$.

It is also clear now that in both the cases the vertices of gap $\tilde{G}$, being the pre-images of endpoints of P and $-P$ are countably infinite.

(b) *Periodic points of $\tilde{G}$ are of period $q > 1$ under the map $\tilde{g}$.* We will construct a new lamination and reduce it to the above case where periodic points of $\tilde{g}$ are fixed points. Consider the boundary of the convex hull L of all periodic points of $\tilde{g}$. By adding L and its pre-images under the map σ_3, original cubic symmetric lamination $\mathcal{L}$ can be subdivided. If $q > 2$, then the convex hull L of the periodic points is a finite gap. By Lemma 3.3.7 L is a 1-transitive or a 2-transitive rotational gap. Thus, the gap L should consist of all the q-period points of the periodic orbit. In other words, the convex hull L of the periodic points must have exactly q sides. Similarly if $q = 2$, convex hull L is a leaf.

In the new subdivided lamination, the original gap $\tilde{G}$ has been split into '$q+1$' sub-gaps namely the gap L and the q gaps $\{\tilde{G}_i\}_{i=1}^{q}$. Each of the q gaps $\{\tilde{G}_i\}_{i=1}^{q}$ share a leaf with L which is the fixed boundary leaf under their first return map $\tilde{g}^q$. Note that all of them are in the same periodic orbit of gaps. There exists a gap $\tilde{G}_k$ among them with a critical leaf P on its boundary, and a fixed leaf on its boundary under the map $\tilde{g}^q$. It follows now from case(a) that the sides of $\tilde{G}_k$ form a one-sided infinite collapsing chain.

□

Below is a theorem about Siegel and Fatou gaps. We have briefly alluded to it in the previous lemma. Proof of it is left to the reader.

THEOREM 3.3.15 (Siegel gaps and Fatou gaps of degree $k > 1$). . *Suppose that G is a periodic infinite gap of period n such that $\partial(G)$ is uncountable. Then there exists a monotonic map $\psi_G : \mathrm{Bd}(G) \to \mathbb{S}$ such that it collapses all edges of G to points. ψ_G semi conjugates $\sigma_3^n|_{\mathrm{Bd}(G)}$ to a map $\hat{\sigma_G} = \hat{\sigma} : \mathbb{S} \to \mathbb{S}$ so that either*

(1) $\sigma_3^n|_{\mathrm{Bd}(G)}$ is of degree $k \geq 2$ and $\hat{\sigma} = \sigma_k : \mathbb{S} \to \mathbb{S}$ (G is a Fatou gap), or

(2) $\sigma_3^n|_{\mathrm{Bd}(G)}$ is of degree 1 and $\hat{\sigma}$ is an irrational rotation. (G is a Siegel gap)

We summarize all the previous lemmas about finite and infinite gaps in Theorem 3.3.16.

THEOREM 3.3.16 (Classification of gaps). *A gap G of a cubic symmetric lamination intersects $\mathbb{S}$ in either a finite set of points, a Cantor set, or a countable set of points.*

If G is not periodic under σ_3, then either

(a) it maps by some iterate of σ_3 to a periodic gap or

(b) it is a triangle or quadrilateral which eventually collapses.

If G is periodic under σ_3, then either *(a) If the first return map g is of degree one, then G may intersect the circle in*

i ***(Periodic polygon)*** *a finite number of points, in which case G is a finite-sided polygon. g acts as a homeomorphism on G, and leaves of G from one or two disjoint orbits of leaves under g. g permutes the sides of G transitively on the orbits(as a rational rotation).*

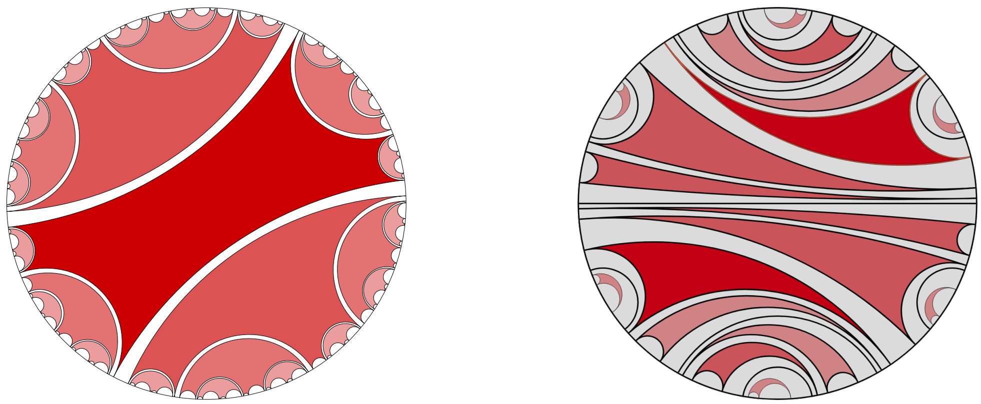
ii ***(Siegel gap)*** *a Cantor set, g acts on ∂G as an irrational rotation.*

iii ***(Caterpillar gap)*** *a countable set of points. At most one side S of G is fixed by g; except for S, all sides of G eventually collapse to points, and they are connected to each other by one-sided infinite chains.*

(b) if the first return map g is of degree two ***(Fatou gap without symmetric orbit)****, in which case G may intersect the circle in a Cantor set.*

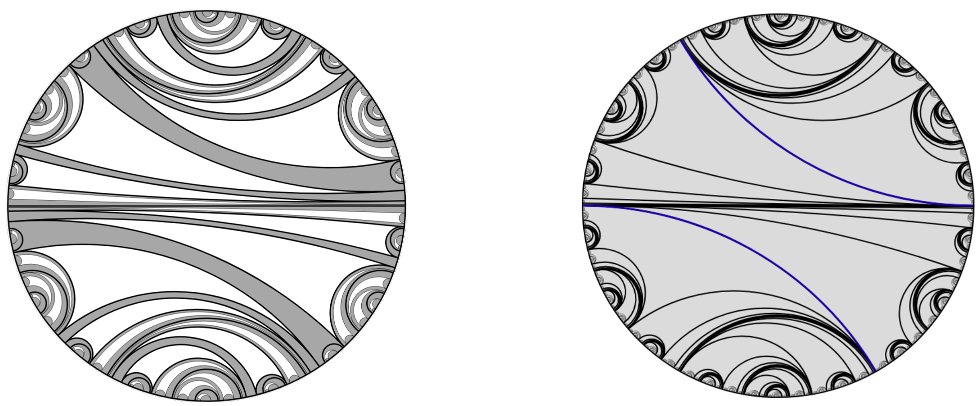
(c) if the first return map g is of degree four, ***(Fatou gap with symmetric orbit)****, in which case G may intersect the circle in a Cantor set.*

We provide some pictures for the gaps mentioned in the previous theorem in figures 3.3-3.5.



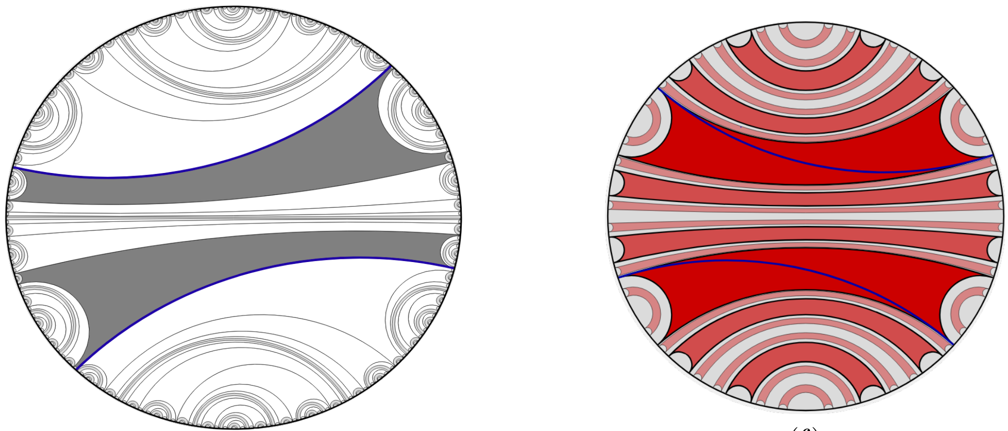
(a) (b)

FIGURE 3.3. Cubic symmetric laminations illustrating gaps in the cases of Theorem 3.3.16.
(a) An invariant central symmetric gap, (b) a pair of symmetric periodic polygons. The shaded gaps are invariant.



(c) (d)

FIGURE 3.4. Cubic symmetric laminations illustrating gaps in the cases of Theorem 3.3.16.
(c) a Fatou gap that maps to itself as a degree two covering, (d) a caterpillar gap with countable number of sides.



(e) (f)

FIGURE 3.5. Cubic symmetric laminations illustrating gaps in the cases of Theorem 3.3.16.
(e) A pair of symmetric Siegel gaps. The shaded gaps are invariant. (f) A pair of collapsing quadrilaterals- gaps with the blue critical chords (not part of the lamination) inside them.

CHAPTER 4

CENTRAL SYMMETRIC ROTATIONAL GAPS

4.1. Basic definitions

DEFINITION 4.1.1. Let A be a subset of the unit circle $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ and f be a map from $A \to \mathbb{S}$ (not necessarily continuous). We say that f is order-preserving if for each triple (a,b,c) in A the triple (f(a),f(b),f(c)) lies in the same cyclic order around $\mathbb{S}$, or else is degenerate (i.e. two, or all three, of the points coincide)

DEFINITION 4.1.2. A subset $A \subset \mathbb{S}$ is invariant under $f\colon \mathbb{S} \to \mathbb{S}$ iff $f(A) = A$. An invariant set A is rotational under f if and only if A is closed and f is order-preserving on A.

DEFINITION 4.1.3. Any order-preserving map $f\colon \mathbb{S} \to \mathbb{S}$ of the circle has a well defined rotation number

$$\rho(f) = \lim_{n\to\infty} \frac{\tilde{f}^n(\hat{x}) - \hat{x}}{n} (\text{mod } 1)$$

where $\tilde{f}\colon \mathbb{R} \to \mathbb{R}$ is any lift satisfying $\tilde{f}(\hat{x}+1) = \tilde{f}(\hat{x}) + 1, \forall \hat{x} \in \mathbb{R}$.

COROLLARY 4.1.1. *If $A \subset \mathbb{S}$ is rotational under f, then A has a well-defined rotation number $\rho(A)$ given by*

$$\rho(A) = \rho(F) = \lim_{n\to\infty} \frac{\tilde{F}^n(\hat{a}) - \hat{a}}{n} (\textit{mod } 1)$$

where $\tilde{F}$ is any lift of an order-preserving extension F of $f|_A$ satisfying $\tilde{F}(\hat{x}+1) = \tilde{F}(\hat{x})+1, \forall \hat{x} \in \mathbb{R}$.
Note that $\hat{a}$ is the lift of a point in A.

4.2. Rotational orbits

Let the circle $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ be parametrized by [0,1) and $\widehat{a\,b}$ denote the arc between the two points determined by the numbers a and b along counter clockwise direction. So whether $a < b$ or $a > b$, $\widehat{a\,b}$ always denotes the arc along a counterclockwise direction from points a to b. And we can associate a point x on the arc $\widehat{a\,b}$ with a number in [0,1).

4.2.1. Quadratic map. First, we will try to understand the rotational behavior of the map σ_2. The following results about the σ_2 map rotational orbits are based on ideas from [**5**] and serve as a basis for our work in the rest of the section.

PROPOSITION 4.2.1. *A is rotational under the map σ_2 if and only if A is a subset of a closed semi circle.*

PROOF. If (a, b, c) is contained in a closed semicircle, then $(\sigma_2(a), \sigma_2(b), \sigma_2(c))$ also has the same cyclic order as (a, b, c) since σ_2 doubles the length. Conversely, if (a, b, c) are not contained in a closed semicircle, then $(a, b + \frac{1}{2}, c)$ are, and hence $(\sigma_2(a), \sigma_2(b), \sigma_2(c)) = (\sigma_2(a), \sigma_2(b + \frac{1}{2}), \sigma_2(c))$ has the same cyclic order as $(a, b + \frac{1}{2}, c)$ which is opposite to that of (a, b, c). □

Let $X = \{x, \sigma_2(x), \sigma_2^2(x), \sigma_2^3(x), ...\}$ denote the forward orbit of a point $x \in \mathbb{S}$. Let X be contained in a semi-circle. We will consider 'a' to be the point having the lowest value assigned in the interval $[0, 1)$ among the closed set of points $\overline{X}$. Now, it is not hard to see that X is contained in the semi-circle $\widehat{a\ \ a + \frac{1}{2}}$. We want to find out a way to compute the rotation number $\rho(X)$. Let us define the lift $\tilde{F}$ required, as mentioned in the Corollary 4.1.1.

$$\tilde{F}(t) = \begin{cases} 2a & 0 \leq t \leq a \\ 2t & a < t \leq a + \frac{1}{2} \\ 2a+1 & a + \frac{1}{2} < t < 1 \end{cases}$$

And it is extended to the rest of real numbers using the property $\tilde{F}(t+1) = \tilde{F}(t) + 1$. It is easy to see that $\tilde{F}$ is the lift of the σ_2 map acting on X. Let $\hat{a} \in \mathbb{R}$ be the lift of a. From the above definition $\hat{a} = 2a$. Also we can observe that $0 \leq \tilde{F}^n(\hat{a}) - \tilde{F}^{n-1}(\hat{a}) < 1$. So the (mod1) in the formula in Corollary 4.1.1 is not needed. Let $\tilde{F}^n(\hat{a}) = \hat{a_n}$, $a = a_0$ and $\sigma_2^n(a_0) = a_n$. Clearly, $\hat{a_n}$ is the lift of a_n.

$$\rho(X) = \lim_{n\to\infty} \frac{\tilde{F}^n(a) - 2a}{n} = \lim_{n\to\infty} \frac{[\tilde{F}^n(a) - 2a]}{n} = \lim_{n\to\infty} \frac{[\hat{a_n} - 2a]}{n}.$$

where [x] denotes the integer part of x. $[\hat{a_n} - 2a]$ increases at most by 1 as n increases by 1. Thus, $[\hat{a_n} - 2a]$ is the number of times that happens. Now, we will define a sequence called *crossing over sequence* which helps us to track this behaviour in the orbit X.

Definition 4.2.1. In the above forward orbit $X = \{a_0, a_1, a_2, a_3, ...\}$, if $0 \in \widehat{a_j\, a_{j+1}}$, then we associate bit 1 with a_j, otherwise we associate bit 0. The sequence $\{y_j\}_{j=0}^{\infty}$ obtained is called the crossing over sequence of the orbit of a.

In other words, the crossing over sequence is assigned '1' whenever the orbit crosses over '0' and is '0' if it does not cross over.

We can now see from the above discussion and the definition of the *crossing over sequence* $\{y_j\}_{j=0}^{\infty}$ that whenever y_j is 1, $[\hat{a_j} - 2a]$ increases by 1. This leads to the following proposition.

PROPOSITION 4.2.2. *Let* $X = \{x, \sigma_2(x), \sigma_2^2(x), \sigma_2^3(x), ...\}$ *denote the forward orbit of a point* $x \in \mathbb{S}$. *Let* a *be the lowest point among the closed set of points* $\overline{X}$. *If the binary expression of the point* a *is denoted by* $\{b_i\}_{i=1}^{\infty}$ *where* $b_i \in \{0, 1\}$ *and* $a = (b_1, b_2, b_3, ...) = \sum_{i=1}^{\infty} \frac{b_i}{2^i}$ *Let* $p_j(a) = b_1 + ... + b_j$. *Then the rotation orbit of* X *is*

$$\rho(X) = \lim_{j \to \infty} \frac{p_j(a)}{j}$$

PROOF. Let $a = a_0$ and $\sigma_2^n(a_0) = a_n$. If $0 \in \widehat{a_j\, a_{j+1}}$, it implies that $a_{j+1} = 2a_j - 1$ or in other words $a_j \in \widehat{\frac{1}{2}\,0}$ (lower half of the circle). It follows that the 'j'th bit of crossing over sequence y_j is same as the '1'st bit of a_j or the 'j'th bit of a, i.e $y_j = b_j$. And from the above discussion, we have seen that $[\hat{a_j} - 2a]$ increases by 1 whenever y_j increases by 1. □

Crossing over sequences are very important in computing rotation numbers. For the next section we will use them to obtain results about the rotation numbers of *central symmetric gaps*. For the σ_2 map, whenever the rotational orbit comes to the lower half of the circle $\widehat{\frac{1}{2}\,0}$, at the next stage the orbit crosses over 0 and 'finishes' a wrap around circle. For a periodic orbit, it is demonstrated below.
Also for points on the circle in the binary expansion, the σ_2 map becomes the shift map for the representation. For a rational rotation number, the numerator can be understood as the number of clicks an orbit rotates or the number of wraps around the circle during the full movement of points along the orbit. As shown in the figure below for the orbit of $\overline{01011} = \frac{11}{31}$ under the map σ_2, number of wraps are 3, hence the rotation number $\frac{3}{5}$.

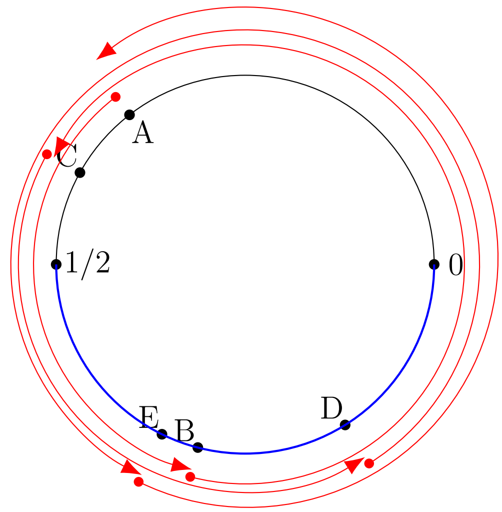


FIGURE 4.1. Periodic orbit of $\overline{01011} = \frac{11}{31}$ ($A- > B- > C- > D- > E$) under the map σ_2 having rotation number $\frac{3}{5}$. Numerator of the rotation number is equal to the number of points on the blue arc.

It can be observed that during every wrap, a unique point of the orbit in the lower half of the circle moves past the point zero (point O) to a point of lesser value (note that circle is parametrized by the quotient space $[0,1]/(0 \sim 1)$. The number of wraps can be associated with the number of distinct points of the orbit on the lower half of the circle. In the figure 2.1, points B,D and E are on the lower half making the rotation number to be $\frac{3}{5}$ for the periodic orbit determined by the point $\overline{01011} = \frac{11}{31}$. In general, the number of distinct points on the lower half of the circle are the number of 1's in a repeating block of the binary expansion of x.

Hence, for σ_2 map, rotation number is the frequency with which the digit '1' occurs in the binary expression of a point in the periodic orbit. For example, $\overline{00101}$ has rotation number $\frac{2}{5}$.

The following important result is proved in [**5**]. We shall provide it without proof here, but it is useful for our results in the next section.

Theorem 4.2.3. *For each $\nu \in [0,1)$ there exists a unique minimal σ_2-invariant rotational subset $A_\nu \subset \mathbb{S}$.*

4.2.2. Cubic map.

Proposition 4.2.4. *A is rotational under the map σ_3 if and only if $\mathbb{S} \setminus A$ contains one interval of length at least $\frac{2}{3}$ or two disjoint intervals of length at least $\frac{1}{3}$ as its components.*

Proof. $\Longleftarrow$ Let M be an interval of length $\frac{2}{3}$ or union of two disjoint intervals of length $\frac{1}{3}$ inside $\mathbb{S} \setminus A$ as the statement suggests. Clearly $\mathbb{S} \setminus M$ is connected or contains two components. In either of the cases it is not hard to see that $\mathbb{S} \setminus M$ maps one to one monotonically onto the full circle under σ_3. It is obvious now that σ_3 preserves order of any tuple in $\mathbb{S} \setminus M$. As $A \subset \mathbb{S} \setminus M$, $\sigma_3|_A$ is order-preserving.
$\Longrightarrow$ Conversely, let us assume three distinct points $\{a, b, c\} \in A$ are such that $\mathbb{S} \setminus \{a, b, c\}$ does not contain an interval of length $\frac{2}{3}$ or two disjoint intervals of length $\frac{1}{3}$. Now $(\sigma_3(a), \sigma_3(b), \sigma_3(c))= (\sigma_3(a), \sigma_3(b \pm \frac{1}{3}), \sigma_3(c \pm \frac{1}{3}))$. One of the four combinations of $(a, b \pm \frac{1}{3}, c \pm \frac{1}{3})$ lie in an interval of length $\frac{1}{3}$ and hence has the same order as $(\sigma_3(a), \sigma_3(b), \sigma_3(c))$ from above. It is clear now that cyclic orders of (a, b, c) and $(\sigma_3(a), \sigma_3(b), \sigma_3(c))$ do not match. □

Remark 4.2.5. As it can be seen there are two kinds of order-preserving periodic orbits possible under σ_3 map, one lying inside an interval of length $\frac{1}{3}$ and the other having two disjoint intervals of length at least $\frac{1}{3}$ in its complement. We will study both of them in the paper.

Consider points on the circle in the ternary expansion i.e $x \in \mathbb{S}$ can be represented as $\overline{b_1 b_2 b_3....}$ where $b_i \in \{0, 1, 2\}$ and $x = \sum\limits_{i=1}^{\infty} \frac{b_i}{3^i}$. σ_3 then becomes a shift map. Now similar to the previous section, it is natural to explore the question of how to compute the rotation number for σ_3 rotational orbits, given a point in the ternary expansion.

