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A Model For Currency Exchange Rates

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A MODEL FOR CURRENCY EXCHANGE RATES

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ABSTRACT

Catastrophic economic events like the oil price shock in 1973, the 9/11 event in 2001, the stock market crash in October, 1987, the financial crash of late 2008 and the Covid-19 epidemic impact the US economy without warning, with varying effect, from sharp downturns to actual market crashes. Current economic theory and the associated statistical models are not able to predict these events. The development of predictive models for the US economy needs at the very least prediction models for its major components: commodities, stocks, bonds and currencies. I am going to model the currency exchange rates using a stochastic differential equation model by predicting the exchange rate trend using a system of delay differential equations, and recovering the future volatility of exchange rates by solving an inverse problem obtained from the Garman-Kohlhagen partial differential equation for foreign exchange options.

DEDICATION

TO MY BELOVED FAMILY: MOTHER SANU TAMANG, LATE FATHER SETE TAMANG, BROTHER PREM LAMA AND LATE BROTHER SUBASH TAMANG AND TO MY SCHOOL TEACHER: RAM BAHADUR BUDHATHOKI

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TABLE OF CONTENTS

LIST OF FIGURES

Figure *Page*

LIST OF TABLES

CHAPTER 1

Introduction

A market is a large collection of people acting individually and collectively, each with their own goals and economic reasons for participating. A market in a typical developed economy such as the US has four primary market components: commodities, bond prices, stocks and currencies. If one has to invest in a market or one wishes to be successful trader, one has to observe how these factors interrelate. Each factor has an important role in a smooth market. There is a relation between bond prices and interest rates, and stocks are correlated to bond prices. Interest rates directly relate to commodity prices. Currencies also depend on commodity prices, however they affect all factors in the market. Commodities directly affect bonds and stocks, but currencies vary inversely with commodity prices. We begin with a very short overview of financial markets and foreign exchange rates.

1. Financial Markets

Markets, traditionally, started out as a places where local people would gather and buy or sell provisions, livestock and other commodities. Nowadays, we can do all these activities online, so a modern market is a medium for purchasing and selling goods, and we no longer need a particular set location. Consequently, the term market in this thesis is used to describe a medium for the exchange of financial securities, stocks or commodities, which could be a physical location such as the NYSE, LSE, BSE or an electronic system such as NASDAQ. There are different kinds of markets. A financial market is a market for trading of financial securities like stock, bonds, precious metals, and derivatives. There are also capital markets such as the stock market and the bond market, commodity markets, money markets, derivatives markets, future markets, foreign exchange markets, and even cryptocurrency markets. Our focus is foreign exchange which is traded in foreign exchange markets, often privately in over-the-counter (OTC) transactions.

2. Currency Markets

Before moving forward, we present a brief history of currencies. The history of currency markets is quite intriguing, and we briefly provide some details of the financial instruments that were traded, and their underlying dynamics. Some 2500 years ago, the Greeks and Egyptians started trading goods and currencies using silver and gold coins as a medium of exchange, and at that time, the value of these currencies was determined by their actual weights and sizes. Around 500 years later, during the Roman empire, currency minting was centralized, and a government-run monopoly on currency trading was established. This structure still exists in today's banking sector in the form of centralized monetary policy. A 1000 years on from this time, the people of the Middle Ages used copper instead of gold as the metal for minting coins and trading, thereby creating coins of lower intrinsic value. Later, the world's oldest bank, Monte dei Paschi, established in Italy, was built to facilitate currency transactions, and 500 years ago, the first foreign exchange (Forex) market was established in Amsterdam to stabilize exchange rates. The so-called Gold Standard was introduced in 1875, and this permitted a country to mint currency, provided that they had sufficient gold in reserve, which was used to guarantee the value of a currency. This system did however not exist for long, and after World War I countries started printing money to manage their expenses.

By 1913, the number of Forex trading firms had risen exponentially, from 3 to 71, within only 10 years in London, and 50 % of all Forex transactions were made in pound sterling. In 2013, the pound was was still the 4th most traded currency after the US-Dollar, the EURO and the Japanese yen. Now, the Forex market is the largest financial market in the world. Currency trading can be very volatile, and the unique characteristics of Forex trading, including leverage and a market that is open 24/7, make it very attractive for retail traders.

Digital crypto currencies such as bitcoins have become very popular in recent years and these currencies may have a bright future because of ongoing global uncertainties and potentially unstable monetary systems. Also, these currencies represent an alternative to centralized and politically controlled currency forms.

The foreign exchange market shows that currency trading is affected by many factors, such as supply and demand, interest rates, economic growth, and speculation. Nonetheless, commodity prices are pre-eminent in currency. Option pricing is the principal tool used to observe the relationship between currencies and commodity prices.

3. Currency Options

In his book *Option Pricing: Mathematical Models and Computation* [**1**], Paul Wilmott defines the simplest financial option, a **European call option**, as a contract with the following conditions:

- at a prescribed time in the future, known as the **expiry date**,the owner of the option may
- purchase a prescribed asset, known as the **underlying asset** or briefly, the **underlying**, for a
- prescribed amount, known as the **exercise price** or **strike price**. A similar definition holds for a **European put option** where the word 'purchase' is replaced by 'sell'.