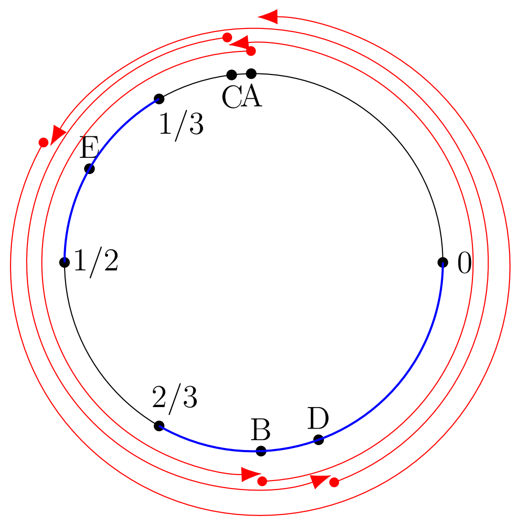


FIGURE 4.2. Periodic orbit of $\overline{02021} = \frac{61}{242}$ $(A- > B- > C- > D- > E)$ under the map σ_3 having rotation number $\frac{3}{5}$. Numerator of the rotation number is equal to the number of points on the blue arcs.

LEMMA 4.2.6. *Rotation number of the orbit of 'x', $X = \{x, \sigma_3(x), \sigma_3^2(x), ...\}$ is given by*

$$\rho(X) = \lim_{n\to\infty} \frac{S(n)}{n}$$

where $S(n)$ is the number of points of the set $\{x, \sigma_3(x), ...\sigma_3^n(x)\}$ on the arcs $\widehat{\frac{1}{3}\ \frac{1}{2}} \cup \widehat{\frac{2}{3}\ 0}$.

PROOF. Consider the *crossing over sequence* $\{y_i\}_{i=1}^{\infty}$ of $X = \{x, \sigma_3(x), \sigma_3^2(x), ...\}$. During every wrap as in the case of σ_2-rotational orbits, whenever the orbit "crosses over 0", a unique point moves past point zero to a point of lesser value. Thus, potential regions on the circle containing those points are determined by solving the inequalities $1 < 3x < x+1$ and $2 < 3x < x+2$ which imply $x \in (\frac{1}{3}, \frac{1}{2}) \cup (\frac{2}{3}, 1)$. Thus, the *crossing over sequence* is '1' whenever the orbit is on the arcs $\widehat{\frac{1}{3}\frac{1}{2}}$ and $\widehat{\frac{2}{3}\ 0}$, i.e

$y_i = 1$ if and only if $\sigma_3^i(x) \in \widehat{\frac{1}{3}\,\frac{1}{2}} \cup \widehat{\frac{2}{3}\,0}$. And similar to the argument for the σ_2 map, the number of '1's in the crossing over sequence is the numerator of the limit fraction in the rotation number formula. □

The figure above demonstrates the case of a finite rotational orbit under the σ_3 map. In the figure below, points B, D are on the arc $\widehat{\frac{2}{3}\,0}$ and point E is on the arc $\widehat{\frac{1}{3}\,\frac{1}{2}}$, making the rotation number to be $\frac{3}{5}$ for the periodic orbit determined by the point $\overline{02021} = \frac{61}{242}$(in ternary expansion).

Let $x \in \mathbb{S}$ be represented in ternary expression as $\{a_i\}_{i=1}^{\infty}$ where $a_i \in \{0, 1, 2\}$ and $x = (a_1, a_2, a_3, ...) = \sum_{i=1}^{\infty} \frac{a_i}{2^i}$. Then, we claim that $S(n)$ is related to the ternary expression $\{a_i\}_{i=1}^{\infty}$ as $S(n) = \#\{1 \leq i \leq n | a_i = 2 \; or \; (a_i = 1 \; and \; a_j = 0 \; where \; j = min(i < k < i + n | a_k \neq 1))\}$, where $\#A$ denote the cardinality of A.
For the points of the orbit X on the arc $\widehat{\frac{2}{3}\,0}$ (one of the blue arcs), their first bit in the ternary expansion is '2'. Thus, points on the arc $\widehat{\frac{2}{3}\,0}$ can be characterized by the number of '2's in $(a_1, a_2, a_3, ...)$.
For the other arc $\widehat{\frac{1}{3}\,\frac{1}{2}}$, it is slightly more complicated. The first bit is a '1' which eventually gets followed by a '0' before a '2'(eg: $\overline{1102} \in \widehat{\frac{1}{3}\,\frac{1}{2}}, \overline{1201} \notin \widehat{\frac{1}{3}\,\frac{1}{2}}$). Hence in computing the rotation number, count only the '1's which will lead to a '0' before a '2'.

4.3. Small rotational gaps

Definition 4.3.1. Small rotational orbits. Small rotational orbits are defined as the kind of rotational σ_3 periodic orbits contained inside an interval of length $\frac{1}{3}$.

Convex-hull of small-rotational orbits form gaps of cubic invariant laminations. We call them *Small rotational gaps.*

We will focus on studying small-rotational orbits first which in turn helps us in studying central symmetric rotational gaps later. Note that the point x forms a rotational orbit if and only if $x \pm \frac{1}{2}$ also forms a rotational orbit.

PROPOSITION 4.3.1. *Small rotational gaps are of two kinds:*

i) Upper small rotational gaps. small rotational gaps that lie in the upper semi-circle $\widehat{0\ \frac{1}{2}}$ *and*

ii) Lower small rotational gaps. small rotational gaps that lie in the lower semi-circle $\widehat{\frac{1}{2}\ 0}$.

PROOF. A simple fact about the small-rotational orbit of a point $x \in \mathbb{S}$ is that it has only two bits among $\{0, 1, 2\}$ in its ternary expansion, since if there were all the three bits in $x = (a_1, a_2, ...)$
,then $\exists i, j, k : \sigma_3^i(x) = x_i = (0..........)$ *and* $\sigma_3^j(x) = x_j = (1..........), \sigma_3^k(x) = x_k = (2..........)$ which implies that the points $\{x_i, x_j, x_k\}$ of the orbit do not lie inside an interval of length $\frac{1}{3}$.
Thus we have three possible cases now for the point $x \in \mathbb{S}$:

(i) *x is made of the bits* $\{0, 1\}$.$X = \{x, \sigma_3(x), \sigma_3^2(x), ...\}$ never enters $\widehat{\frac{2}{3}\ 0}$. It follows that X should be a subset of the cantor set obtained by removing the $\widehat{\frac{2}{3}\ 0}$ and its iterated pre-images under the σ_3 map from the unit circle $\mathbb{S}$. Also it is not hard to see that the cantor set $\subset \widehat{0\ \frac{1}{2}}$. Thus, in this case $X = \{x, \sigma_3(x), \sigma_3^2(x), ...\}$ lies in the upper-semicircle $\widehat{0\ \frac{1}{2}}$. We call X as an upper small rotational orbit and the convex hull of X is called an upper small rotational gap.

(ii) *x is made of the bits* $\{1,2\}$. Note that the point $x-\frac{1}{2}$ diametrically opposite to x is made of the bits $\{0,1\}$ now. From the above case, the forward orbit of $x-\frac{1}{2}$ is a small-rotational orbit lying in the upper semi-circle. Thus, the original orbit X lies in the lower-semicircle $\widehat{\frac{1}{2}\ 0}$ and we call X as a lower small rotational orbit. Convex hull of X is called a lower small rotational gap. Lower and upper small rotational gaps always exist in *symmetric pairs.*

(iii) *x is made of the bits* $\{0,2\}$. We claim that this case does not exist. There exists at least one bit combination of '20' in the ternary expansion of x. In other words, there exists a point $\sigma_3^k(x) = x_k = (2,0,...)$ in the orbit $X = \{x, \sigma_3(x), \sigma_3^2(x), ...\}$. $x_k \in \widehat{\frac{2}{3}\ 0}$ and $x_{k+1} \in \widehat{0\ \frac{1}{3}}$.
Let δ be the length of the arc $\widehat{\frac{2}{3}\ x_k}$. Then the length of the arc $\widehat{0\ x_{k+1}}$ is 3δ. Remember for X to be a small-rotational orbit, the complement of X, $\mathbb{S}\setminus X$ should consist of an arc of length $\frac{2}{3}$. And we know that X does not enter the arc $\widehat{\frac{1}{3}\ \frac{2}{3}}$. It follows that $\mathbb{S}\setminus X \subset \widehat{x_{k+1}\ x_k}$. But the length of the arc $\widehat{x_{k+1}\ x_k}$ from the above is $\frac{2}{3}-2\delta$ smaller than $\frac{2}{3}$. Thus, it is not possible for X to be a small-rotational orbit.

□

Lemma 4.3.2. *There exists exactly two small rotational gaps symmetric to each other for every rotation number.*

Proof. By Theorem 4.2.3, for each $\nu \in \mathbb{R}$ there exists a unique minimal σ_2-invariant rotational subset $A_\nu \subset \mathbb{S}$. If we interpret the points of the set A_ν in ternary expansion, we get an upper small-rotational orbit having the same rotation number ν. That is because from the above proposition, the upper small rotational orbits are made of the bits $\{0,1\}$. Thus, we get a 1-1 correspondence between σ_2-invariant rotational sets and upper small-rotational sets. The convex hull of the upper small-rotational set gives us the upper small rotational gap G_ν. The symmetric gap $-G_\nu$ is a lower small rotational gap and has the same rotation number ν. $\square$

Recall the definition of central symmetric gaps from the Definition 3.2.2 in the previous chapter. Central symmetric gaps have two disjoint symmetric intervals of length $\frac{1}{3}$ at least. So they are rotation of a different kind as mentioned in remark 4.2.5. The main focus of this chapter is to study the rotational behavior of central symmetric gaps. It turns out small rotational gaps are very useful for that purpose. Also recall the following from the earlier chapter.

(1) The central symmetric gap G is 1-transitive if G has one periodic orbit on its boundary.

(2) The central symmetric gap G is 2-transitive if G has two periodic orbits on its boundary.

The main relation between central symmetric and small rotational gaps is based on the fact that A is a σ_3 periodic orbit iff $\sigma_2(A)$ is periodic under σ_3 since

$$a_1 \longrightarrow a_2 \longrightarrow \ldots\ldots \quad a_n \quad (a_n \to a_1)$$

$$\Longleftrightarrow$$

$$2a_1 \longrightarrow 2a_2 \longrightarrow \ldots\ldots \quad 2a_n \quad (2a_n \to 2a_1)$$

Note that $\sigma_2(A)$ is either n periodic if A is 1-transitive central-symmetric gap or $\frac{n}{2}$ periodic if A is 2-transitive central-symmetric gap.
A similar thing is true for infinite orbits too.

LEMMA 4.3.3. *Gap G is a central symmetric gap if and only if $\sigma_2(G)$ is a small rotational gap. And if $\rho(G) = \alpha, \rho(\sigma_2(G) = \beta$. Then $\alpha = \frac{\beta}{2}$ or $\alpha = \frac{1}{2} + \frac{\beta}{2}$.*

PROOF. Let us prove the first part of the lemma.

The σ_2 map linearly stretches any arc of $\mathbb{S}$ having length less than $\frac{1}{2}$. Thus, if G is the central symmetric gap, the σ_2 map stretches the two disjoint symmetric arcs of at least length $\frac{1}{3}$ each in two components of $\mathbb{S} \setminus \partial(G)$ to form an arc of length at least $\frac{2}{3}$ in a component of $\mathbb{S} \setminus \partial(\sigma_2(G))$ making $\sigma_2(G)$ a small rotational gap. On the other hand if we have $\sigma_2(G) = \tilde{G}$ a small rotational gap, then the full pre-image of $\partial(\tilde{G})$ under the σ_2 map consists of pairs of diametrically opposite points. And the full pre-image of the two-third length arc in a component of $\mathbb{S} \setminus \partial(\tilde{G})$ consists of two disjoint symmetric arcs of length $\frac{1}{3}$ each in two components of $\mathbb{S} \setminus \partial(G)$. Thus, G is a central symmetric gap.

To understand the relation between the rotation numbers of G and $\sigma_2(G)$, we need to understand the relation between their respective lifts used in the definition of rotation number.

The rotational action of $\sigma_3 : G \to G$ can be extended to $f : \mathbb{S} \to \mathbb{S}$ and then lifted to the universal cover $\tilde{F} : \mathbb{R} \to \mathbb{R}$ which determines the rotation number $\rho(G) = \alpha$. Similarly we can do the same thing for the small rotational gap $\sigma_2(G)$ and get its corresponding lift $\tilde{G} : \mathbb{R} \to \mathbb{R}$ which determines $\rho(\sigma_2(G)) = \beta$. Now using the path lifting property we get that the maps $\tilde{F}$ and $\tilde{G}$ are conjugated by the lift of the σ_2 map which is $x \to 2x$ acting on the universal cover $\mathbb{R}$ as shown in the diagram.

$$\begin{array}{ccc} G & \xrightarrow{\sigma_2} & \sigma_2(G) \\ {\scriptstyle \alpha\curvearrowleft}\downarrow & & {\scriptstyle \beta\curvearrowleft}\downarrow \\ G & \xrightarrow{\sigma_2} & \sigma_2(G) \end{array} \qquad \begin{array}{ccc} \mathbb{R} & \xrightarrow{x\to 2x} & \mathbb{R} \\ {\scriptstyle \tilde{F}}\downarrow & & {\scriptstyle \tilde{G}}\downarrow \\ \mathbb{R} & \xrightarrow{x\to 2x} & \mathbb{R} \end{array}$$

The actual expressions for the lifts $\tilde{F}$ and $\tilde{G}$ are provided in the remarks after the lemma.

Now ,

$$\rho(G) = \alpha = \lim_{n\to\infty} \frac{\tilde{F}^n(x) - x}{n} (\text{mod } 1), \quad \rho(\sigma_2(G)) = \beta = \lim_{n\to\infty} \frac{\tilde{G}^n(x) - x}{n} (\text{mod } 1).$$

And from the above commutative diagram $\tilde{F}(x) = \frac{1}{2}\tilde{G}(2x)$.

$\implies \tilde{F}^2(x) = \tilde{F}(\tilde{F}(x)) = \frac{1}{2}\tilde{G}[2(\tilde{F}(x))] = \frac{1}{2}\tilde{G}[2\frac{1}{2}\tilde{G}(2x)] = \frac{1}{2}\tilde{G}^2(2x)$.

Inductively we can prove $\tilde{F}^n(x) = \frac{1}{2}\tilde{G}^n(2x).\forall n \in \mathbb{N}$.

$$\lim_{n\to\infty} \frac{\tilde{F}^n(x) - x}{n} = \lim_{n\to\infty} \frac{\frac{\tilde{G}^n(2x)}{2} - x}{n} = \lim_{n\to\infty} \frac{\tilde{G}^n(2x) - 2x}{2n} = \frac{1}{2} \lim_{n\to\infty} \frac{\tilde{G}^n(2x) - 2x}{n}.$$

Now we can conclude that $(2\alpha - \beta)$ mod $1 = 0$, i.e $2\alpha - \beta$ should be an integer. As both $0 \leq \alpha, \beta < 1$, we have $-1 < 2\alpha - \beta < 2$. It follows that $2\alpha - \beta = 0$ or $2\alpha - \beta = 1$. Thus, $\alpha = \frac{\beta}{2}$ or $\alpha = \frac{1}{2} + \frac{\beta}{2}$. □

REMARK 4.3.4. (1) Let G be a central symmetric gap. There will be two symmetric sides on G having the same length, bigger than any other side of G. Let a pair of endpoints of one such side be $\{a, a+\frac{1}{2}\}$ such that $0 \leq a \leq \frac{1}{2}$. We will define the lift $\tilde{F}$ map as follows:

$$\tilde{F}(t) = \begin{cases} 3a & t \in [0, a] \\ 3t & t \in (a, a+\frac{1}{6}) \\ 3a + \frac{1}{2} & t \in [a+\frac{1}{6}, a+\frac{1}{2}] \\ 3t - 1 & t \in (a+\frac{1}{2}, a+\frac{2}{3}) \\ 3a + 1 & t \in [a+\frac{2}{3}, 1) \end{cases}$$

For real numbers lying outside $[0, 1)$ we extend using the relation $\tilde{F}(t+1) = \tilde{F}(t) + 1$. The small rotational gap $\sigma_2(G)$ has '$2a$' as one of its vertices. We will define the lift $\tilde{G}$ map for it as follows:

$$\tilde{G}(t) = \begin{cases} 6a & t \in [0, 2a] \\ 3t & t \in (2a, 2a + \frac{1}{3}) \\ 6a + 1 & t \in [2a + \frac{1}{3}, 1) \end{cases}$$

For real numbers lying outside $[0, 1)$ we extend using the relation $\tilde{G}(t + 1) = \tilde{G}(t) + 1$.

(2) It is easy to see that $\tilde{F}$ and $\tilde{G}$ are order-preserving extensions of $\sigma_3|_G$ and $\sigma_3|_{\sigma_2(G)}$ respectively.

(3) And also $\tilde{G}(x) = 2\tilde{F}(\frac{x}{2})$.

Lemma 4.3.5. *Let G be a central symmetric gap. Then $\partial(G)$ has two possibilities.*

(a) $\partial(G) \in \widehat{0\ \frac{1}{4}} \cup \widehat{\frac{1}{2}\ \frac{3}{4}}$. In which case $\rho(G) = \frac{1}{2}\rho(\sigma_2(G))$.

(b) $\partial(G) \in \widehat{\frac{1}{4}\ \frac{1}{2}} \cup \widehat{\frac{3}{4}\ 0}$. Here $\rho(G) = \frac{1}{2} + \frac{1}{2}\rho(\sigma_2(G))$.

Leaves $\overline{0\frac{1}{2}}$ and $\overline{\frac{1}{4}\frac{3}{4}}$ are the only central symmetric gaps (degenerate) which are in both the above regions.

Proof. By the above Lemma, $\sigma_2(G)$ is a small rotational gap. Using the Proposition 4.3.1, when $\sigma_2(G)$ is an upper small rotational gap, we have $\partial(G) \in \widehat{0\ \frac{1}{4}} \cup \widehat{\frac{1}{2}\ \frac{3}{4}}$ and when $\sigma_2(G)$ is a lower small rotational gap, we have $\partial(G) \in \widehat{\frac{1}{4}\ \frac{1}{2}} \cup \widehat{\frac{3}{4}\ 0}$.

Now we will show that for the central symmetric gaps lying in the first region rotation numbers are smaller than $\frac{1}{2}$ and for the gaps lying in the second region rotation numbers are bigger than $\frac{1}{2}$.

(a) $\partial(G) \in \widehat{0\ \frac{1}{4}} \cup \widehat{\frac{1}{2}\ \frac{3}{4}}$. Consider a point $x \in \partial(G) \in \widehat{0\ \frac{1}{4}} \cup \widehat{\frac{1}{2}\ \frac{3}{4}}$. Let us look at the *crossing over sequence* $\{y_n\}_{n=0}^{\infty}$ for the orbit of x, $X = \{x, \sigma_3(x), \sigma_3^2(x), ...\}$. We claim that there cannot be two consecutive '1's in the sequence y_n.

Clearly $y_k = 1$ if and only if $x_k = \sigma_3^k(x) \in \widehat{\frac{2}{3}\ \frac{3}{4}}$ because that's the only way X can cross over 0. And $x_{k+1} \in \widehat{0\ \frac{1}{4}}$ as σ_3 linearly stretches the arc $\widehat{\frac{2}{3}\ \frac{3}{4}}$ to the arc $\widehat{0\ \frac{1}{4}}$. Thus $y_k = 1 \implies y_{k+1} = 0$. It implies that at least half of the bits in the *crossing over sequence* y_n are '0's. And $\rho(G)$ is nothing but the "frequency" of '1's in the sequence y_n. Thus, $\rho(G) \leq \frac{1}{2}$ and from the above Lemma $\rho(G) = \frac{1}{2}\rho(\sigma_2(G))$.

(b) $\partial(G) \in \widehat{\frac{1}{4}\ \frac{1}{2}} \cup \widehat{\frac{3}{4}\ 0}$. In this case, there cannot be two consecutive '0's in the sequence y_n. $y_k = 0$ if and only if $x_k = \sigma_3^k(x) \in \widehat{\frac{1}{4}\ \frac{1}{3}}$ and $x_{k+1} \in \widehat{\frac{3}{4}\ 0}$ as σ_3 linearly stretches the arc $\widehat{\frac{1}{4}\ \frac{1}{3}}$ to the arc $\widehat{\frac{3}{4}\ 0}$. The points on the arc $\widehat{\frac{3}{4}\ 0}$ will cross over '0'. Thus $y_k = 0 \implies y_{k+1} = 1$. It implies that at least half of the bits in the *crossing over sequence* y_n are '1's. And $\rho(G)$ is nothing but the "frequency" of '1's in the sequence y_n. Thus, $\rho(G) \geq \frac{1}{2}$ and by the above Lemma $\rho(G) = \frac{1}{2} + \frac{1}{2}\rho(\sigma_2(G))$.

□

We provide the pictures of *1-transitive* and *2-transitive* finite central symmetric gaps below. The pictures show the relation between the rotation numbers of central symmetric gaps and small rotational gaps.

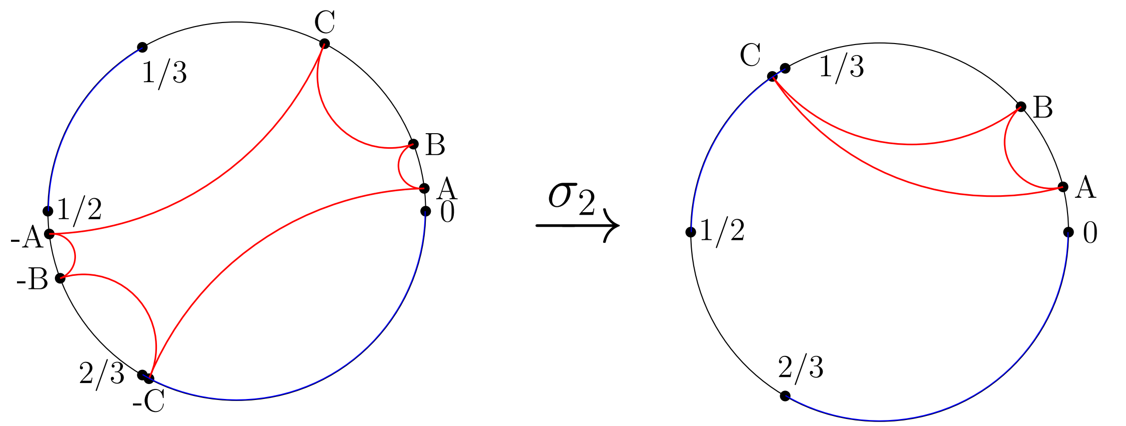


FIGURE 4.3. On the left, a 1-transitive central symmetric gap formed by the periodic orbit of $\overline{000112} = \frac{1}{52}$ $(A- > B- > C- > -A- > -B- > -C)$ having rotation number $\frac{1}{6}$. On the right, an upper small rotational gap formed by the periodic orbit of $\overline{001} = \frac{1}{26}$ $(A- > B- > C)$ having rotation number $\frac{1}{3}$.

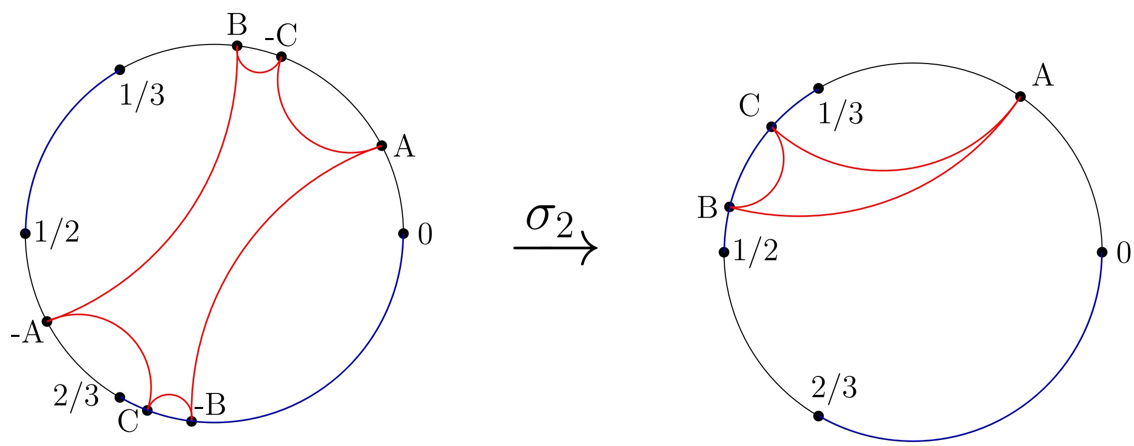


FIGURE 4.4. On the left, a 2-transitive central symmetric gap formed by two periodic orbits of $\overline{002} = \frac{1}{13}$ $(A- > B- > C)$ and its symmetric one $\overline{012} = \frac{5}{26}(-A- > -B- > -C)$ each having rotation number $\frac{1}{3}$. On the right, an upper small rotational gap formed by the periodic orbit of $\overline{011} = \frac{2}{13}$ $(A- > B- > C)$ having rotation number $\frac{2}{3}$.

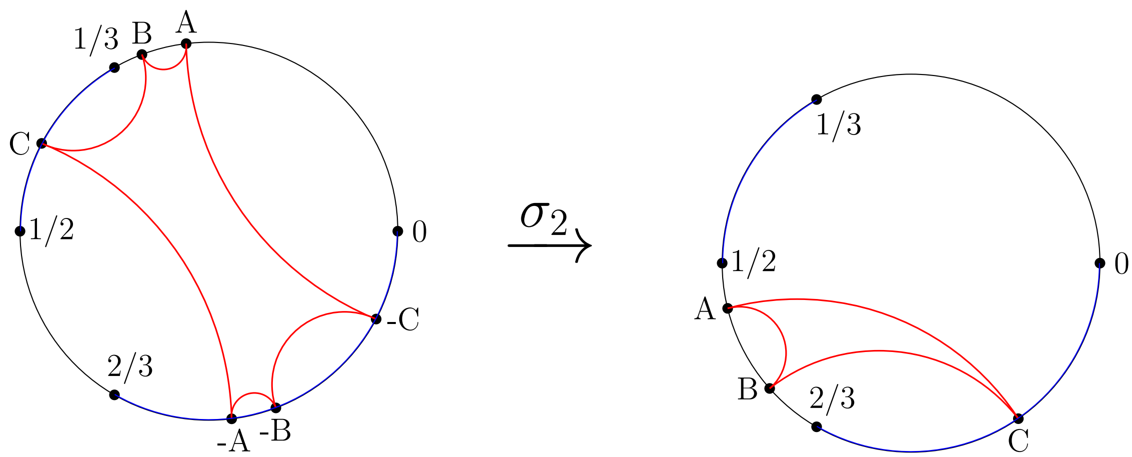


FIGURE 4.5. On the left, a 2-transitive central symmetric gap formed by two periodic orbits of $\overline{021} = \frac{7}{26}$ $(A- > -B- > C)$ and its symmetric one $\overline{202} = \frac{5}{26}(-A- > B- > -C)$ each having having rotation number $\frac{2}{3}$. On the right, a lower small rotational gap formed by periodic orbit of $\overline{112} = \frac{7}{13}$ $(A- > B- > C)$ having rotation number $\frac{1}{3}$.

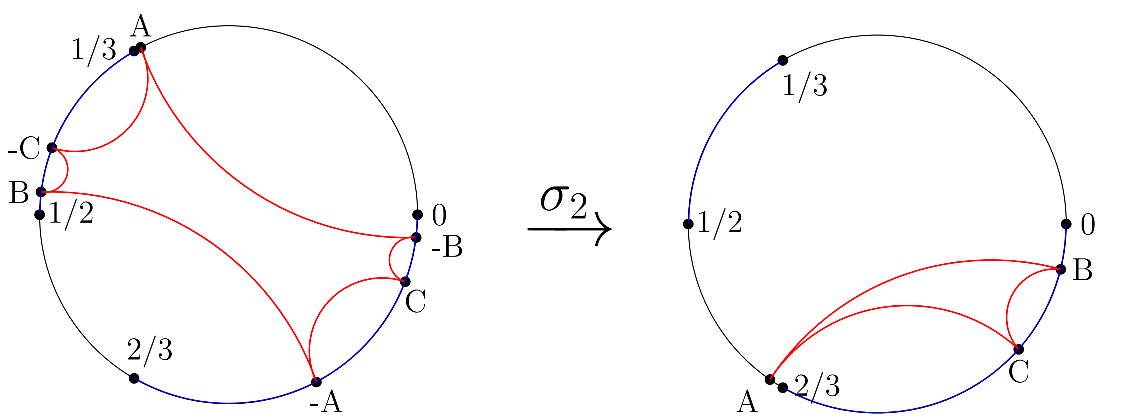


FIGURE 4.6. On the left, a 1-transitive central symmetric gap formed by the periodic orbit of $\overline{022211} = \frac{17}{52}$ $(A- > -B- > C- > -A- > B- > -C)$ having rotation number $\frac{5}{6}$. On the right, a lower small rotational gap formed by the periodic orbit of $\overline{001} = \frac{17}{26}$ $(A- > B- > C)$ having rotation number $\frac{2}{3}$.

And finally we have the main result of the section.

THEOREM 4.3.6. *There exists a unique central symmetric gap for every rotation number.*

PROOF. Follows by Lemma 4.3.2 and Lemma 4.3.5. □

The family of central symmetric gaps also helps us in understanding the parameter space of cubic symmetric laminations. In the next chapter a whole subsection is devoted to that. Central symmetric gaps will be understood in relation to the critical chords of the circle $\mathbb{S}$. Note that every cubic symmetric lamination is obtained by taking the closure of pre-images of leaves. And critical chords are necessary to determine the three inverse branches of the σ_3 map which helps us in taking pre-images of leaves.

CHAPTER 5

CUBIC SYMMETRIC COMAJOR LAMINATION C_sCL

5.1. Comajors and their properties

In the previous two sections, we gained some insight into properties of leaves and gaps of *cubic symmetric laminations*. In the current section, we will work towards understanding the structure of the family of *cubic symmetric laminations*.

Every *cubic symmetric lamination* has three important leaves: major, comajor and minor. They tell us all the information needed to build the lamination. Formal definitions are given below. Consider a *cubic symmetric lamination* $\mathcal{L}$ for the rest of the chapter.

DEFINITION 5.1.1. **Major -** A leaf ℓ of $\mathcal{L}$ such that $|\|\ell\| - \frac{1}{3}|$ has the least value among all the leaves of $\mathcal{L}$, or in other words the leaf closest to the critical chord of the circle is defined as a major of $\mathcal{L}$. Let us denote a major of $\mathcal{L}$ as M. There are multiple major leaves corresponding to $\mathcal{L}$ since,

(1) Long/medium sibling $\widehat{M}$ of leaf M also satisfy the above definition and
(2) Symmetric pair of leaves $-M$ and $-\widehat{M}$ also satisfy the above definition.

DEFINITION 5.1.2. **Comajor -** The short sibling c of the major leaves M and $\widehat{M}$ is defined as the comajor of the lamination $\mathcal{L}$. Similarly, the short sibling $-c$ of the other two major leaves $-M$ and $-\widehat{M}$ is also a comajor. $\{c, -c\}$ form a pair of symmetric comajors of $\mathcal{L}$.

Definition 5.1.3. **Minor -** Images of the majors are defined as the minors of the lamination. Similar to the comajors, there are two symmetric minors $\{m, -m\}$ of a cubic symmetric lamination.

Below are some useful results about these leaves.

Lemma 5.1.1. *Consider the minor leaves $\{m, -m\}$ of the cubic symmetric lamination $\mathcal{L}$. We can make the following conclusions.*

(i) No forward image of m (and $-m$) is shorter than m.

(ii) Any leaf ℓ of $\mathcal{L}$ shorter than m also possesses the same property that no forward image of ℓ is shorter than ℓ.

(iii) If a leaf ℓ of $\mathcal{L}$ is such that it is longer than m, then no forward image of the leaf ℓ can be smaller than m.

Proof. (i) The length of the minor leaf $m = \sigma_3(M)$ is $\|m\| = |1 - 3\|M\||$. Consider the forward images of the minor leaf $\{m_i\}_{i=1}^{\infty}$ in $\mathcal{L}$. By the way of contradiction, let us assume that the first time a forward image of m gets shorter than m is leaf m_k. As all short leaves expand by a factor of three, the pre-image leaf m_{k-1} of m_k has to be a long/medium leaf for otherwise m_k would not be the first time a forward image gets shorter than m. Thus, $\|m_k\| = |1 - 3\|m_{k-1}\||$.

We know that no leaf of $\mathcal{L}$ gets closer to the critical chord of length $\frac{1}{3}$ than the major leaf M. In particular it is true for the leaf m_{k-1}, i.e $|\|m_{k-1}\| - \frac{1}{3}| \geq |\|M\| - \frac{1}{3}|$. It follows that $|3\|m_{k-1}\| - 1| \geq |3\|M\| - 1| \implies \|m_k\| \geq \|m\|$ giving us a contradiction. We get a similar contradiction with the other minor leaf $-m$.

(ii) We will prove the second statement of the lemma in a similar manner. Consider a leaf ℓ which is shorter than minor leaf m, i.e $\|\ell\| < \|m\|$. Consider the forward images of the leaf ℓ, $\{l_i\}_{i=1}^{\infty}$ in $\mathcal{L}$. By the way of contradiction, let us assume that the first time a forward image of ℓ gets shorter than ℓ is the leaf ℓ_k. Similar to the argument above, ℓ_{k-1} has to be a long/medium leaf, i.e $\|\ell_k\| = |1 - 3\|\ell_{k-1}\||$. $||\|\ell_{k-1}\| - \frac{1}{3}| \geq ||\|M\| - \frac{1}{3}|$. It follows that $|3\|\ell_{k-1}\| - 1| \geq |3\|M\| - 1| \implies \|\ell_k\| \geq \|m\| \implies \|\ell\| > \|\ell_k\| \geq \|m\|$ giving us a contradiction.

(iii) By the way of contradiction, let us assume that the first time a forward image of ℓ gets shorter than m is the leaf ℓ_k. Similar to the argument above, ℓ_{k-1} has to be a long/medium leaf, i.e $\|\ell_k\| = |1 - 3\|\ell_{k-1}\||$. As, $\|\ell_k\| < \|m\|$, it follows that $||\|\ell_{k-1}\| - \frac{1}{3}| < ||\|M\| - \frac{1}{3}|$. But major leaves M and $-M$ are the closest leaves to a critical chord of $\mathbb{S}$ in the lamination $\mathcal{L}$ giving us a contradiction.

□

Corollary 5.1.2. *Non-degenerate comajor leaves are non-periodic.*

Proof. The only way for a non-degenerate comajor leaf to be periodic would be if a minor leaf maps to comajor leaf in the future. We know that comajor leaves are short leaves and the length of a minor equals 3∗length of comajor. By the above lemma, the forward images of minor leaves have to be at least as long as the minor leaves. It follows that there is no possibility of a minor leaf mapping to a comajor leaf in the future. Thus, non-degenerate comajor leaves cannot be periodic. □

Lemma 5.1.3. *The endpoints of a non-degenerate comajor leaf of a cubic symmetric lamination $\mathcal{L}$ are both preperiodic with the same preperiod (and the same period) or both not eventually periodic. In the latter case, the comajor leaf is approximated from both sides by leaves of $\mathcal{L}$ that have no common endpoints with the comajor.*

Proof. Consider a *cubic symmetric lamination* $\mathcal{L}$ with a non-degenerate comajor leaf c. It follows that $\mathcal{L}$ has two symmetric critical gaps G and $-G$ bounded by the major leaves M,$\widehat{M}$,$-M$ and $-\widehat{M}$. By Theorem 3.3.15, G (and $-G$) is either a Fatou gap (with or without a symmetric orbit) or a finite gap.

(1) *If G (and $-G$) is a Fatou gap with or without a symmetric orbit:* By the Lemma 3.3.12, one of the major leaves M and $\widehat{M}$ are invariant under the first return map of G or, in other words, a major leaf is periodic, which makes the comajor leaf c preperiodic with both endpoints having the same preperiod 1 and the same period.

(2) *If G (and $-G$) is a finite gap with at least 6 sides:* By the *No Wandering Triangles* lemma, it follows that G and its image gap must be preperiodic. By Lemma 3.3.7, all the vertices of periodic polygons are in one periodic orbit or two symmetric periodic orbits. In both cases, the adjacent vertices of a periodic polygon stay adjacent within their periodic orbits, and have the same period. It follows that both the endpoints of the major leaf M have the same preperiod which is at least 2 and the same period. Same is the case with the comajor leaf c.

(3) *If G (and $-G$) is a collapsing quadrilateral.* We distinguish three sub-cases here:

   (a) *The major leaf M is adjacent to a finite gap with at least 6 sides:* Proceeding the same way as we have done in case(2), the finite gap G attached to M is (pre)periodic and we get that the endpoints of the major leaf M have the same preperiod and period. Here major leaves can be periodic. Thus, we can conclude that both the endpoints of comajor leaf c are preperiodic with the same preperiod (at least 1) and the same period.

(b) *The major leaf M is adjacent to an eventually collapsing quadrilateral G'. G'* eventually maps to collapsing quadrilaterals G or $-G$.

Let us first consider the case when G' eventually maps to G. Consider the minor leaf m on $\sigma_3(G')$, m should eventually map to a boundary leaf of G. By Lemma 5.1.1, the minor leaf m cannot map to the short leaves of G which have a third of its length. It follows that the minor leaf m returns to one of the two major leaves $\{M, \widehat{M}\}$ on G, which makes the minor leaf periodic. Thus, the comajor leaf c, one of the pre-image leaves of minor leaf is preperiodic with both endpoints having the same preperiod (at least 1) and the same period.

The other case is when G' eventually maps to $-G$ not G. Using the same argument as above, the minor leaf $m = \sigma_3(M)$ has to map to one of the two major leaves $\{-M, -\widehat{M}\}$ on $-G$. Which means that the minor leaf m eventually maps to the other minor leaf $-m$. It follows that both the minor leaves m and $-m$ will be in one periodic orbit. Thus, in this case too, the comajor leaf, one of the pre-image leaves of minor leaf is preperiodic with both the endpoints having the same preperiod (at least 1) and the same period.

(c) *Major leaf M is adjacent to no other gap.* This means that the major leaf M is a limit of other leaves. Let ℓ be a short boundary leaf of G.

If ℓ is adjacent to a finite gap with at least 6 sides, we conclude as case (2).

If ℓ is adjacent to an eventually collapsing quadrilateral, then we may erase the short edges $\{\ell, \hat{\ell}\}$ on G and their corresponding symmetric leaves on $-G$ together with their backward orbits, and obtain a lamination with the same minor leaves $\{m, -m\}$. The symmetric critical gaps of the new lamination are Fatou gaps. Now we conclude as case (1).

We may thus assume that ℓ is a limit of other leaves as well. It turns out that the leaves approximating M and ℓ have no common endpoints with G.

If all leaves approximating M (or those approximating ℓ) have a common endpoint with G and thus with M (or with ℓ), then a vertex of G is an isolated vertex of infinitely many gaps accumulating at a side of G. The lengths of two adjacent sides of these gaps are close to the length of M (or ℓ). By Lemma 3.3.10, it follows that none of the accumulating gaps can be eventually collapsing quadrilaterals.

The accumulating gaps as mentioned above cannot be preperiodic polygons either, since by Corollary 3.3.9, two different preperiodic polygons have disjoint sets of vertices, unless they share a common boundary leaf that eventually maps to a critical leaf (which we don't have here).

We now conclude that both M and ℓ are limits of leaves that have no common endpoints with G. Thus, the comajor leaf c is a limit of leaves from both sides by leaves that have no common endpoints with the comajor. This case happens when comajor leaves have non-periodic endpoints, since a wandering leaf cannot be part of any gap, it stays non-isolated.

□

Our goal is to describe the set of *cubic symmetric laminations* in terms of their comajor leaves. This goal is complicated by the fact that it is possible for two different laminations to have the same comajor leaf.

Given a pair of chords or boundary points $\{c, -c\}$ of the disk, when can they be the comajor leaves of a *cubic symmetric lamination*?

LEMMA 5.1.4. *There exists a cubic symmetric lamination $\mathcal{L}(c)$ with the pair of boundary points $\{c, -c\}$ $\forall c \in \mathbb{S}$ as degenerate comajors.*

PROOF. Consider the pair of symmetric critical leaves $\{\ell, -\ell\}$ such that $\sigma_3(\ell) = \sigma_3(c)$ and $\sigma_3(-\ell) = \sigma_3(-c)$. We will try to construct a *cubic symmetric lamination* $\mathcal{L}(c)$ out of these critical leaves.
Critical leaves $\{\ell, -\ell\}$ split the open disk $\mathbb{D}$ into three regions. Unit circle $\mathbb{S}$ is split among the boundaries of the three regions equally as two arcs of length $\frac{1}{3}$ on the left and the right and a union of two symmetric arcs of length $\frac{1}{6}$ in the middle. Thus, a branch of the inverse map σ_3^{-1} can be defined on each of the three regions of $\mathbb{S}$ above. The inverse map σ_3^{-1}, thus obtained can be extended to leaves and gaps similar to the extension of map σ_3 on leaves and gaps(definition 2.2.7). Now consider the collection $\mathcal{C}$ of pre-images of the critical leaves $\{\sigma_3^{-i}(\ell), \sigma_3^{-i}(-\ell)\}_{i=1}^{\infty}$. We observe two different scenarios here.

(a) *The critical leaves ℓ and $-\ell$ do not have periodic endpoints.* Pre-images at every stage in the collection $\mathcal{C}$ do not share an endpoint with the critical leaves ℓ and $-\ell$ since the leaves $\{\ell, -\ell\}$ do not have periodic endpoints i.e $\forall i \in \mathbb{N}, \sigma_3^{-i}(\pm\ell) \cap \pm\ell = \emptyset$. Thus the pre-images are unambiguously defined and no leaf ℓ in the collection $\mathcal{C}$ crosses with the leaf ℓ or $-\ell$.
Let $\mathcal{L}_n = \bigcup_{i=0}^{n}\{\sigma_3^{-i}(\ell), \sigma_3^{-i}(-\ell)\} \bigcup_{m=0}^{\infty}\{\sigma_3^{m}(c), \sigma_3^{m}(-c)\}$. Clearly, $\mathcal{L}_n$ is forward invariant. Finally, we define $\mathcal{L}(c) = \overline{\bigcup_{n=0}^{\infty} \mathcal{L}_n}$.

It is clear that no two leaves in $\mathcal{L}(c)$ cross each other since no two leaves of $\mathcal{L}_n$ cross each other $\forall n \in \mathbb{N}$. It is also not hard to see that $\mathcal{L}(c)$ is both *forward* (D1) and *backward* invariant (D2)(Definition 2.2.4). Forward invariance follows easily because it is satisfied at each stage, whereas backward invariance needs some work to show. Symmetric condition (D3)(definition 3.1.1) also follows easily from the fact that it is satisfied at each stage. Thus, we obtain a *cubic symmetric lamination* $\mathcal{L}(c)$ with the pair of comajors $\{c, -c\}$.

(b) *The critical leaves ℓ and $-\ell$ have periodic endpoints.* Let $\{p, -p\}$ be the periodic endpoints of the critical leaves ℓ and $-\ell$ respectively of period k. We have two subcases here.

   (i) *The points p and $-p$ have disjoint orbits.* Collections of pullback leaves containing pre-images $\{\sigma_3^{-i}(\ell)\}$ and $\{\sigma_3^{-i}(-\ell)\}$ are clearly defined for all i from 0 to $k-1$. When $i = k$, i.e for the collections $\{\sigma_3^{-k}(\ell)\}$ and $\{\sigma_3^{-k}(-\ell)\}$, there will be two possible pre-image leaves sharing an endpoint with the critical leaf ℓ, one long and one short leaf. Similarly there will be two possible leaves sharing an endpoint with $-\ell$. We shall consider only the two short pull back leaves attached to ℓ and $-\ell$.

   It is clear to see that all the pull back leaves $\{\sigma_3^{-i}(\ell)\}$ and $\{\sigma_3^{-i}(-\ell)\}$ are uniquely defined for $i > 1$. Let $\mathcal{L}_n = \bigcup_{i=0}^{n}\{\sigma_3^{-i}(\ell), \sigma_3^{-i}(-\ell)\} \bigcup_{m=0}^{\infty}\{\sigma_3^{m}(c), \sigma_3^{m}(-c)\}$. Clearly, $\mathcal{L}_n$ is forward invariant. Finally, we define $\mathcal{L}(c) = \overline{\bigcup_{n=0}^{\infty} \mathcal{L}_n}$.

   It is clear that no two leaves in $\mathcal{L}(c)$ cross each other since no two leaves of $\mathcal{L}_n$ cross each other $\forall n \in \mathbb{N}$. It is also not hard to see that $\mathcal{L}(c)$ is both *forward*(D1) and *backward* invariant(D2). Forward invariance follows easily because it is satisfied at each stage, backward invariance needs some work to show. Symmetric condition (D3)(definition 3.1.1) also follows easily from the fact that it is satisfied at each stage. Thus, we obtain a *cubic symmetric lamination* $\mathcal{L}(c)$ with $\{c, -c\}$ as a pair of degenerate comajors.

(b) *p and −p are part of one periodic orbit.* In such a case, there will be pullback leaves in the collection $\{\sigma_3^{-\frac{k}{2}}(\ell)\}$ sharing an endpoint with the critical leaf $-\ell$. Same as the earlier case, there will be two choices, one long and one short. We shall only use the short leaf. Similar thing is done for the collection $\{\sigma_3^{-\frac{k}{2}}(-\ell)\}$. The rest of the construction follows exactly as the previous case and we proceed to construct a *cubic symmetric lamination* $\mathcal{L}(c)$ with $\{\ell, -\ell\}$ as a pair of degenerate comajors.

□

PROPOSITION 5.1.5. *Suppose that a pair of non-degenerate leaves $\{c, -c\}$ satisfies the following conditions:*

(a) all forward images of c and $-c$ have disjoint interiors,

*(b) the length of all forward images of c(and $-c$) is never less than 3*length of c,*

(c) c must be a short leaf (of length at most $1/6$) and all leaves on the forward orbit of c are disjoint from the interiors of the two sibling leaves of c of length at least $1/4$ (these would be major leaves). Similar conditions for leaf $-c$.

Then, there exists a cubic symmetric lamination $\mathcal{L}(c)$ with the pair of comajors $\{c, -c\}$.

PROOF. Condition (a) guarantees that the set $\bigcup_{m=0}^{\infty}\{\sigma_3^m(c), \sigma_3^m(-c)\}$ does not contain any crossing leaves.

Condition (b) is in compliance with the Lemma 5.1.1 that the minor leaf $m = \sigma_3(c)$ has to be the shortest leaf in its orbit.

Condition (c) helps us to build major leaves and its pre-images thereof needed to build the lamination. Since c cannot have length greater than $1/6$, the four siblings to endpoints of c can be joined in pairs in exactly one way by leaves M and $\widehat{M}$ of length at least $1/4$. Similarly from $-c$, we can get the pair of leaves $-M$ and $-\widehat{M}$ symmetric to the ones above. These will be the major leaves of the lamination. Construction of $\mathcal{L}(c)$ is very similar to what we did in the previous lemma. Imagine a set of critical chords $\{\ell, -\ell\}$ inside the short strips $C(M) \cup -C(M)$ from one endpoint of the major leaf M (and $-M$) to the other endpoint of its sibling $\widehat{M}$ (and $-\widehat{M}$). These critical chords split the open disk $\mathbb{D}$ into three regions. Similar to the construction in the previous lemma, a branch of the inverse map σ_3^{-1} can be defined on each of the three corresponding regions of $\mathbb{S}$ and later extend to the leaves and gap. Now consider the collection $\mathcal{C}$ of pre-images of the major leaves $\{\sigma_3^{-i}(M), \sigma_3^{-i}(\widehat{M}), \sigma_3^{-i}(-M), \sigma_3^{-i}(-\widehat{M})\}_{i=1}^{\infty}$. Condition (c) gives us that no leaf ℓ in the collection $\mathcal{C}$ crosses with the major leaves because a pre-image of a major leaf crossing a major leaf would imply forward image of a comajor leaf crossing one of the major leaves contradicting (c). We observe two different scenarios here.

(i) *The major leaves do not have periodic endpoints.* Like the case(a) of the previous lemma, pre-images are unambiguously defined

We define $\mathcal{L}_n = \bigcup_{i=0}^{n}\{\sigma_3^{-i}(M), \sigma_3^{-i}(\widehat{M}), \sigma_3^{-i}(-M), \sigma_3^{-i}(-\widehat{M})\} \bigcup_{m=0}^{\infty}\{\sigma_3^{m}(c), \sigma_3^{m}(-c)\}$ and $\mathcal{L}(c) = \overline{\bigcup_{n=0}^{\infty} \mathcal{L}_n}$.

Using very similar arguments as that of the above lemma, we can show that $\mathcal{L}(c)$ is a *cubic symmetric lamination* with the pair of comajors $\{c, -c\}$.

(ii) *The major leaves have periodic endpoints.* Similar to the previous lemma, we have two sub-cases where $\{p, -p\}$, the endpoints of the major leaves M and $-M$ have disjoint orbits or the endpoints belong to one periodic orbit. We will take the short pullbacks at the first stage and then the pullbacks become unique. Details are left to the reader.

It follows from the similar arguments as that of the previous lemma that $\mathcal{L}(c)$ constructed in this fashion is a *cubic symmetric lamination* with the pair of comajors $\{c, -c\}$.

□

We call the *cubic symmetric lamination* $\mathcal{L}(c)$ as *pull-back lamination*. Pull-back laminations are useful in helping us to parametrize all cubic symmetric laminations.

For a non-diameter chord $n = \overline{ab}$, the smaller of the two arcs into which n divides $\mathbb{S}$, is denoted by $H(n)$.

DEFINITION 5.1.4. Let us denote the closed subset of the disk $\overline{\mathbb{D}}$ bounded by n and $H(n)$ as $R(n)$. Given two comajors m and n, we write $m \succ n$ if $n \subset R(m)$, and say that n is *under* m.

The partial order $\succ$ has the property that if $m \succ n$ and $m' \succ n$ for leaves $m' \neq m$, then either $m \succ m'$ or $m' \succ m$, that is to say that the set of comajors greater than any given comajor is linearly ordered. If $\ell \succ m \succ n$, we say that m separates ℓ and n.

LEMMA 5.1.6. *Let $\{c, -c\}$ and $\{c', -c'\}$ be two distinct pairs of comajors of two different cubic symmetric laminations. If $c' \succ c$, then $c' \in \mathcal{L}(c)$.*

PROOF. Let us first take the case when the comajor leaf c is non-degenerate. First, we claim that the leaf c' does not intersect any leaf in the cubic symmetric lamination $\mathcal{L}(c)$. Let $\mathcal{C}(c)$ and $\mathcal{C}(c')$ be the critical strips in the laminations $\mathcal{L}(c)$ and $\mathcal{L}(c')$ respectively. As $c' \succ c$, we have $\mathcal{C}(c) \subset \mathcal{C}(c')$. By the *Short Strip Lemma*, Corollary 3.2.5, no image of c' enters the strip $\mathcal{C}(c')$. It follows that no image of c' enters the strip $\mathcal{C}(c)$ too, which means that no image of c' crosses the major leaves of $\mathcal{L}(c)$. As the pull-backs of major leaves are dense in $\mathcal{L}(c)$, we can now conclude that the leaf c' does not intersect any leaf in the lamination $\mathcal{L}(c)$.

So c' has to be either a leaf of $\mathcal{L}(c)$ or a diagonal in a finite gap G in $\mathcal{L}(c)$. By the *No Wandering Triangles Theorem*, Theorem 3.2.6, every finite gap is (pre)periodic or (pre)critical. We have already established that c' does not enter critical strips of $\mathcal{L}(c)$. It follows that finite gap G is not (pre)critical. We can now use Corollary 3.3.8 to conclude that c' cannot be a diagonal in the (pre)periodic polygon G of $\mathcal{L}(c)$. Therefore, c' is a leaf in $\mathcal{L}(c)$.

The proof for the degenerate case is similar and we leave it to the reader. $\square$

THEOREM 5.1.7. *Comajors of distinct cubic symmetric laminations do not cross.*

PROOF. Consider two distinct pairs of comajors $\{c_1, -c_1\}$ and $\{c_2, -c_2\}$ of two distinct cubic symmetric laminations $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. Without loss of generality, let us assume c_1 and c_2 lie within a semicircle. Clearly, if $H(c_1) \cap H(c_2) = \emptyset$, then c_1 does not cross c_2.

Let us assume that $H(c_1) \cap H(c_2) \neq \emptyset$. Now, using the fact that periodic points of the σ_3 map are dense in the circle, we can find a periodic point p in the region $H(c_1) \cap H(c_2)$.

By Proposition 5.1.4, there exists a cubic symmetric lamination $\mathcal{L}(p)$ with the points $\{p, -p\}$ as degenerate periodic comajors. Note that from the choice of selection of point p, it follows that $c_1 \succ p$ and $c_2 \succ p$. By Lemma 5.1.6, c_1 and c_2 are leaves of the lamination $\mathcal{L}(p)$. Thus, the comajors c_1 and c_2 do not cross. $\square$

The following result follows by Theorem 5.1.7 and Theorem 2.2.3.

THEOREM 5.1.8. *The space of all cubic symmetric laminations is compact. The set of all their comajors is a lamination.*

Proof. By Theorem 2.2.3 if a sequence of cubic symmetric laminations converges, then the limit is a σ_3- invariant lamination. Now we need to check the symmetric property (D3). Consider a sequence of cubic symmetric laminations $\mathcal{L}_i$ converging to $\mathcal{L}$. For every leaf $\ell \in \mathcal{L}$, there exists a sequence of leaves $\ell_i \in \mathcal{L}_i$ such that $\ell_i \to \ell$. Since all of the laminations $\mathcal{L}_i$ satisfy property (D3), there exists leaves $-\ell_i \in \mathcal{L}_i$. Finally, it is obvious that $-\ell_i \to -\ell$ as $\ell_i \to \ell$ which implies that $-\ell$ should be in the limit lamination, $-\ell \in \mathcal{L}$.

Now, to prove the second claim of the theorem, what remains is to prove that the comajors of cubic symmetric laminations form a closed family of chords. Indeed, if a sequence of comajors converges to a chord c, then we can choose a subsequence so that the corresponding cubic symmetric laminations converge too. Their limit lamination has c as its comajor and by the first claim of the theorem, a cubic symmetric lamination. This proves the theorem. □

The next definition is similar to Thurston's definition of QML.

Definition 5.1.5. The set of all chords in $\mathbb{D}$ which are comajors of some cubic symmetric lamination is a lamination called the *Cubic symmetric Comajor Lamination*, denoted by C_sCL.

Note that C_sCL satisfies symmetric property (D3) as all comajors come in symmetric pairs.

5.2. Properties of C_sCL

5.2.1. C_sCL is a q-lamination. We will first prove that *Cubic symmetric Comajor Lamination C_sCL is a q-lamination.* In order to do that we need to prove a few more things about comajor leaves. By Corollary 5.1.2, all non-degenerate comajor leaves are non-periodic. Hence, there are three kinds of non-degenerate comajor leaves: *preperiodic of preperiod 1, preperiodic of preperiod bigger than 1 and not eventually periodic*. We will talk about each case separately.

LEMMA 5.2.1. *Comajor leaves of preperiod 1 are disjoint from all other comajors in C_sCL.*

PROOF. By Theorem 5.1.7, as comajors do not cross, the only way they can intersect is if they share an endpoint. And by Lemma 5.1.3, if two comajors share an endpoint, they have the same preperiod and period. Thus, a comajor leaf of preperiod 1 can only share an endpoint with a comajor of the same kind. By the way of contradiction, let us assume that there exists two distinct pairs of comajors $\{c, -c\}$ and $\{c', -c'\}$ of preperiod 1 and period k that share an endpoint in C_sCL. By Proposition 5.1.5, let $\mathcal{L}(c)$ and $\mathcal{L}(c')$ be the pull-back laminations of the pairs of comajors $\{c, -c\}$ and $\{c', -c'\}$ respectively. As comajors are of preperiod 1, the minor leaves m and m' of $\mathcal{L}(c)$ and $\mathcal{L}(c')$ respectively, are periodic of period k and share an endpoint, too. We observe two cases here.

(a) If the leaves m and m' belong to the same periodic orbit, then consider the leaf (or a pair of symmetric leaves) on their orbit closest to a critical chord of $\mathbb{S}$. This particular leaf has to be the major leaf (or a pair of symmetric major leaves) of the lamination and it has to be the same for both $\mathcal{L}(c)$ and $\mathcal{L}(c')$. So there is a clear contradiction that $\mathcal{L}(c)$ and $\mathcal{L}(c')$ are two distinct cubic symmetric laminations having the same set of majors but different comajors.

(b) If the leaves m and m' do not belong to the same periodic orbit, we have two possibilities here.

First possibility is when all the three endpoints of m and m' are in three disjoint orbits. Consider a triangle T formed by the three endpoints of m and

m' and the map $g = \sigma_3^k$. As the map g fixes all the three endpoints of m and m', we have $g(T) = T$. Also, the images of T under smaller iterates of the σ_3 map compared to the map g are disjoint from T, i.e $\sigma_3^i(T) \cap T = \emptyset \; \forall 1 \leq i < k$. It follows that we get a fixed return triangle T. Similarly we get the triangle $-T$. The presence of the pair of symmetric fixed return triangles $\{T, -T\}$ contradicts Proposition 3.3.5.

The other possibility is that the three endpoints of m and m' form two disjoint orbits with the common endpoint in a different orbit than the orbit containing the other two endpoints. We have two cases here.

(i) *If c and c' are $\succ$ comparable.* Without loss of generality, let us assume that $c' \succ c$. Then by Lemma 5.1.6, $c' \in \mathcal{L}(c)$. It follows that the leaf m' is also in $\mathcal{L}(c)$. We know that the leaves m and m' are periodic of period k and the three endpoints of m and m' form two disjoint orbits. It follows that the leaves m and m' form a 2-transitive periodic polygon in $\mathcal{L}(c)$. By Lemma 3.3.7 part(b), the leaf m eventually maps to the leaf $-m'$ and the leaf $-m$ eventually maps to m'. In other words, the minor leaves $\{m, -m\}$ of $\mathcal{L}(c)$ eventually map into the minor leaves $\{-m', m'\}$ of $\mathcal{L}(c')$ respectively and vice-versa. We can now conclude that all the four leaves $\{m, m', -m, -m'\}$ are present in both the laminations $\mathcal{L}(c)$ and $\mathcal{L}(c')$. Using a similar argument as case (a), there is a clear contradiction that $\mathcal{L}(c)$ and $\mathcal{L}(c')$ are two distinct cubic symmetric laminations having the same set of majors but different minors.

(ii) *If c and c' are not $\succ$ comparable.* Consider the pair of major leaves $\{M, -M\}$ and $\{M', -M'\}$ of the laminations $\mathcal{L}(c)$ and $\mathcal{L}(c')$ respectively. As comajors share an endpoint, major leaves also share an endpoint, too. Let us denote the endpoints of major leaves as follows: $\partial(M) =$

$\{a, p\}, \partial(M') = \{b, p\}$ and $\partial(-M) = \{a + \frac{1}{2}, p + \frac{1}{2}\}, \partial(-M') = \{b + \frac{1}{2}, p + \frac{1}{2}\}$.

It is not hard to see that if c and c' are not $\succ$ comparable, the major leaves M and M' are located in the open disk $\mathbb{D}$ in such a way that there is a critical chord ℓ of the circle $\mathbb{S}$ coming out of their common endpoint p 'separating' them, i.e M and M' are in two different components of $\mathbb{D} \setminus \ell$. In other words, one of the major leaves $\{M, M'\}$ has to be a long leaf and the other one has to be a medium leaf.

Because of the presence of the critical chord ℓ between M and M', the circular order of their endpoints gets reversed in their images, i.e if the endpoints of M and M' are arranged around the circle $\mathbb{S}$ in the anticlockwise direction as $\{a \to b \to p\}$ (from left to right), then their images in the anticlockwise direction of $\mathbb{S}$ will be $\{\sigma_3(b) \to \sigma_3(a) \to \sigma_3(p)\}$ (from left to right).

We will arrive at the contradiction just by following the circular order of the endpoints $\{a, b, p\}$ in the periodic orbit of the leaves M and M'. Consider the endpoints $\{\sigma_3^i(a), \sigma_3^i(b), \sigma_3^i(p)\}$ of the leaves $\sigma_3^i(M)$ and $\sigma_3^i(M')$ in the orbit. The circular order of the points $\{\sigma_3^i(a), \sigma_3^i(b), \sigma_3^i(p)\}$ can be reversed in their images only if there is a critical chord of $\mathbb{S}$ 'separating' the leaves $\sigma_3^i(M)$ and $\sigma_3^i(M')$. And clearly, the above situation happens only when the leaves $\sigma_3^i(M)$ and $\sigma_3^i(M')$ are major leaves themselves. Also, because the major leaves M and M' are periodic, they have to reverse the circular order of their endpoints at least once more in the above mentioned orbit in order to get back to themselves. From the earlier argument, the reversal of the circular order can happen at the most two times.

It follows that the major leaves M and M' have to eventually map to the major leaves $-M$ and $-M'$ before getting back to themselves. The circular order of the leaves M and M' is $\{a \to b \to p\}$ followed by their images $\{\sigma_3(b) \to \sigma_3(a) \to \sigma_3(p)\}$. The circular order of the endpoints of the leaves $-M$ and $-M'$ is $\{a+\frac{1}{2} \to b+\frac{1}{2} \to p+\frac{1}{2}\}$ which if represented as the images of the endpoints of the leaves M and M' will be $\{\sigma_3^{\frac{k}{2}}(b) \to \sigma_3^{\frac{k}{2}}(a) \to \sigma_3(p)\}$ (same as $\{\sigma_3(b) \to \sigma_3(a) \to \sigma_3(p)\}$). Thus, $\sigma_3^{\frac{k}{2}}(b) = a + \frac{1}{2}$ and $\sigma_3^{\frac{k}{2}}(a) = b + \frac{1}{2}$. It follows that the leaf M eventually maps to the leaf $-M'$ and the leaf M' eventually maps to the leaf $-M$. Finally, using the same argument as case (i), we get a similar contradiction that $\mathcal{L}(c)$ and $\mathcal{L}(c')$ are two distinct cubic symmetric laminations having the same set of majors but different comajors.

□

Now, to study *not eventually periodic* comajors, we need to explore the conditions when a leaf of the pull-back lamination $\mathcal{L}(c)$ could be comajor of a cubic symmetric lamination. Let us call a leaf of a lamination a *stand-alone* leaf if it is not a part of any gap. For instance, *not eventually periodic* comajors are stand-alone leaves. There could be periodic and preperiodic stand-alone leaves of a lamination.

We will prove an important result that if a comajor c is a stand-alone leaf in a cubic symmetric lamination $\mathcal{L}(c)$, then it is a stand-alone leaf in the *Cubic symmetric Comajor Lamination* C_sCL, too. The following two lemmas describe situations when a comajor leaf c in C_sCL is approximated by other comajor leaves in C_sCL.

LEMMA 5.2.2. *Let $c \in C_sCL$ be a non-degenerate comajor. If there is a sequence of leaves $c_i \in \mathcal{L}(c)$ with $c \succ c_i$ and $c_i \to c$, then $\exists N \in \mathbb{N}$ such that $\forall n > N, c_n \in C_sCL$.*

PROOF. Consider a leaf ℓ in the sequence of leaves $c_i \in \mathcal{L}(c)$ such that $\|\ell\| > \frac{\|c\|}{3}$. We will show that leaves $\{\ell, -\ell\}$ satisfy the three conditions of Proposition 5.1.5 to be a pair of comajors of a cubic symmetric lamination.

(a) Clearly all forward images of ℓ and $-\ell$ have disjoint interiors as they are leaves of cubic symmetric lamination $\mathcal{L}(c)$.

(b) As c is a short leaf, ℓ is a short leaf too. It follows that $\|\sigma_3(\ell)\| = 3\|\ell\|$. And as leaf ℓ is shorter than the comajors c and $-c$, $\sigma_3(\ell)$ is shorter than the minor leaves of $\mathcal{L}(c)$. We can use Lemma 5.1.1, to deduce that no forward image of $\sigma_3(\ell)$ is shorter than $\sigma_3(\ell)$. Thus, no forward image of ℓ has length smaller than $3\|\ell\|$.

(c) Consider the long and medium sibling leaves M_ℓ and $\hat{M}_\ell$ of ℓ of length at least $\frac{1}{4}$. These leaves cannot be a part of a cubic symmetric lamination $\mathcal{L}(c)$ because they are closer to the critical chord of $\mathbb{S}$ than the majors $\{M, .-M\}$ of $\mathcal{L}(c)$. Also note that M_ℓ and $\hat{M}_\ell$ would have to stay inside the short strips $C(M) \cup -C(M)$.

Now as all forward images of ℓ have length at least $3\|\ell\| > \|c\|$, they cannot intersect the short strips $C(M) \cup -C(M)$ which are $\|c\|$ wide. Thus, the leaves on the forward orbit of ℓ cannot intersect the interior of sibling leaves M_ℓ and $\hat{M}_\ell$ either.

Thus, there exists a cubic symmetric lamination $\mathcal{L}(\ell)$ with the pair of comajors $\{\ell, -\ell\}$. In other words $\ell \in C_sCL$.

Now we consider all those leaves in the sequence of leaves $c_i \in \mathcal{L}(c)$ having length at least $\frac{\|c\|}{3}$ and conclude the lemma. □

The situation when a comajor leaf is approximated from the other side is slightly trickier.

The following lemma is a simplification of Proposition 5.1.5 under certain conditions.

LEMMA 5.2.3. *Let $\mathcal{L}$ be a cubic symmetric lamination with $\{c, -c\}$ as a pair of comajors. If there exists a short leaf $\ell_s \in \mathcal{L}$ satisfying the conditions below.*

(i) $\ell_s \succ c$,

(i) the leaf $\ell_m = \sigma_3(\ell_s)$ does not ever map under itself, i.e $\ell_m \not\succ \sigma_3^k(\ell_m)$ for all k and

(ii) the leaf $\ell_m = \sigma_3(\ell_s)$ does not ever map under the leaf $-\ell_m$ i.e $-\ell_m \not\succ \sigma_3^k(\ell_m)$ for all k.

Then, there exists a cubic symmetric lamination $\mathcal{L}(\ell_s)$ with $\{\ell_s, -\ell_s\}$ as a pair of comajors.

PROOF. (a) All forward images of ℓ_s and $-\ell_s$ are in $\mathcal{L}$ and hence disjoint.

(b) Consider the siblings of the short leaf ℓ_s in the cubic symmetric lamination $\mathcal{L}$. The siblings either will be both short or one long and one medium leaf. Since $\ell_s \succ c$, a short sibling leaf of ℓ_s would intersect the major leaves of $\mathcal{L}$. Thus, the sibling leaves of ℓ_s are long and medium. It follows that all leaves on the forward orbit of ℓ_s are disjoint from the interiors of the long and medium sibling leaves of ℓ_s.

(c) Let us assume that a forward image $\sigma_3^k(\ell_s)$ of ℓ_s has length smaller than 3∗length of ℓ_s. It implies that $\sigma_3^{k-1}(\ell_s)$ is closer to a critical chord of $\mathbb{S}$ than the long and medium sibling leaves of ℓ_s. It follows that the leaf $\sigma_3^k(\ell_s)$ is under the leaf $\ell_m = \sigma_3(\ell_s)$ or the leaf $-\ell_m = \sigma_3(-\ell_s)$ contradicting the conditions given in the lemma.

Thus, by Proposition 5.1.5, we can conclude that there exists a cubic symmetric lamination $\mathcal{L}(\ell_s)$ with $\{\ell_s, -\ell_s\}$ as a pair of comajors. $\square$

Definition 5.2.1. Let ℓ be a leaf of a cubic symmetric lamination $\mathcal{L}$ such that $\sigma_3^k(\ell) \neq \ell$ for some k. Then we have the following two possibilities:
If the leaf $\sigma_3^k(\ell)$ is under ℓ, then we say that the leaf ℓ *moves in* under the map σ_3^k and if $\sigma_3^k(\ell)$ is not under ℓ, then we say that the leaf ℓ *moves out* under the map σ_3^k.

Let $\ell' \neq \ell$ be a leaf of $\mathcal{L}$ such that $\ell' \succ \ell$ and $\sigma_3^k(\ell') \neq \ell'$ for the same k as before. If both the leaves ℓ and ℓ' move in or both move out under the map σ_3^k, then we say that the leaves are *moving in the same direction.* And if one among the pair of leaves $\{\ell, \ell'\}$ moves in and the other leaf moves out, then we say that the leaves are *moving in opposite directions.* Following are the two possible ways in which leaves ℓ and ℓ' can move in opposite directions.

(a) If the leaf ℓ moves out and the leaf ℓ' moves in, then we say that the leaves ℓ and ℓ' *move towards each other.*

(a) If ℓ moves in and ℓ' moves out, then we say that the leaves ℓ and ℓ' *move away from each other.*

We will show that if two leaves of a cubic symmetric lamination move in opposite directions under an iterate of the σ_3 map (with a few extra conditions), then there exists an invariant leaf separating them. Idea is that there exists a leaf in between where the direction of the movement switches.
In the following lemma we will consider the case when two leaves move towards each other. Recall the definition of strip between the disjoint chords ℓ and ℓ' from chapter 3 (before Lemma 3.3.2). The closed subset of $\overline{\mathbb{D}}$ bounded by both the leaves ℓ and ℓ' is denoted by $S_b(\ell, \ell')$, i.e $S_b(\ell, \ell') = R(\ell) \cap R(\ell')$. The strip $S_b(\ell, \ell')$ is bounded by

the leaves ℓ and ℓ' and two arcs of $\mathbb{S}$. We will define the width of the strip $S_b(\ell, \ell')$ to be the length of the shorter arc among the two arcs bounding the strip.

LEMMA 5.2.4. *Let $\ell' \neq \ell$ be two non-periodic leaves in a cubic symmetric lamination $\mathcal{L}$ such that $\ell' \succ \ell$. If the following conditions are true,*

(i) $\mathcal{L}$ has a non-degenerate pair of comajor leaves $\{c, -c\}$,
(ii) The leaves ℓ and ℓ' move towards each other under the map σ_3^k,
(iii) Neither the leaves ℓ and ℓ' nor a leaf separating them can map into the arcs of the strip $S_b(\ell, \ell')$.

Then, there exists a periodic leaf y (under the map σ_3) that separates ℓ and ℓ'.

PROOF. Consider the family of leaves $\mathcal{C}$ in $\mathcal{L}$ that separates ℓ and ℓ'. Let the leaves ℓ and ℓ' be also included in $\mathcal{C}$. We claim that the $\mathcal{C}$ has at least one leaf that separates ℓ and ℓ' and $\mathcal{C}$ is a closed subset of $\overline{D}$. If there is no leaf in $\mathcal{C}$ that separates ℓ and ℓ', then there exists a gap G of $\mathcal{L}$ with ℓ and ℓ' on its boundary. It is given in the lemma that the leaves ℓ and ℓ' cannot map into the arcs of the strip $S_b(\ell, \ell')$. It follows that $\sigma_3^k(\ell) = \ell'$ and $\sigma_3^k(\ell') = \ell$, otherwise the gap $\sigma_3^k(G)$ would strictly cover the gap G, which is a contradiction. Now, it is clear that $\sigma_3^{2k}(\ell) = \ell$ and $\sigma_3^{2k}(\ell') = \ell'$, a contradiction with the fact that the leaves ℓ and ℓ' are non-periodic. Thus, $\mathcal{C}$ has at least one leaf that separates ℓ and ℓ'.

It is not hard to see that $\mathcal{C}$ is closed, since if there is a sequence of leaves $\{\ell_i\}_{i=1}^{\infty}$ in $\mathcal{C}$, then the limit leaf ℓ_∞ is either ℓ or ℓ' or a leaf that separates the leaves ℓ and ℓ' in $\mathcal{L}$.

Let us consider a subset A of leaves in $\mathcal{C}$ as follows. Every leaf m in the set A moves out under the map σ_3^k. And for every leaf $m \in A$, if a leaf n separates the leaves ℓ and m, then n also moves out under the map σ_3^k. In other words, all the leaves in A move in the same direction as the leaf ℓ.

There are clearly two leaves in $\mathrm{Bd}(A)$, one is leaf ℓ and the other, say a leaf y ($y \succ \ell$). By continuity, either the leaf y moves out like the leaves of A or it is invariant. If it is the latter, we are done.

Let us say y moves out under the map σ_3^k. We claim that y should be part of a gap G in $\mathcal{L}$ such that y is the smallest side of G in the sense of $\succ$. If there were a sequence of leaves t_i in $\mathcal{C}$ approaching y such that $t_i \succ y$, all of them would move in (if they are chosen close enough to the leaf y) as opposed to the leaf y which is moving out. That is because none of the leaves t_i are in A or in $\mathrm{Bd}(A)$. By continuity, the limit leaf $t = y$ of the leaves t_i cannot move out as y does, a contradiction. Thus, there exists a gap G with the sides y and t in the collection $\mathcal{C}$ such that $t \succ y$.

If the leaf t is invariant, we are done. Let us assume t is not invariant, it follows that t has to move in. Now, like the argument in the first paragraph as the leaves y and t cannot map into the arcs of the strip $S_b(\ell, \ell')$, we can argue that $\sigma_3^k(y) = t$ and $\sigma_3^k(t) = y$. It follows that $\sigma_3^{2k}(y) = y$ and $\sigma_3^{2k}(t) = t$ and we get our desired periodic leaves.

One last thing to consider about the periodic leaf y we got is that it can be an eventually collapsing leaf and map to one of its endpoints. In a sense, y doesn't move in or move out and is still invariant. However, the above case can happen only if y eventually maps to a critical leaf. And by the first condition in the lemma that $\mathcal{L}$ has a non-degenerate pair of comajor leaves $\{c, -c\}$, it follows that $\mathcal{L}$ has no critical leaves in it. Thus, y cannot be (pre)critical and y is indeed a periodic leaf. □

LEMMA 5.2.5. *Let $c \in C_sCL$ be a non-degenerate comajor such that c is either not eventually periodic or preperiodic of preperiod bigger than 1. If there is a sequence of leaves $c_i \in \mathcal{L}(c)$ with $c_i \succ c$ and $c_i \to c$, then c is the limit of preperiodic comajors c'_j of preperiod 1 with $c'_j \succ c$ for all j.*

Proof. Let $\{m, -m\}$ and $\{M, -M\}$ be the pairs of minors and majors of $\mathcal{L}(c)$ respectively. As c is either not eventually periodic or preperiodic of preperiod bigger than 1, we have that minor leaves m and $-m$ are not periodic. Consider the sequence of leaves $m_i \in \mathcal{L}(c)$ such that $m_i = \sigma_3(c_i)$. By continuity, we get that $m_i \to m$ and also $m_i \succ m$ for all i. We know that pre-images of major leaves are dense in a pull-back lamination. In particular, we can choose m_i's such that all of them are pre-images of the major leaf M. Hence, there exists a sequence of numbers $k_i \to \infty$ such that $\sigma_3^{k_j}(m_j) = m$ for all j. Because no forward image of m can be shorter than m, it is clear that $\sigma_3^{k_j}(m)$ cannot be under m. Also, since m is not periodic, $\sigma_3^{k_j}(m) \neq m$ for all j. Thus, we conclude that $\sigma_3^{k_j}$ maps m_j and m towards each other.

Also we can choose the leaves in the sequence $\{m_j\}_{j=1}^{\infty}$, close enough to m such that the width of the strip $S_b(m, m_j)$ is less than $\|m\|$ for all j. We claim that neither the leaf m nor the leaf m_j maps into the arcs of the strip $S_b(m, m_j)$. Clearly, by Lemma 5.1.1(i) as m is the shortest leaf in its orbit, it cannot map into the arcs of the strip $S_b(m, m_j)$. And by Lemma 5.1.1(iii), no forward image of m_j can be smaller than m. It follows that m_j cannot map into the arcs of the strip $S_b(m, m_j)$ either.

Clearly m_j is not a periodic leaf. All the conditions needed for the previous lemma are met. By Lemma 5.2.4, there exists a periodic leaf y_j that separates m and m_j. Let us choose the shortest leaf y'_j in the orbit of y_j. Consider the short leaf $c'_j = \sigma_3^{-1}(y'_j)$. Clearly as y'_j is the shortest leaf in its orbit, it can neither map under itself nor under the leaf $-y'_j$.

By Lemma 5.2.3, c'_j is a comajor. Thus, we obtain a sequence of preperiodic comajors of preperiod 1, $\{c'_j\}_{j=1}^{\infty}$ converging to c such that $c'_j \succ c$ for all j. $\square$

Corollary 5.2.6. *Every not eventually periodic comajor c is a stand-alone leaf in the Cubic symmetric Comajor Lamination C_sCL. The leaf c can be approximated by both sides by a sequence of comajors in C_sCL. None of the leaves in C_sCL share an endpoint with c, in particular aforementioned approximating leaves.*

Proof. We know that by Lemma 5.1.3, every *not eventually periodic* comajor c is a stand-alone leaf in $\mathcal{L}(c)$ approximated by other leaves of $\mathcal{L}(c)$ not sharing an endpoint with c. It also follows that no leaf of $\mathcal{L}(c)$ shares an endpoint with c. Now using Lemmas 5.2.2 and 5.2.5, we can conclude that c can be approached from both sides by a sequence of comajors in C_sCL too. The approximating leaves in C_sCL also do not share an endpoint with c because they are all part of the original lamination $\mathcal{L}(c)$. Thus, we can conclude that none of the leaves in C_sCL share an endpoint with c. □

Now, the final kind of comajors are *preperiodic of preperiod bigger than 1.* We claim that they are either stand-alone leaves and behave exactly like *not eventually periodic* ones or they are a side of a finite gap in C_sCL with leaves approaching from all sides.

Lemma 5.2.7. *Every preperiodic comajor c of preperiod bigger than 1 is a side of a finite gap or a stand-alone leaf in C_sCL. In the case we have a gap, the sides of it are approximated by a sequence of comajors in C_sCL that do not share an endpoint with c.*

Proof. Consider the cubic symmetric lamination $\mathcal{L}(c)$. First, we claim that the critical gaps G and $-G$ of $\mathcal{L}(c)$ have to be finite gaps. They cannot be Fatou gaps because in that case comajors c and $-c$ will be of preperiod equal to 1. They cannot be Siegel gaps either because c is non-degenerate. Thus, G and $-G$ are finite. Note that the entire lamination $\mathcal{L}(c)$ contains only finite gaps.

(i) *If G and $-G$ have at least six sides each.* Then, the corresponding sibling gaps $\hat{G}$ and $-\hat{G}$ containing $c, -c$ have at least 3 sides each. We shall prove that every side of $\hat{G}$ (and $-\hat{G}$) is a comajor of a cubic symmetric lamination. Consider $\ell \in \hat{G}$. There are two sibling leaves to ℓ in G. With these two sibling leaves, let us form a quadrilateral inside G by adding two long leaves. In other words, we subdivide G by adding a collapsing quadrilateral. We do a similar construction for $-G$. Now, by adding all pre-images of these new leaves inside the pre-images of G, we obtain a new cubic symmetric lamination with ℓ and $-\ell$ as comajors. We can do the similar process for all the n sides of $\hat{G}$ by adding a collapsing quadrilateral in G in n ways. Observe that one of these subdivisions has the same comajor leaves as $\mathcal{L}$.

Now, we claim that the leaves of $\hat{G}$ (and $-\hat{G}$) indeed form a gap of C_sCL since by Corollary 3.3.8, no diagonal of the polygon $\hat{G}$ can be a leaf (let alone comajor!) of a cubic symmetric lamination.

For the second part of the theorem, we notice that all the sides of $\hat{G}$ (and $-\hat{G}$) are non-isolated and have to be approached by leaves of $\mathcal{L}(c)$. That is because if there was another finite gap attached to the sides of $\hat{G}$, we would contradict Corollaries 3.3.8 and 3.3.9. Thus using Lemmas 5.2.2 and 5.2.5, all the sides of $\hat{G}$ (and $-\hat{G}$) are approximated by a sequence of leaves in C_sCL, too. Finally, we claim that none of these approximating comajors share an endpoint with sides of $\hat{G}$ (and $-\hat{G}$). If they did, they would all have the same preperiod and same period by Lemma 5.1.3. Now, consider the situation when $\hat{G}$ becomes periodic and shares an endpoint with an approximating leaf (say ℓ). We can make a fixed return triangle with ℓ and the side of the periodic gap it is sharing an endpoint with, contradicting Proposition 3.3.5.

(ii) *If G and $-G$ have four sides each.* Then we observe two scenarios.

  (a) *If G and $-G$ are isolated.* It would mean that there are finite gaps attached to G and $-G$. Consider the biggest finite gaps containing G and $-G$ in $\mathcal{L}(c)$. The bigger sides of G and $-G$ (majors of $\mathcal{L}(c)$) are inside this larger finite gaps and by deleting them and their pre-images in $\mathcal{L}(c)$, we get a new cubic symmetric lamination $\mathcal{L}'$ with central gaps now containing more than 6 sides each. We return to case (i).

  (b) *If G and $-G$ are not isolated.* In this case, the comajor leaf c would remain non-isolated too in $\mathcal{L}(c)$. Now, using the Lemmas 5.2.2, 5.2.5 and Proposition 3.3.5 again as it is done in case (i), we conclude that c is a stand-alone leaf and is approximated by a sequence of comajors in C_sCL that do not share an endpoint with c.

□

Finally the main theorem of the section is as follows.

THEOREM 5.2.8. Cubic symmetric Comajor Lamination *C_sCL is a q-lamination.*

PROOF. By Lemma 5.2.1, Corollary 5.2.6 and Lemma 5.2.7, it is clear that no more than two comajors meet at a single point in C_sCL. Hence, C_sCL is a q-lamination. □

5.2.2. Cubic symmetric main cardioid $\mathcal{C}_s\mathcal{MC}$. This subsection is devoted to understanding the properties of a special gap of the *Cubic symmetric Comajor Lamination C_sCL* called as *Cubic symmetric main cardioid $\mathcal{C}_s\mathcal{MC}$*. It is understood to represent the set of the first group of laminations called *canonical laminations* in the family of cubic symmetric laminations. Every other gap G of C_sCL will be under this gap $\mathcal{C}_s\mathcal{MC}$. That is why they are considered "first" in the sense of $\succ$.

Let $\mathcal{L}$ be a cubic symmetric lamination with a non-degenerate comajor c. There can be many possible cubic symmetric laminations with c as a comajor leaf. *Pull-back lamination* $\mathcal{L}(c)$ as discussed in the previous section is one among those laminations. Let us denote the union of the symmetric critical gaps in the lamination $\mathcal{L}$ by $C(\mathcal{L})$. As discussed in Lemma 5.1.3, $C(\mathcal{L})$ is either a pair of Fatou gaps (with or without a symmetric orbit) or a pair of finite gaps, each of them with at least 6 sides, or a pair of collapsing quadrilaterals. The following lemma is a different way to approach the result obtained in Lemma 5.2.7.

LEMMA 5.2.9. *If $C(\mathcal{L})$ is a pair of symmetric finite gaps each of them having at least 6 sides, then the collection $\widehat{C(\mathcal{L})}$ containing the sibling leaves to the gaps of $C(\mathcal{L})$ is a pair of symmetric gaps of C_sCL.*

PROOF. Consider a critical gap G of $C(\mathcal{L})$. The gap G is a polygon of $2n$ sides, where $n \geq 3$. The corresponding sibling gap $\hat{G}$ of $\widehat{C(\mathcal{L})}$ has n sides.

First, we shall prove that every side of $\hat{G}$ (and $-\hat{G}$) is a comajor of a cubic symmetric lamination. Consider $\ell \in \hat{G}$. There are two sibling leaves to ℓ in G. With these two sibling leaves, let us form a quadrilateral inside G by adding two long leaves. In other words, G has been subdivided by adding a collapsing quadrilateral. We can do similar construction for $-G$. Now, by adding all pre-images of these new leaves inside the pre-images of G, we obtain a new cubic symmetric lamination with ℓ and $-\ell$ as comajors. We can do the similar process for all the n sides of $\hat{G}$ by adding a collapsing quadrilateral in G in n ways. Observe that one of these subdivisions has the same comajor leaves as $\mathcal{L}$.

Finally, we claim that the leaves of $\hat{G}$ (and $-\hat{G}$) indeed form a gap of C_sCL. By

Corollary 3.3.8, no diagonal of the polygon $\hat{G}$ can be a leaf(let alone comajor!) of a cubic symmetric lamination. Thus, the gaps $\hat{G}$ (and $-\hat{G}$) cannot be further subdivided in C_sCL. It follows that they are gaps of C_sCL. $\square$

The case when $C(\mathcal{L})$ is a pair of symmetric Fatou gaps (with or without symmetric orbit) is a little trickier as a Fatou gap G can be subdivided into smaller Fatou gaps using the help of the map ψ_G in Theorem 3.3.15 in many different ways. The process is complicated and will yield infinitely many cubic symmetric laminations with distinct Fatou gaps. We will instead focus on studying a subset of cubic symmetric laminations with a specific kind of symmetric Fatou gaps.

The following lemma helps us in understanding a subset of the cubic symmetric laminations in relation to the pair of symmetric critical chords $\{\ell, -\ell\}$ on circle $\mathbb{S}$. For the chords $\{\ell, -\ell\}$, let us denote $\mathbb{S} \setminus (H(\ell) \cup H(-\ell))$ as $S(\ell)$. Note that $S(\ell)$ is composed of two disjoint symmetric arcs in the circle $\mathbb{S}$.

LEMMA 5.2.10. *Let $\{\ell, -\ell\}$ be a pair of symmetric critical chords. Then there exists a unique symmetric rotational set $I(\ell) \subset \overline{S(\ell)}$. And there exists a cubic symmetric lamination $\mathcal{L}_\ell$ which contains the convex hull $G(\ell)$ of $I(\ell)$ as its central symmetric gap.*

PROOF. We will consider two different cases here when critical chords $\{\ell, -\ell\}$ have periodic endpoints or not.

(a) *The critical chords ℓ and $-\ell$ do not have periodic endpoints.* Consider the set of points $I(\ell)$ that stay forever in $\overline{S(\ell)}$, i.e $I(\ell) = \{p \in \overline{S(\ell)} \,|\sigma_3^k(p) \in \overline{S(\ell)}\ \forall k \in \mathbb{N}\}$. $I(\ell)$ is clearly a forward invariant set because for every point p that stays forever in $\overline{S(\ell)}$, i.e $p \in I(\ell)$, its image $\sigma_3(p)$ also stays forever in $\overline{S(\ell)}$, i.e $\sigma_3(p) \in I(\ell)$. The set $I(\ell)$ is backward invariant too. Since $\sigma_3(\partial(\overline{S(\ell)})) = \mathbb{S}$, it follows that for every $p \in I(\ell)$, $\sigma_3^{-1}(p)$ is in $\overline{S(\ell)}$, and thereby in $I(\ell)$. By Proposition 4.1.1, $I(\ell)$ is rotational under the map σ_3 since $\mathbb{S} \setminus I(\ell)$ contains two disjoint intervals $H(\ell)$ and $H(-\ell)$ of length $\frac{1}{3}$ each. By Corollary 4.1.3, the set $I(\ell)$ has a well-defined rotation number $\rho(\ell)$. If $I(\ell)$ is finite, $\rho(\ell)$ is rational and if $I(\ell)$ is infinite, $\rho(\ell)$ is irrational. Also we claim that $I(\ell)$ is symmetric with respect to the center of the circle i.e $\forall p \in I(\ell)$, $-p$ is also in $I(\ell)$. Since orbits of p and $-p$ under the σ_3 map are always symmetric with respect to the center of the circle $\mathbb{S}$, it implies that if orbit of p stays forever in $\overline{S(\ell)}$, then orbit of $-p$ has to stay in a region symmetric to $\overline{S(\ell)}$ which is $\overline{S(\ell)}$ itself. Thus, the point $-p$ is also in the set $I(\ell)$.

We have three possibilities for the set $I(\ell)$. If $I(\ell)$ is finite, we can use the argument used in Corollary 3.3.6 and deduce that $I(\ell)$ can contain at most two periodic orbits. First possibility is $I(\ell)$ is made of one single periodic orbit. The second possibility is that when $I(\ell)$ contains two periodic orbits, both the orbits are symmetric with respect to the center of the circle and they combine to form a symmetric rotational set. The last case is when $I(\ell)$ is infinite, it contains two disjoint orbits. This case is similar to the Lemma 3.3.14 case (i) Siegel gaps. So, here too, both the disjoint orbits combine to form a symmetric rotational set. We can conclude that the set $I(\ell)$ is minimal in the sense that no proper subset of it can be a symmetric rotational set.

Finally, $I(\ell)$ is unique since if there was any other symmetric rotational subset $I'(\ell)$ of $\overline{S(\ell)}$, then just like the points of $I(\ell)$, the points of the set $I'(\ell)$ would stay forever in $\overline{S(\ell)}$. It follows that $I'(\ell)$ would have to be a proper subset of $I(\ell)$ contradicting the above argument.

Consider the convex hull $G(\ell)$ of $I(\ell)$. From the above discussion, $G(\ell)$ is an invariant rotational closed subset of $\overline{\mathbb{D}}$ and the sides of $G(\ell)$ form a part of one periodic orbit or two disjoint periodic orbits or two disjoint infinite orbits under the σ_3 map. We will try to build a *cubic symmetric lamination* $\mathcal{L}_\ell$ with $G(\ell)$ as its *central symmetric gap.*

Consider the longest edges M and $-M$ of $G(\ell)$. They are also closest to the critical leaves ℓ and $-\ell$. Short sibling leaves $\{c, -c\}$ to M and $-M$ can be constructed in $R(\ell) \cup -R(\ell)$. We claim that the leaves $\{c, -c\}$ satisfy all conditions of Proposition 5.1.5.

(i) Forward images of $\{c, -c\}$ being the same as images of the leaves M and $-M$ are part of the sides of $G(\ell)$. Thus, forward images have disjoint interiors.

(ii) Consider the leaves $m = \sigma_3(M) = \sigma_3(c)$ and $-m = \sigma_3(-M) = \sigma_3(-c)$ on $G(\ell)$. We claim that m and $-m$ are the shortest edges of $G(\ell)$. No forward image of leaves m and $-m$ can get shorter than themselves because for a leaf ℓ' to become shorter than m (and $-m$), its pre-image $\sigma_3^{-1}(\ell')$ has to get closer to the critical leaves ℓ and $-\ell$ than M(and $-M$) which cannot be the case.

The leaves m and $-m$ have 3 times the length of leaves c and $-c$. Thus, the length of forward images of c(and $-c$) is never less than 3*length of c.

(iii) The leaves c and $-c$ are chosen to be short leaves among the part of full sibling collection $\{M, \widehat{M}\}$ and $\{-M, -\widehat{M}\}$ respectively. Clearly the leaves on the forward orbit of c (and $-c$) being sides of $G(\ell)$ are disjoint from the interior of the sibling leaves $\{M, \widehat{M}\}$(and $\{-M, \widehat{-M}\}$)

Thus, by Proposition 5.1.5, there exists a cubic symmetric lamination $\mathcal{L}(c)$. Finally, we claim that $\mathcal{L}(c)$ has $G(\ell)$ as its *central symmetric gap.* All sides of $G(\ell)$ are either in the forward or in the backward orbit of leaves c and $-c$. Thus, sides of $G(\ell)$ are the leaves of lamination $\mathcal{L}(c)$.

If $G(\ell)$ has finitely many sides, we use the argument in Corollary 3.3.8 to claim that no diagonal ℓ' of $G(\ell)$ can be a leaf of a cubic symmetric lamination. In other words $G(\ell)$ cannot be subdivided or $G(\ell)$ is the central symmetric gap of $\mathcal{L}(c)$. From the earlier discussion, it follows that $G(\ell)$ either can be a 1-transitive rotational gap or a 2-transitive rotational gap.

If the sides of $G(\ell)$ are a part of two disjoint infinite orbits, then critical chords ℓ and $-\ell$ have to be part of $G(\ell)$ (Lemma 3.3.12) and all other sides of $G(\ell)$ eventually map to them and collapse. In other words, $G(\ell)$ is the *Siegel gap*. Thus, $G(\ell)$ is the central symmetric gap of $\mathcal{L}(c)$.

$\mathcal{L}_\ell = \mathcal{L}(c)$ is the cubic symmetric lamination we referred to in the statement of the lemma.

(b) *The critical chords ℓ and $-\ell$ have periodic endpoints.*

We need to consider two sub cases here.

If all points on the periodic orbit of the critical chords ℓ and $-\ell$ do not stay in $\overline{S(\ell)}$, then we define the rotational set $I(\ell) \subset \overline{S(\ell)}$ in the same way as case (a) and proceed to get the corresponding cubic symmetric lamination $\mathcal{L}_\ell$.

In the case where all points on the periodic orbits of the critical chords ℓ and $-\ell$ stay in $\overline{S(\ell)}$, we need to discard some points in forming the rotational set $I(\ell) \subset \overline{S(\ell)}$. Among all points which stay forever in $\overline{S(\ell)}$, there will be preperiodic points mapping to the endpoints of the critical leaves $\{\ell, -\ell\}$, we discard them because they do not exhibit rotational behavior. Consider the set $I(\ell) = \{p \in \overline{S(\ell)}\ |\sigma_3^k(p) \in \overline{S(\ell)} \wedge \{p \textit{ is periodic}\},\ \forall k \in \mathbb{N}\}$. Once we get the rotational set $I(\ell)$, cubic symmetric lamination $\mathcal{L}_\ell$ is constructed in the same way as in case (a).

Note that in both subcases of case (b), the central symmetric gap $G(\ell)$ is finite. Every infinite (Siegel) gap has the critical leaves on their boundary and the endpoints of the critical leaves are a part of two disjoint infinite orbits. In case (b), the endpoints of the critical leaves ℓ and $-\ell$ are a part of periodic orbits.

□

Lemma 5.2.10 gives us a correspondence between the family of critical chords $\{\ell, -\ell\}$ and the cubic symmetric laminations $\mathcal{L}_\ell$. We call $\mathcal{L}_\ell$, the *canonical* lamination of critical chords $\{\ell, -\ell\}$. Let us denote the comajors of the canonical lamination $\mathcal{L}_\ell$ by $\{c(\ell), -c(\ell)\}$. The following corollary, however, shows that there is a whole interval of critical chords corresponding to the same canonical cubic symmetric lamination.

Corollary 5.2.11. *Consider a pair of symmetric critical chords $\{\ell, -\ell\}$ and its corresponding cubic symmetric lamination $\mathcal{L}_\ell$ with its majors $\{M(\ell), -M(\ell)\}$. For any pair of symmetric critical chords $\{\ell', -\ell'\}$ inside the short strips $C(M(\ell)) \cup -C(M(\ell))$, the corresponding canonical cubic symmetric lamination $\mathcal{L}_{\ell'}$ is the same as $\mathcal{L}_\ell$.*

Proof. If the leaves $\{\ell', -\ell'\}$ are inside the short strips $C(M(\ell)) \cup -C(M(\ell))$, it means the original symmetric rotational set $I(\ell)$ is a subset of $\overline{S(\ell')}$, too. By Lemma 5.2.10, the symmetric rotational sets are unique, which forces $I(\ell)$ to be the same as $I(\ell')$. Thus, both canonical cubic symmetric laminations $\mathcal{L}_{\ell'}$ and $\mathcal{L}_\ell$ are the same. □

Definition 5.2.2. Comajors of *canonical* cubic symmetric laminations are called *prime comajors*. If $G(\ell)$ is a finite gap then the corresponding comajors are said to be *rational*. If $G(\ell)$ is a Siegel gap then the corresponding degenerate comajors are said to be *irrational*.

Note that the *rational prime comajors* are non-degenerate and preperiodic of preperiod 1 whereas *irrational prime comajors* are degenerate and not eventually periodic. Let us denote the set of *prime comajors* in the *Cubic symmetric Comajor Lamination C_sCL*, *Cubic symmetric Main Cardioid $\mathcal{C}_s\mathcal{MC}$*. We shall prove that $\mathcal{C}_s\mathcal{MC}$ is a gap of C_sCL. The following lemma about the location of prime comajors is useful in doing that.

LEMMA 5.2.12. *The set of prime comajors $\mathcal{C}_s\mathcal{MC}$ forms the boundary of a closed convex set in $\overline{\mathbb{D}}$. Moreover, $\partial(\mathcal{C}_s\mathcal{MC})$ is a Cantor set.*

PROOF. First we claim that prime comajors are pairwise disjoint. By Theorem 5.1.7, the only way two comajors intersect is when they have a common endpoint. Clearly, rational prime comajors are disjoint with irrational prime comajors as one of them is preperiodic and the other one is not. Let us assume that two rational prime comajors c and c' have a common endpoint. Then, it follows that there will be a set of corresponding majors M and M' that are meeting at a common endpoint, say p, too. We can draw a pair of symmetric critical chords $\{\ell, -\ell\}$ coming out of p and $-p$ such that they are inside both short strips $C(M) \cup -C(M)$ and $C(M') \cup -C(M')$. By Corollary 5.2.11, the critical chords $\{\ell, -\ell\}$ should correspond to a unique canonical cubic symmetric lamination $\mathcal{L}_\ell$ with the pair of prime comajors $\{c(\ell), -c(\ell)\}$. It follows that $c = c' = c(\ell)$.

Using a similar argument, we will prove that prime comajors are pairwise non-comparable in the sense of $\succ$. Let us assume that there exists two prime comajors c and c' such that $c \succ c'$. Consider the corresponding majors M and M'. The short strips $C(M) \cup -C(M)$ will contain the short strips $C(M') \cup -C(M')$ in their interior. We can draw a pair of symmetric critical chords $\{\ell, -\ell\}$ inside the smaller short strips $C(M') \cup -C(M')$. Again by Corollary 5.2.11, the critical chords $\{\ell, -\ell\}$ should correspond to a unique canonical cubic symmetric lamination $\mathcal{L}_\ell$ with the pair of prime comajors $\{c(\ell), -c(\ell)\}$. It follows that $c = c' = c(\ell)$.

Let us show that every point $p \in \mathbb{S}$ is either an irrational prime comajor, or has a rational prime comajor c such that $p \in H(c)$. Consider the pair of symmetric critical chords $\{\ell, -\ell\}$ whose endpoints have the same images as that of points p and $-p$, i.e $\sigma_3(\ell) = \sigma_3(p)$ and $\sigma_3(-\ell) = \sigma_3(-p)$. Consider the corresponding canonical cubic symmetric lamination $\mathcal{L}_\ell$. If its central symmetric gap $G(\ell)$ is infinite, then the critical chords $\{\ell, -\ell\}$ are major leaves of $\mathcal{L}_\ell$ and the comajors $\{c(\ell), -c(\ell)\}$ are the same as points $\{p, -p\}$. If $G(\ell)$ is finite, the major leaves of $\mathcal{L}_\ell$ are disjoint from the critical chords $\{\ell, -\ell\}$ with the endpoints in $S(\ell)$. It follows that comajors $\{c(\ell), -c(\ell)\}$ are located in such a way that $p \in H(c)$ and $-p \in H(-c)$. This proves the fact that $\mathcal{C}_s\mathcal{MC}$ forms the boundary of a closed set in $\overline{\mathbb{D}}$. Since the prime comajors are both pairwise non-comparable and pairwise disjoint, we can now claim that they form the boundary of a closed convex set in $\overline{\mathbb{D}}$.

Finally, we claim that $\partial(\mathcal{C}_s\mathcal{MC})$, the union of irrational prime comajors and the endpoints of rational prime comajors is a Cantor set. If a sequence of prime comajors converges to a point $p \in \mathbb{S}$, then by the above paragraph, either p is an irrational prime comajor or there exists a rational prime comajor c such that $p \in H(c)$. The point p cannot be in the interior of $H(c)$ otherwise there would exist a prime comajor c' in the converging sequence close to p such that c' is also in the interior of $H(c)$ resulting in $c \succ c'$ contradicting the above claim that prime comajors are pairwise non-comparable in the sense of $\succ$. Thus, either p is an irrational prime comajor or p is an endpoint of a rational prime comajor c. It follows now that $\partial(\mathcal{C}_s\mathcal{MC})$ is closed. There are no isolated points in $\partial(\mathcal{C}_s\mathcal{MC})$ because it would mean two rational prime comajors meeting at a point. The set $\partial(\mathcal{C}_s\mathcal{MC})$ cannot contain sub-segments either. Thus, $\partial(\mathcal{C}_s\mathcal{MC})$ is a Cantor set. □

Theorem 5.2.13. *The set of prime comajors $\mathcal{C}_s\mathcal{MC}$ is a gap of C_sCL.*

Proof. To show that $\mathcal{C}_s\mathcal{MC}$ is a gap of C_sCL, it suffices to show that there are no comajors inside $\mathcal{C}_s\mathcal{MC}$. By the way of contradiction, let us suppose that there exists a comajor c inside $\mathcal{C}_s\mathcal{MC}$. Which means that there is a prime comajor smaller than c in the sense of $\succ$ and also as prime comajors are pairwise non-comparable in the sense of $\succ$, there does not exist any prime comajors bigger than c in the sense of $\succ$.

Let us denote the symmetric majors of the pull-back cubic symmetric lamination $\mathcal{L}(c)$ as M and $-M$. To avoid confusion, let us assume that when we talk about the major leaves, we talk about the long leaves (not medium) close to the center of the circle $\mathbb{S}$. Imagine a pair of symmetric critical chords $\{\ell, -\ell\}$ inside the short strips $C(M) \cup -C(M)$ such that ℓ (and $-\ell$) share an endpoint p (and $-p$) with the major leaf M (and $-M$) essentially splitting each short strip into two halves. Let the other endpoint of major leaf M not shared by the critical chord ℓ be q. By Lemma 5.2.10, there exists a unique canonical cubic symmetric lamination $\mathcal{L}_\ell$ with the pair of prime comajors $\{c(\ell), -c(\ell)\}$. Both the pairs of majors $\{M, -M\}$ of $\mathcal{L}(c)$ and majors $\{M(\ell), -M(\ell)\}$ of $\mathcal{L}_\ell$ lie in between the critical chords $\{\ell, -\ell\}$. And by Theorem 5.17, as comajors do not cross, it implies in this particular scenario that majors also do not cross because both sets of majors lie in between the same critical chords. It follows that either $M \succ M(\ell)$ or $M(\ell) \succ M$. If $M(\ell) \succ M$, it follows that $c(\ell) \succ c$ which contradicts the fact that there does not exist any prime comajors bigger than c in the sense of $\succ$ (see the first paragraph). Thus, $M \succ M(\ell)$ and the only way that happens is if the leaf $M(\ell)$ shares the endpoint p with M (and ℓ). Also, it is clear that $M(\ell)$ and M are two distinct leaves as c is not a prime comajor. And by Corollary 3.3.8, the leaf M cannot be a diagonal in the central symmetric gap $G(\ell)$.

In other words, the other endpoint q of the leaf M is not a vertex of the gap $G(\ell)$. We can now conclude that all forward images of the endpoint p stay forever in $\overline{S(\ell)}$, but the forward images of the other endpoint q of M does not. That is, there exists a natural number N such that $\sigma_3^N(q) \notin \overline{S(\ell)}$. For this choice of N, we get the contradiction since the leaf $\sigma_3^N(M)$ crosses the leaf M. □

Lemma 5.2.10 tells us that all prime comajors of the *Cubic symmetric Main Cardioid* $\mathcal{C}_s\mathcal{MC}$ can be characterized by central symmetric gaps $G(\ell)$. And in Chapter 4, we have seen that every central symmetric gap has a unique rotation number. Thus, the first gap $\mathcal{C}_s\mathcal{MC}$ of the parameter space C_sCL can be characterized by rotation numbers of central symmetric gaps of canonical cubic symmetric laminations.

5.3. Constructing C_sCL

In the final section of our work, we provide an algorithm that generates a dense subset of C_sCL. We need to prove a few lemmas before setting up the algorithm. The following two lemmas give us more insight into the preperiodic leaves of C_sCL that have endpoints of preperiod 1.

LEMMA 5.3.1. *Every preperiodic point $p \in \mathbb{S}$ of preperiod 1 is an endpoint of a non-degenerate comajor c of a cubic symmetric lamination $\mathcal{L}$.*

PROOF. Consider the following things: a preperiodic point $p \in \mathbb{S}$ of preperiod 1 and period k, the cubic symmetric lamination $\mathcal{L}(p)$ with $\{p, -p\}$ as a pair of degenerate comajors and the critical leaves $\{\ell, -\ell\}$ whose images are same as the images of $\{p, -p\}$, respectively.
By modifying the lamination $\mathcal{L}(p)$, we obtain the required lamination $\mathcal{L}$ in the lemma. Note that as we have seen in Lemma 5.1.4 case (b), all the pre-images to the critical leaves $-\ell$ (and ℓ) are connected to each other in one-sided infinite chains in $\mathcal{L}(p)$ as

follows. The leaves adjacent to the critical leaf ℓ (and ℓ) in the chain are chosen to be a pair of short leaves $\{\ell_s, \hat{\ell}_s\}$ and all other leaves of the chain are uniquely determined by the inverse map σ_3^{-1}. Let us call the infinite chain of leaves attached to the critical leaf ℓ as $\mathcal{C}(\ell)$. Similarly we get $\mathcal{C}(-\ell)$.

Claim: The leaves in $\mathcal{C}(\ell)$ converge to a point $x \in \mathbb{S}$ which is not an endpoint of the critical leaf ℓ.

Proof. Clearly, the leaves in $\mathcal{C}(\ell)$ converge to a point $x \in \mathbb{S}$ as there are infinitely many of them forming a chain in a finite arc of $\mathbb{S}$. The interesting argument is that the leaves in $\mathcal{C}(\ell)$ do not go all the way back and converge to one of the endpoints of the critical leaf ℓ. By the way of contradiction, let us say that happens, i.e the leaves in $\mathcal{C}(\ell)$ converge to an endpoint of the critical leaf ℓ.

Since every cubic symmetric lamination has a central symmetric gap, $\mathcal{L}(p)$ has such a gap G, too. Note that this is same as the gap $G(\ell)$ discussed in the Lemma 5.2.9. The two sibling gaps $\widehat{G(\ell)}$ and $-\widehat{G(\ell)}$ are also present in $\mathcal{L}(p)$. Let us say the gap $\widehat{G(\ell)}$ lies in the region $R(\ell)$. As the leaves in the chain $\mathcal{C}(\ell)$ converge to an endpoint of the critical leaf ℓ, there should be a medium leaf ℓ_m in $\mathcal{C}$ bigger than all leaves of $\widehat{G(\ell)}$ in the sense of $\succ$. Same is the case with the chain $\mathcal{C}(-\ell)$ and the gap $\widehat{-G(\ell)}$, and we get a similar medium leaf $-\ell_m$.

We obtain the contradiction as follows:

Consider the short strips $C(\ell_m) \cup -C(\ell_m)$ bounded by the leaves $\{\ell_m, -\ell_m\}$ and their long sibling leaves. Note that the only other long leaves of $\mathcal{L}(p)$ inside the short strips $C(\ell_m) \cup -C(\ell_m)$ are critical leaves ℓ and $-\ell$.

The adjacent sides of the chains $\mathcal{C}(\ell) \cup -\mathcal{C}(\ell)$ behave as follows. Consider two adjacent leaves m and n in the chain $\mathcal{C}(\ell)$ with the leaf n more closer to the critical leaf ℓ than the leaf m. We know that the leaf m eventually maps to either the leaf n or $-n$

before it can map to any other leaves in the chains $\mathcal{C}(\ell) \cup -\mathcal{C}(\ell)$. It is the case for every pair of adjacent leaves in the chains $\mathcal{C}(\ell) \cup -\mathcal{C}(\ell)$.
It follows that the leaf ℓ_m eventually maps to one of the leaves among $\{\ell_s, -\ell_s\}$ before they map to the critical leaf ℓ or $-\ell$. In other words, the medium leaf ℓ_m is mapping inside its short strips $C(\ell_m) \cup -C(\ell_m)$ as short leaves before mapping as a long leaf contradicting Proposition 3.2.3.

Now, going ahead with the rest of the proof of the lemma, we observe that the point x obtained from the above process is periodic and it has the same period k as one of the endpoints of critical leaves. Now, we can construct our required lamination $\mathcal{L}$ as follows: Join the points x and the periodic endpoint of the critical leaf ℓ. By the above claim, x is closer to the non-periodic endpoint than the periodic endpoint, so we get a medium leaf M. We do the same thing with $-x$ and $-\ell$ to obtain the leaf $-M$. We can build a cubic symmetric lamination by taking the pre-images of the leaves M and $-M$ inside the gaps of the original lamination $\mathcal{L}(p)$. Finally, we obtain a new collection of leaves $\mathcal{L}$ by removing the critical leaves ℓ and $-\ell$ and all its pre-images from the above lamination. It is not hard to see that $\mathcal{L}$ is a cubic symmetric lamination with $\{M, -M\}$ as its periodic majors. The short siblings to $\{M, -M\}$ in $\mathcal{L}$ form a non-degenerate pair of preperiodic comajors of preperiod 1 $\{c, -c\}$. Also the leaves c and $-c$ share endpoints with the original points p and $-p$ respectively. $\square$

Definition 5.3.1. We say a gap G 'separates' two leaves ℓ_1 and ℓ_2 if ℓ_1 and ℓ_2 are in two different components of $\mathbb{S} \setminus G$. Similarly we say a leaf ℓ 'separates' two leaves ℓ_1 and ℓ_2 if ℓ_1 and ℓ_2 are in two different components of $\mathbb{S} \setminus \ell$

Lemma 5.3.2. *Preperiodic comajors of preperiod 1 are dense in C_sCL.*

Proof. Consider a non-degenerate comajor $c \in C_sCL$. We will assume that the leaf c is not a preperiodic comajors of preperiod 1. We have two cases here.

(a) *If there is a sequence of leaves $c_i \in \mathcal{L}(c)$ such that $c_i \succ c$ and $c_i \to c$.* Then, by Lemma 5.2.5, c is the limit of preperiodic comajors c_i' of preperiod 1 such that $c_i' \succ c$.

(b) If the leaf c is not the limit of leaves $c_i \in \mathcal{L}(c)$ such that $c_i \succ c$, then by Corollary 5.2.6, c cannot be an eventually periodic comajor. It follows that the leaf c is a preperiodic comajor of preperiod bigger than 1 and c is an edge of a finite preperiodic gap G of preperiod bigger than 1 in the cubic symmetric lamination $\mathcal{L}(c)$. Also note that there are no leaves under c in the gap G. We can now use Lemma 5.2.7 and show that c is approached by the leaves $c_i \in \mathcal{L}(c)$ such that $c \succ c_i$ for all i and $c_i \to c$. And by Lemma 5.2.2, all of the leaves c_i can be chosen to be comajors themselves. Thus, we get $c_i \in C_sCL$ all of which are 'under' c, i.e $c \succ c_i$ for all i.

Lemma 5.2.7 also tells us that each $c_i \in \mathcal{L}(c)$ does not share an endpoint with c. It follows that there are infinitely many leaves 'separating' each c_i and c in the lamination $\mathcal{L}(c)$. For a particular i, consider the sequence of leaves $\{c_{i(j)}\}_{j=1}^{\infty}$ such that $c \succ c_{i(j)} \succ c_i$ and $c_{i(j)} \in \mathcal{L}(c)$ for all j. And the leaves $c_{i(j)} \to c_i$. Now, we use the previous argument given by Lemma 5.2.5 to show that the comajor c_i in itself is a limit of the sequence of preperiodic comajor leaves $\{c'_{i(j)}\}_{j=1}^{\infty}$, each of preperiod 1 such that $c'_{i(j)} \in \mathcal{L}(c)$ and $c \succ c'_{i(j)} \succ c_i$ for all j.

Using the argument above we can build a sequence of preperiodic comajor leaves $\{c'_{i(j)}\}_{j=1}^{\infty}$ of preperiod 1 such that $c \succ c'_{i(j)} \succ c_i$ $\forall j$ and $c'_{i(j)} \to$

c_i for all i. Now consider the following sequence of preperiodic comajor leaves $\{c'_{1(1)}, c'_{2(2)}, c'_{3(3)}, ...\}$. Each leaf $c'_{n(n)}$ in the above sequence satisfies the following conditions, $c \succ c'_{n(n)}$ and $c'_{n(n)}$ is preperiodic comajor leaf of preperiod 1. Finally, it is not hard to see that $c'_{n(n)} \to c$. Thus, even in this case we get c as limit of preperiodic comajor leaves $\{c'_{n(n)}\}_{n=1}^{\infty}$, each of preperiod 1.

□

The goal is to provide an algorithm to construct all preperiodic comajor leaves of preperiod 1. By the previous lemma, they are dense in C_sCL. The quadratic version of this algorithm was given by Lavaurs in [**6**]. We need a few more lemmas before setting up the algorithm.

LEMMA 5.3.3. *For every preperiodic point $p \in \mathbb{S}$ of preperiod bigger than 1, there exists a cubic symmetric q-lamination $\mathcal{L}$ such that either p is a degenerate comajor in C_sCL and $\mathcal{L} = \mathcal{L}(p)$ or the following is true:*

(a) $\mathcal{L}$ has a pair of finite critical gaps Δ and $-\Delta$.

(b) $\mathcal{L}$ has finite sibling gaps $\widehat{\Delta}$ and $-\widehat{\Delta}$ with p and $-p$ as one of the vertices respectively.

(c) $\widehat{\Delta}$ and $-\widehat{\Delta}$ are gaps in C_sCL.

PROOF. Consider the pull back lamination $\mathcal{L}(p)$ and the critical chords ℓ and $-\ell$ of the circle $\mathbb{S}$ that have the same images as the points p and $-p$ respectively. It follows that ℓ and $-\ell$ are leaves of the lamination $\mathcal{L}(p)$ and their endpoints are preperiodic as the points p and $-p$ are preperiodic of preperiod bigger than 1 . If the critical leaves ℓ and , $-\ell$ are stand-alone leaves in $\mathcal{L}(p)$, then $\mathcal{L}(p)$ is q-lamination itself and the required lamination $\mathcal{L} = \mathcal{L}(p)$ and p is a degenerate comajor in C_sCL.

If the critical leaves ℓ and $-\ell$ are part of the gaps G and $-G$ in $\mathcal{L}(p)$ respectively, then the gaps G and $-G$ have to be preperiodic, too, as the critical leaves are themselves preperiodic. By Theorem 3.3.16, G and $-G$ have to be finite gaps. We claim that no side of the gap G ever maps to the critical leaf ℓ or $-\ell$ under a forward iterate of the map σ_3. Clearly the leaf ℓ cannot map to the leaf $-\ell$ as the endpoints of the leaves $\ell, -\ell$ are not periodic. Consider a side m of G other than the critical leaf ℓ. Note that the gap $\sigma_3(G)$ has one one fewer vertices than any of the four gaps $\{G, \hat{G}, -G, -\hat{G}\}$, where $\hat{G}$ and $-\hat{G}$ are the other two gaps sharing critical leaves ℓ and $-\ell$ on their boundaries with the gaps G and $-G$ respectively. Now, the leaf m cannot ever map to the critical leaves ℓ or $-\ell$ because no forward image of the gap $\sigma_3(G)$ can ever intersect any of the four gaps $\{G, \hat{G}, -G, -\hat{G}\}$.

We will try to build a cubic symmetric lamination $\mathcal{L}$ by cleaning out critical leaves $\ell, -\ell$ and their pre-images, i.e $\mathcal{L} = \mathcal{L}(p) \setminus \bigcup_{i=0}^{\infty}\{\sigma_3^{-i}(\ell), \sigma_3^{-i}(-\ell)\}$. From the earlier argument about the sides of G and $-G$, it follows that $\mathcal{L}$ is a non-empty collection of chords in $\mathbb{S}$. We got $\mathcal{L}$ by taking a full pre-image collection from $\mathcal{L}(p)$. Hence, it is not hard to see that $\mathcal{L}$ is a cubic symmetric lamination. And $\mathcal{L}$ has a pair of finite critical gaps Δ and $-\Delta$ obtained by merging the gaps on either side of the critical leaves ℓ and $-\ell$ respectively. Also, two of the vertices of Δ being the endpoints of ℓ have the same image as p. Same is the case with $-\Delta$ and $-p$. Thus, $\mathcal{L}$ has finite sibling gaps $\widehat{\Delta}$ and $-\widehat{\Delta}$ with p and $-p$ as one of the vertices respectively. By Lemma 5.2.9, it follows that $\widehat{\Delta}$ and $-\widehat{\Delta}$ are gaps in C_sCL.

Finally we have to prove that $\mathcal{L}$ is a q-lamination. Note that $\mathcal{L}(p)$ has only finite gaps in it. So if there were three leaves sharing an endpoint in $\mathcal{L}(p)$, then at least one of them has to be a diagonal of a finite gap. We claim that every diagonal of a finite gap in $\mathcal{L}(p)$ eventually maps to the critical leaf ℓ or $-\ell$. There are broadly two kinds of leaves in $\mathcal{L}(p)$, the pre-image leaves that eventually map to the critical leaves

ℓ and $-\ell$ and the limit leaves that are obtained by the accumulation of pre-image leaves. Clearly, limit leaves cannot be in the interior of a gap. Thus, all diagonals of finite gaps in $\mathcal{L}(p)$ eventually map to the critical leaves and during the cleaning process, we remove all such diagonal leaves and end up with a q-lamination $\mathcal{L}$. $\square$

Recall the notion of leaves moving in and out from Definition 5.2.1. When two leaves move towards each other, Lemma 5.2.4 proved the existence of periodic leaves separating them. We will consider the case now when the leaves move away from each other. The proofs are mostly similar. The following lemma is very useful in proving the main theorem of the current section.

LEMMA 5.3.4. *Let $\ell' \neq \ell$ be two leaves in a cubic symmetric lamination $\mathcal{L}$ such that $\ell' \succ \ell$. If the following conditions are true,*

(i) the leaves ℓ and ℓ' move away from each other under the map σ_3^k and

(ii) no leaf separating ℓ and ℓ' maps to a critical chord of $\mathbb{S}$ under the map σ_3^{k-1}.

Then, there exists a periodic leaf y of period k under the map σ_3 that separates ℓ and ℓ'.

PROOF. Consider the family of leaves $\mathcal{C}$ in $\mathcal{L}$ that separate ℓ and ℓ'. Let the leaves ℓ and ℓ' be also included in $\mathcal{C}$. We claim that the $\mathcal{C}$ has at least one leaf that separates ℓ and ℓ' and $\mathcal{C}$ is a closed subset of $\overline{D}$. If there is no leaf in $\mathcal{C}$ that separates ℓ and ℓ', then there exists a gap G of $\mathcal{L}$ with ℓ and ℓ' on its boundary. It follows that the gap $\sigma_3^k(G)$ would strictly cover the gap G, which is a contradiction. Thus, $\mathcal{C}$ has at least one leaf that separates ℓ and ℓ'.

It is not hard to see that $\mathcal{C}$ is closed, because if there is a sequence of leaves $\{\ell_i\}_{i=1}^{\infty}$ in $\mathcal{C}$, then the limit leaf ℓ_∞ is either ℓ or ℓ' or a leaf that separates the leaves ℓ and ℓ' in $\mathcal{L}$. Let us consider a subset A of leaves in $\mathcal{C}$ as follows. Every leaf m in the set A

moves in under the map σ_3^k. And for every leaf $m \in A$, if a leaf n separates the leaves ℓ and m, then n also moves in under the map σ_3^k. In other words, all the leaves in A move in the same direction as the leaf ℓ.

There are clearly two leaves in $\mathrm{Bd}(A)$, one is leaf ℓ and the other, say a leaf y ($y \succ \ell$). By continuity, either the leaf y moves in like the leaves of A or it is invariant. We claim that the case when the leaf y moves in is not possible. Let us assume that y moves in under the map σ_3^k. We claim that y should be part of a gap G in $\mathcal{L}$ such that y is the smallest side of G in the sense of $\succ$. Note that as y is in $\mathrm{Bd}(A)$, y is locally maximal (in the sense of $\succ$) leaf that moves in. Since y is maximal, there exists a sequence of leaves t_i in $\mathcal{C}$ approaching y such that $t_i \succ y$ and all the leaves t_i move out. But then by continuity, the limit leaf $t = y$ of the leaves t_i would also have to move out, a contradiction. It follows that there exists a gap G with the sides y and t in the collection $\mathcal{C}$ such that $t \succ y$.

Now, the gap $\sigma_3^k(G)$ would strictly cover the gap G, which is a contradiction.

Thus, the leaf y is invariant under the map σ_3^k, i.e $\sigma_3^k(y) = y$. We claim that the map σ_3^k actually fixes the endpoints of the leaf y. By the way of contradiction, let us assume that $\sigma_3^k(a) = b$ and $\sigma_3^k(b) = a$ where a and b are the endpoints of the leaf y. In other words, σ_3^k flips the leaf y. We have two sub-cases here.

If y is a stand-alone leaf, choose a leaf $t \in A$ very close to the leaf y. Since the map σ_3^k flips the leaf y, the leaf t would move out under the map σ_3^k contradicting the fact that $t \in A$.

If y is part of a gap G, then the gap $G' = \sigma_3^k(G)$ also shares the leaf y because of the fact that σ_3^k flips the leaf y. In other words, the gaps G and G' are on both sides of the leaf y. Now, we can find a leaf $t \in A$ in the gap G or G' and get a similar contradiction as before.

One last thing to consider about the periodic leaf y we got is that it can be an eventually collapsing leaf and map to one of its endpoints. In a sense, y doesn't move in or move out and is still invariant. However, the above case can happen only if y maps to a critical chord of $\mathbb{S}$ under the map σ_3^{k-1}. And by the second condition in the lemma, no leaf in $\mathcal{C}$ maps to a critical chord of $\mathbb{S}$ under the map σ_3^{k-1}. Thus, y is a periodic leaf of period k under the map σ_3. $\square$

Comajors of preperiod 1 have periodic majors in their pull-back laminations. It follows that they have Fatou gaps as their critical gaps. In complex dynamics, cubic polynomials with *Fatou gaps of degree 4* are said to be of type B (Bi-transitive) and cubic polynomials with *Fatou gaps of degree 2* are said to be of type D (Disjoint). We classify comajors of preperiod 1 in the similar fashion as follows:

DEFINITION 5.3.2. Preperiodic comajor c of preperiod 1 is said to be of type B if $\mathcal{L}(c)$ has a pair of symmetric Fatou gaps of degree 4 and c is said to be of type D if $\mathcal{L}(c)$ has a pair of symmetric Fatou gaps of degree 2.

Note that for a *Fatou gap of degree 4* in $\mathcal{L}(c)$, the major leaf M maps to $-M$ such that the endpoints a and b of M eventually map to the endpoints $-a$ and $-b$ of $-M$ respectively . In other words, majors rotate by a 180° when they get to their symmetric counterparts.
And for a *Fatou gap of degree 2* in $\mathcal{L}(c)$, the orbits of major leaves are disjoint from each other. We assign a number to type B and type D comajors as follows:

DEFINITION 5.3.3. A preperiodic comajor c of preperiod 1 of type D is said to be of number n if the leaf $\sigma_3(c)$ is periodic of period n. And a comajor c of type B is said to be of number n if the leaf $\sigma_3(c)$ maps to the leaf $-\sigma_3(c)$ under the map σ_3^n.

We can similarly define type B and type D periodic minors of a cubic symmetric lamination. A type B periodic minor m of number n in the cubic symmetric lamination $\mathcal{L}$ is such that $\sigma_3^n(m) = -m$. The idea here is to look at type B periodic minors and type D periodic minors of a given number n in a similar vein.

Consider the rotation map of the unit circle: $\pi(x) = x + \frac{1}{2}\ \forall x \in \mathbb{S}$. The map π rotates points on the unit circle by 180°. π can be extended to the leaves and gaps of the lamination as follows.

$\pi(\ell) = -\ell\ \forall \ell \in \mathcal{L}$, $\pi(G) = -G$. Note that π is clearly a bijection from $\mathcal{L}$ to itself, which motivates us to look at the following class of continuous maps from $\mathcal{L}$ to itself.

Let $g = \pi : \mathcal{L} \to \mathcal{L}$. Note that the forward images of a leaf $\ell \in \mathcal{L}$ under the map σ_3 not only cross each other, also do not cross the leaf $-\ell$ and its forward images. It follows that, like the map σ_3, the map g is also clearly defined on $\mathcal{L}$. Also, $\mathcal{L}$ is clearly both forward and backward invariant under the map g.

For the map σ_3, let us define $g_k = \pi \circ \sigma_3^k : \mathcal{L} \to \mathcal{L}$ for some k. We would like to use the map g_k in further exploring the parameter space C_sCL.

The following lemma is an equivalent to the Lemma 5.3.4 for the map g_k. The proof is exactly the same. We state without proof.

Lemma 5.3.5. *Let $\ell' \neq \ell$ be two leaves in a cubic symmetric lamination $\mathcal{L}$ such that $\ell' \succ \ell$. If the following conditions are true,*

(i) the leaves ℓ and ℓ' move away from each other under the map $g_k = \pi \circ \sigma_3^k$ and

(ii) no leaf separating ℓ and ℓ' maps to a critical chord of $\mathbb{S}$ under the map $g_{k-1} = \pi \circ \sigma_3^{k-1}$.

Then, there exists a periodic leaf y of period 1 under the map $g_k = \pi \circ \sigma_3^k$ that separates ℓ and ℓ'.

Now, the main theorem needed for the algorithm is as follows.

THEOREM 5.3.6. *If two preperiodic comajors of preperiod 1, c and c' satisfy the following conditions:*

(i) $c \succ c'$,

(ii) c and c' are of the same type (both are either of type B or type D) and

(iii) c and c' have the same number n.

Then there exists a preperiodic comajor d of preperiod 1 separating c and c' ($c \succ d \succ c'$) such that d is of a number $j < n$.

PROOF. Choose a preperiodic point p of preperiod bigger than 1 and period bigger than n in the arc $H(c')$. We can find infinitely many such points in any arc of $\mathbb{S}$. By Lemma 5.3.3, there exists a cubic symmetric q-lamination $\mathcal{L}$ with a pair of finite critical gaps $\{\Delta, -\Delta\}$ or a pair of critical leaves as critical gaps. We will assume that $\mathcal{L}$ has Δ and $-\Delta$ as their critical gaps for the rest of the discussion. Same exact argument works when they are critical leaves. Note that the sibling gaps $\widehat{\Delta}$ and $-\widehat{\Delta}$ in $\mathcal{L}$ contains p and $-p$ as one of their vertices respectively. By Lemma 5.1.6, c and c' are leaves of $\mathcal{L}$.

Consider the leaves $m = \sigma_3(c)$ and $m' = \sigma_3(c')$. As $c \succ c'$, we have $m \succ m'$, too. Both m and m' have periodic orbits. We claim that somewhere along their periodic orbits, at least one among the finite gaps Δ and $-\Delta$ separate a forward image of m and m'.

If not, we get the contradiction as follows- The assumption implies that there are no pre-images of Δ and $-\Delta$ separating m and m'. We also know that pre-images of finite critical gaps Δ and $-\Delta$ are dense in $\mathcal{L}$. Thus, it follows that the convex hull of m and m' should be a part of a gap G in $\mathcal{L}$. As Δ and $-\Delta$ never separate a forward image of m and m', we have that the four endpoints of the periodic leaves m and

m' always stay in the same circular order along their periodic orbit. It implies that the gap G has to be a periodic gap and it has to be finite because $\mathcal{L}$ has only finite gaps. By Lemma 3.3.7, if G is *1-transitive*, then the leaf m will eventually map to the leaf m', contradicting that m is the shortest leaf in its orbit. And if G is *2-transitive*, then the leaf m will eventually map to the leaf $-m'$, again contradicting that m is the shortest leaf in its orbit.

Now we need to consider two cases.

(i) *If c and c' are of type D*. Then the periodic orbits of m and $-m$ are disjoint and have the period n. Same things can be said about the leaves m' and $-m'$. Let k be the smallest number between 0 and n such that at least one among the finite gaps Δ and $-\Delta$ separates the leaves $\sigma_3^k(m)$ and $\sigma_3^k(m')$. First thing is to see that k cannot be equal to $n-1$. If $k = n-1$, the circular order of the four endpoints of m and m' is preserved by their corresponding images in the leaves $\sigma_3^{n-1}(m)$ and $\sigma_3^{n-1}(m')$. Now at the last stage in the orbit, i.e for leaves $\sigma_3^n(m)$ and $\sigma_3^n(m')$, exactly one of the two leaves flip because of the presence of critical gap between them. Without loss of generality, let us say the leaf $\sigma_3^{n-1}(m)$ flips its endpoints when it maps to the leaf $m = \sigma_3^n(m)$. Let $\partial(m) = \{a, b\}$ and $\partial(\sigma_3^{n-1}(m)) = \{\sigma_3^{n-1}(a), \sigma_3^{n-1}(b)\}$. The above situation forces the point $\sigma_3^{n-1}(a)$ to map to the point b. It follows that $\sigma_3^n(a) = b$, a contradiction with the fact that the leaf m is of period n under the map σ_3. Thus, $0 < k < n-1$. Now we have two sub-cases.

   (a) *If Δ separates the leaves $\sigma_3^k(m)$ and $\sigma_3^k(m')$.* All the leaves and gaps separating m and m' map 1-1 in the region of the closed disk $\overline{D}$ separating the leaves $\sigma_3^k(m)$ and $\sigma_3^k(m')$. It follows that there exists a pre-image $\Delta^* = \sigma_3^{-k}(\Delta)$ separating m and m'. It follows that $\sigma_3^{k+1}(\Delta^*) = \sigma_3(\Delta)$ is

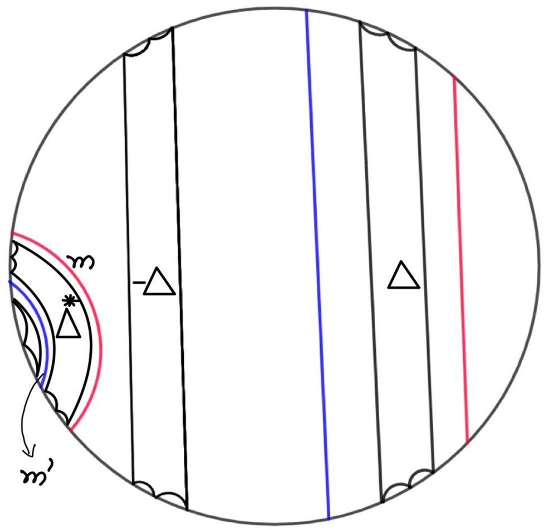


FIGURE 5.1. Cubic symmetric lamination $\mathcal{L}$ with its gaps $\Delta, -\Delta$ and Δ^* separating the leaves m and m' illustrating the proof of case (i) part (a).

under the leaf m', i.e $m' \succ \sigma_3^{k+1}(\Delta^*)$. Let ℓ^* be the side of Δ^* closest to the leaf m. Clearly, ℓ^* separates the leaves m and m' and it moves in under the map σ_3^{k+1}.

We claim that the leaves m and ℓ^* move away from each other under the map σ_3^{k+1}. The leaf $\sigma_3^{k+1}(m)$ is neither under the leaf m nor under the leaf $-m$ because of the fact that minor is always the shortest leaf in its orbit. It follows that m moves out under the map σ_3^{k+1}. Thus, the leaves m and ℓ^* move away from each other under the map σ_3^{k+1}.

Let us check the condition (ii) in the Lemma 5.3.4 statement. Note that the leaf ℓ^* is mapping to the major of the lamination $\mathcal{L}$ (say M) under the map σ_3^k. And all the leaves that separate ℓ^* and m in the strip $S_b(\ell^*, m)$ map 1-1 to the leaves that separate M and $\sigma_3^k(m)$ in the strip

$S_b(M, \sigma_3^k(m))$ under the map σ_3^k. As there are no critical chords of $\mathbb{S}$ in the strip $S_b(M, \sigma_3^k(m))$, no leaf separating ℓ^* and m maps to a critical chord of $\mathbb{S}$ under the map σ_3^k. Hence, by Lemma 5.3.4, there exists a periodic leaf y of period $k+1 < n$ in the lamination $\mathcal{L}$ that separates m and ℓ^*. Let $\mathcal{C}$ be the collection of all the leaves that separate m and m'. Consider two sub-collections in $\mathcal{C}$ as follows.

Let C_1 be the collection of all periodic leaves of period smaller than n under the map σ_3 in $\mathcal{C}$.

Let C_2 be the collection of all fixed leaves under the map $g_k = \pi \circ \sigma_3^k$ where k can be any number smaller than n in $\mathcal{C}$. Let us associate k as the iterate number with all fixed leaves under the map g_k in $\mathcal{C}$. Every leaf in C_2 can be associated with an iterate number.

We have shown above that $y \in C_1$. Thus, C_1 is non-empty but C_2 could very well be empty.

Let us choose a periodic leaf y_1 of the least period j_1 in C_1. Clearly $j_1 \leq k+1 < n$. Moreover, we may assume that y_1 is chosen among all such leaves to be closest to the leaf m in the lamination $\mathcal{L}$.

Similarly, let us choose a fixed leaf y_2 under the map $g_{j_2} = \pi \circ \sigma_3^{j_2}$ in C_1 such that j_2 is the smallest iterate number of all leaves in C_1. Moreover, we may assume that y_2 is chosen among all such leaves to be closest to the leaf m in the lamination $\mathcal{L}$.

If $j_1 < j_2$, then we claim that the short leaf $d = \sigma_3^{-1}(y_1)$ in $\mathcal{L}$ is the desired comajor of number $j = j_1 < n$. By Lemma 5.2.3, it suffices to prove that the leaf y_1 neither maps under itself nor under the leaf $-y_1$ under the map σ_3^i where i can be any number smaller than j_1.

(1) If the leaf y_1 maps under itself under the map σ_3^i , $i < j_1$, then the leaves y_1 and m move away from each other under σ_3^i. By Lemma 5.3.4, we get a periodic leaf y_1' (under the map σ_3) of period i ($i < j_1$) separating m and y_1. This is in contradiction with the fact that j_1 is the smallest such period.

(2) If the leaf y_1 maps under the leaf $-y_1$ under the map σ_3^i , $i < j_1$, it implies that the leaf $g_i(y_1)$ is under the leaf y_1. Now, the leaves y_1 and m move away from each other under $g_i = \pi \circ \sigma_3^i$. By Lemma 5.3.5, we get a fixed leaf y_1' under the map $g_i = \pi \circ \sigma_3^i$ separating m and y_1. Clearly y_1' separates m and m', too. The iterate number for this fixed leaf is i where $i < j_1 < j_2$. This is in contradiction with the fact that j_2 is the smallest such iterate number.

Thus, the short leaf $d = \sigma_3^{-1}(y_1)$ in $\mathcal{L}$ is the desired comajor of number $j = j_1 < n$.

Similarly if $j_2 < j_1$, then we get that the short leaf $d = \sigma_3^{-1}(y_2)$ in $\mathcal{L}$ is the desired comajor of number $j = j_2 < n$.

Finally, there is a possibility that both the critical sets Δ and $-\Delta$ separate the leaves $\sigma_3^k(m)$ and $\sigma_3^k(m')$. We can get the gap $\Delta^* = \sigma_3^{-k}(\Delta)$ using the same argument as before. And all other arguments proceed exactly like before. So this case is not essentially different from what we discussed earlier.

(b) *If Δ does not separate the leaves $\sigma_3^k(m)$ and $\sigma_3^k(m')$ but $-\Delta$ does.* We use the same arguments as in case (a) and get the gap $\Delta^* = \sigma_3^{-k}(-\Delta)$ separating m and m'. In this case we have the gap $\sigma_3^{k+1}(\Delta^*)$ going under the leaf $-m'$. The only difference in the arguments is that we use Lemma 5.3.5 first to get a leaf y separating m and m' such that $g_{k+1}(y) = y$.

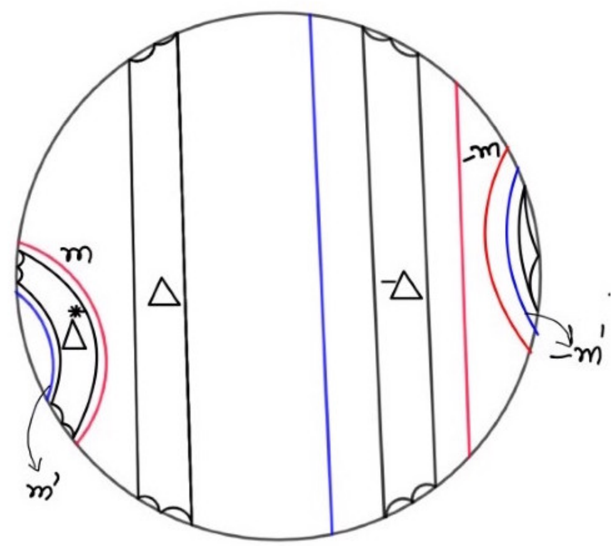


FIGURE 5.2. Cubic symmetric lamination $\mathcal{L}$ with its gaps $\Delta, -\Delta$ and Δ^* separating the leaves m and m' illustrating the proof of case (i) part (b).

Thus, the collection C_2 is non-empty here whereas collection C_1 could be empty. Rest of the argument follows exactly as before and we end up with a comajor d between c and c' of a number $j < n$.

(ii) *c and c' are of type B.* The leaves m and m' are now periodic of period $p = 2n$ and have symmetric orbits, i.e the orbits of m and $-m$ are the same. Similarly, the orbits of the leaves m' and $-m'$ are the same as well. Proofs in this case are very similar to that of case (i).

First, we will show that there exists a number k such that $0 < k < n-1$ such that at least one among the gaps $\{\Delta, -\Delta\}$ separate the leaves $\sigma_3^k(m)$ and $\sigma_3^k(m')$. Let k be the smallest number between 0 and $p = 2n$ such that one among the finite gaps Δ and $-\Delta$ separate the leaves $\sigma_3^k(m)$ and $\sigma_3^k(m')$. As the orbits of both the leaves m and m' are symmetric, the strips formed by the leaves $\sigma_3^k(m)$ and $\sigma_3^i(m')$ where $0 < i \leq n-1$ are symmetric to the strips formed by the leaves $\sigma_3^r(m)$ and $\sigma_3^r(m')$ where $n \leq r < 2n$. It follows that for the first time, separation by one of the critical gaps Δ and $-\Delta$ happens during the first half of the orbit itself, i.e $0 < k \leq n-1$.

We claim that k cannot be equal to n. The circular order of the four endpoints of m and m' are preserved in the leaves $\sigma_3^{n-1}(m)$ and $\sigma_3^{n-1}(m')$ and exactly one of them flips because of a critical gap between them. Without loss of generality, let us say the leaf $\sigma_3^{n-1}(m)$ flips its endpoints when it maps to the leaf $-m = \sigma_3^n(m)$. Let $\partial(-m) = \{a + \frac{1}{2}, b + \frac{1}{2}\}$ and $\partial(\sigma_3^{n-1}(m)) = \{\sigma_3^{n-1}(a), \sigma_3^{n-1}(b)\}$. The above situation forces the point $\sigma_3^{n-1}(a)$ to map to the point $b + \frac{1}{2}$. It follows that $\sigma_3^n(a) = b + \frac{1}{2}$. But we know that the type B leaf m is of number n, In other words, the point $\sigma_3^{n-1}(a)$ should map to the point $a + \frac{1}{2}$ not the point $b + \frac{1}{2}$. Thus, $0 < k < n-1$. We have two subcases here similar to the case (i).

(a) *If Δ separates the leaves $\sigma_3^k(m)$ and $\sigma_3^k(m')$.* Then, following the similar arguments as in case(i) part(a), we get a comajor leaf d separating the leaves c and c' of a number $j < k+1 = n$.

(b) *If Δ does not separate the leaves $\sigma_3^k(m)$ and $\sigma_3^k(m')$ but $-\Delta$ does.* Then, following the similar arguments as in case(i) part(b), we get a comajor leaf d separating the leaves c and c' of a number $j < k+1 = n$.

□

For the rest of this section, whenever we refer to comajor leaves, we mean comajor leaves of preperiod 1.

PROPOSITION 5.3.7. *All type D comajor leaves of the odd number $n = 2p+1$ can be obtained by rotating all type B comajors leaves having the same number n by $90°$.*

PROOF. Consider a type D comajor leaf c of number $n = 2p + 1$. Let a leaf d be obtained by rotating the leaf c by $90°$, i.e $\partial(d) = \partial(c) + \frac{1}{4}$.
Note that $\sigma_3^{2p+1}(\frac{1}{4}) = \frac{3}{4}$ and $\sigma_3^{2p}(\frac{1}{4}) = \frac{1}{4}$. Consider two points $a,\ b \in \mathbb{S}$ that are $90°$ apart , i.e, the arc $\widehat{a\,b}$ is of the length $\frac{1}{4}$. It follows that there will be an arc of length $\frac{1}{4}$ joining the points $\sigma_3^k(a)$ and $\sigma_3^k(b)$ in $\mathbb{S}$ for every iterate σ_3^k. It follows that every forward image $\sigma_3^k(d)$ of the leaf d can either be obtained by rotating the corresponding image $\sigma_3^k(c)$ of the leaf c by $90°$ either clockwise or anti-clockwise. Same is the case with the backward iterates $\sigma_3^{-k}(c)$ and $\sigma_3^{-k}(d)$.
Thus, if we rotate the leaves of pull-back lamination $\mathcal{L}(c)$ by $90°$, we get a collection of chords of $\mathbb{S}$ that is a pull-back cubic symmetric lamination $\mathcal{L}(d)$ with $\{d, -d\}$ as a pair of comajor leaves.
Consider the minor leaves $m = \sigma_3(c)$ and $m' = \sigma_3(d)$. We have $\sigma_3^{2p+1}(m) = m$, it follows that $\sigma_3^{2p+1}(m') = -m'$ because of the fact that $\partial(d) = \partial(c) + \frac{1}{4}$. Thus, the comajor leaves d and $-d$ are of type D and is of the number $n = 2p + 1$.

On the other hand if we start with a pair of type B comajor leaves of number $n = 2p+1$ and rotate them by $90°$, we can show that we get a pair of type D comajor leaves of the same number $n = 2p + 1$ by using similar arguments. □

Based on Theorem 5.3.5 and Proposition 5.3.6, the *algorithm* for Cubic symmetric Comajor Lamination C_sCL can be stated as follows:

(1) **Level 1:** At the end of this level all comajor leaves of number 1 will be constructed.
   (a) *Type D leaves.* Consider the angles in $\mathbb{S}$ of preperiod 1 and period(number) 1 (namely, $\frac{1}{3}, \frac{1}{6}, \frac{2}{3}, \frac{5}{6}$).

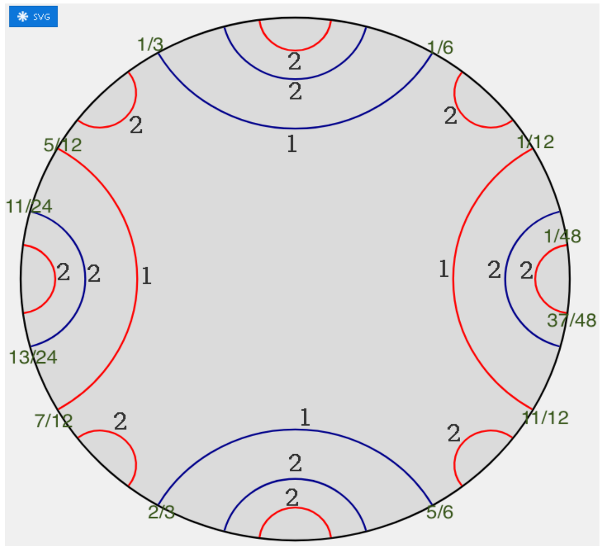


FIGURE 5.3. Comajors of numbers 1 and 2. Type D leaves are shown blue in color and type B leaves are shown red in color.

As all the comajor leaves are short leaves having length smaller than $\frac{1}{6}$, there is only one way to connect them and we get the leaves $\overline{\frac{1}{6}\frac{1}{3}}$ and $\overline{\frac{2}{3}\frac{5}{6}}$.

(b) *Type B leaves.* As we are in odd number level, rotate type D leaves by $90°$ (adding $\frac{1}{4}$ to the endpoints) to obtain type B leaves $\overline{\frac{5}{12}\frac{7}{12}}$ and $\overline{\frac{11}{12}\frac{1}{12}}$.

(2) **Level 2:** At the end of this level all comajor leaves of number 2 will be constructed.

(a) *Type D leaves.* Consider the angles in $\mathbb{S}$ of preperiod 1 and period (number) 2. Some of them are $\frac{1}{24}, \frac{5}{24}, \frac{7}{24}, \frac{11}{24}$.... Connect the new angles pairwise as follows. Start at a lowest unused angle bigger than $\frac{1}{12}$ and connect it to the lowest angle that can be reached without crossing the leaves that have already been constructed. Some of them are shown in Figure 5.3.

(b) *Type B leaves.* To find the angles θ for this case, we do the following.

1) First let us compute the periodic angles x satisfying the condition

$\sigma_3^2(x) = x + \frac{1}{2}$ or $3^2 x = x + \frac{1}{2}$ $(mod 1)$. More specifically $x = \frac{2n+1}{16}$ where n varies from 0 to 7.

Now, we need preperiodic images of x, i.e $\theta = \sigma_3^{-1}(x)$ and θ has to be preperiodic. For every periodic angle x, there will be two pre-images of x which are preperiodic and one pre-image will be periodic. We need to ignore the periodic angle. In total there are 16 type B angles of number 2. Some of them are $\frac{1}{48}, \frac{5}{48}, \frac{7}{48}, \frac{11}{48}$... Repeat the same process used in case (a) to connect new angles. Some of them are shown in Figure 5.3.

(3) **Level k:** Assume that all comajors of preperiod 1 and number less than $k-1$ are constructed.

   (a) *Type D leaves.* Consider the new angles in $\mathbb{S}$ of preperiod 1 and period (number) k. Connect them by leaves as follows:

   Start at a lowest unused angle bigger than $\frac{1}{12}$ and connect it to the lowest angle that can be reached without crossing the leaves that have already been constructed.

   (b) *Type B leaves.* If k is odd, rotate type D leaves by 90°, otherwise do the following.

   The type B angles θ of preperiod 1 in $\mathbb{S}$ of number k are obtained by taking the preperiodic images of the angles x satisfying the condition $3^k x = x + \frac{1}{2}$ $(mod 1)$.

   Repeat the same process used in case (a) to connect new angles.

At every stage, the new set of leaves that are added form a collection which is invariant by 90° rotation. It also makes sense for that to be the case as we know that the parameter space C_sCL is invariant by 90° rotation.

Theorem 5.3.8. *The above proposed algorithm generates all comajors of preperiod 1 in Cubic symmetric Comajor Lamination C_sCL.*

Proof. By Lemma 5.1.4, every point of $\mathbb{S}$ is either an endpoint of a non-degenerate comajor leaf or a degenerate comajor in C_sCL. Consider a periodic point $p \in \mathbb{S}$ of preperiod 1. By Lemma 5.1.3, p is either an endpoint of a non-degenerate preperiodic comajor leaf of preperiod 1 or a degenerate comajor in C_sCL. By Lemma 5.3.1, p must be an endpoint of a non-degenerate preperiodic comajor leaf of preperiod 1, in other words p has to be connected to a preperiodic point $q \neq p \in \mathbb{S}$ of preperiod 1 to form a comajor leaf $\overline{pq}$. By Lemma 5.2.1, the point q is unique. In other words, there is exactly one point $q \neq p \in \mathbb{S}$ such that the leaf $\overline{pq}$ is a preperiodic comajor leaf of preperiod 1.

Suppose that all preperiodic comajors of preperiod 1 up to level $n-1$ denoted by $\mathcal{CO}_{n-1}$ have been constructed using the algorithm and they coincide with actual comajors of C_sCL up to number $n-1$. Assume without loss of generality, that algorithm fails at some preperiodic point p of type D and number n, i.e p is a preperiodic point of preperiod 1 and period n. Then p is located in a component U of $\overline{\mathbb{D}} \setminus \mathcal{C}\prime_{n-1}$. Since comajors cannot cross, the actual comajor $c = \overline{pq}$ connects the point p to a point $q \in \mathbb{S}$ in U. And as we assumed the algorithm fails at the point p, there exists a point $q' \in \mathbb{S} \cap U$ of preperiod 1 and period n which is under c. From the earlier arguments, q' has to be connected to an available preperiodic point $q'' \in \mathbb{S} \cap U$ of preperiod 1 to form a comajor leaf $c' = \overline{qq''}$. As comajors do not cross, it is clear that $c \succ c'$. Since all the points $\{p, q, q', q''\}$ are in U, it is clear that there are no comajors of type D and smaller numbers between c and c', a contradiction with Theorem 5.3.6.

A similar contradiction exists with a type B comajor of number n. $\square$

LIST OF REFERENCES

[1] Sourav Bhattacharya, Alexander Blokh and Dierk Schleicher, *Unicritical Laminations*, preprint arXiv:2101.08101 (2021)

[2] A. Blokh, D. Mimbs, L. Oversteegen, K. Valkenburg, *Laminations in the language of leaves*, Trans. Amer. Math. Soc., **365** (2013), 5367–5391.

[3] A. Blokh, L. Oversteegen, R. Ptacek, V. Timorin, *Laminations from the Main Cubioid*, preprint arXiv:1305.5788 (2013)

[4] A. Blokh, L. Oversteegen, R. Ptacek, V. Timorin, *Combinatorial Models for Spaces of Cubic Polynomials*, preprint arXiv:1405.4287 (2014)

[5] Shaun Bullett and Pierrette Sentenac. *Ordered orbits of the shift, square roots, and the devils staircase,* Math. Proc. Cambridge Philos. Soc. 115 (1994), no. 3, 451481 (English, with English and French summaries). MR 1269932.

[6] P.Lavaurs, *Systemes dynamics holomorphes*, Explosion de points pe- riodiques paraboliques, These de doctorat, Universite Paris-Sud, Orsay(1989)

[7] J. Milnor, *Dynamics in One Complex Variable*, Annals of Mathematical Studies **160**, Princeton (2006).

[8] D. Schleicher, *Appendix: Laminations, Julia sets, and the Mandelbrot set*, in: "Complex dynamics: Families and Friends", ed. by D. Schleicher, A K Peters (2009), 111–130.

[9] W. Thurston, H. Baik, Y. Gao, J. Hubbard, T. Lei, K. Lindsey, D. Thurston, *Degree d invariant laminations*, preprint arXiv:1906.05324 (2019)

[10] W. Thurston. *The combinatorics of iterated rational maps* (1985), with appendix by D. Schleicher, *Laminations, Julia sets, and the Mandelbrot set*, published in: "Complex dynamics: Families and Friends", ed. by D. Schleicher, A K Peters (2009), 1–137.