Here the word 'may' in the description implies that for the holder of the option, this contract is a right and not an obligation, whereas, the other party to the contract, who is known as the writer, does have a potential obligation: he must sell the asset if the holder chooses to buy it. Since the option confers on its holder a right with no obligation it has some value. So, a **currency option** (also known as a **Forex option**) is a contract that gives the buyer the right, but not the obligation, to buy or sell a certain currency at a specified exchange rate on or before a specified date. For this right, a premium is paid to the seller. The main keys are: a) currency options give investors the right, but not the obligation, to buy or sell a particular currency at a pre-specified exchange rate before the option expires; b) currency options allow traders to hedge currency risk or to speculate on currency moves; c) currency options come in two main varieties, so-called vanilla options, and over-the-counter SPOT options. A SPOT option is a type of option contract that allows an investor to set not only the conditions that need to be met in order to receive the desired payout, but also the size of the payout he or she wishes to receive if those conditions are met.

We introduce some notation which is used consistently throughout this thesis. $v = v(S,t)$ is the value of an option that is a function of the current value of the underlying asset, S, and time, t. An option also depends on various parameters: the volatility (σ) of the underlying asset, the exercise price (E), the expiry (T), and the interest rates (r_D and r_F) of the two countries representing currency pair. We use $C(S,t)$ and $P(S,t)$ to denote call and put options respectively.

If $S > E$ at expiry, it makes financial sense to exercise the call option, handing over an amount E, to obtain an asset worth S. The profit from such a transaction is then $S - E$. On the other hand, if $S < E$ at expiry, one should not exercise the option because one would make loss of $E - S$. In this case the option expires valueless. Thus the value of the call option at expiry can be written as

$$C(S,T) = \max(S - E, 0).$$

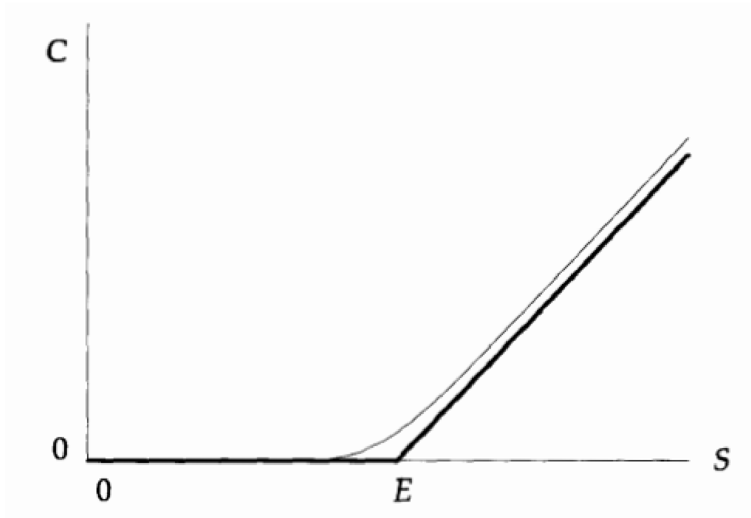


FIGURE 1.1. The payoff diagram for a call $C(S,T)$, and the option value $C(S,t)$ prior to expiry, as a function of S.

On the other hand, at expiry the put option is worthless if $S > E$ but has the value $E - S$ for $S < E$. Thus the payoff at expiry for a put option is

$$P(S,T) = \max(E - S, 0).$$

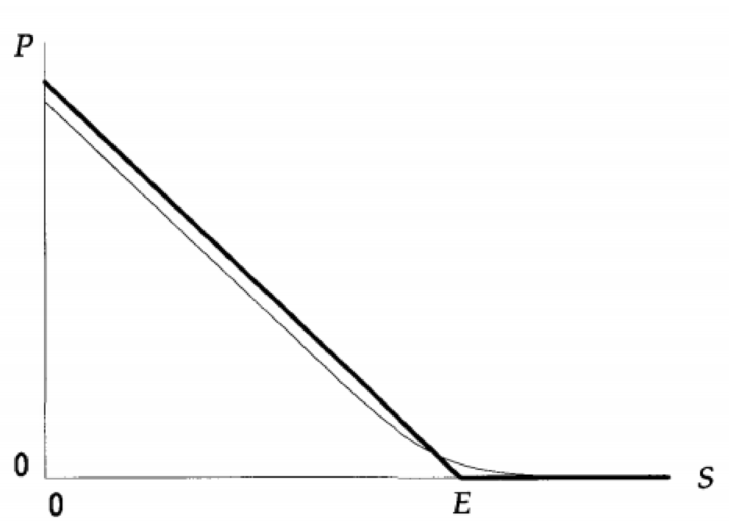


FIGURE 1.2. The payoff diagram for a put $P(S,T)$, and the option value $P(S,t)$ prior to expiry, as a function of S.

Finally, we quote the exact solution of the European call option problem when the interest rates and volatilities are considered constant. The explicit solution for the

European call is given by

$$C(S,t) = SN(d_1) - Ee^{-r(T-t)}N(d_2) \tag{1.1}$$

And the explicit solution for the European put option is given by

$$P(S,t) = Ee^{-r(T-t)}N(-d_2) - SN(-d_1) \tag{1.2}$$

where $N(\cdot)$ is the cumulative distribution function for a standardised normal random variable, given by

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy$$

where,

$$d_1 = \frac{\ln(S/E) + r + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_2 = \frac{\ln(S/E) + r - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

So the explicit solution for currency option is obtained using the formula (1.1) and (1.2) where r is replaced by difference of the interest rates $(r_F - r_D)$, given below:

$$C(S,t) = SN(d_1) - Ee^{(r_D - r_F)(T-t)}N(d_2)$$

and

$$P(S,t) = Ee^{(r_D - r_F)(T-t)}N(-d_2) - SN(-d_1),$$

where

$$d_1 = \frac{\ln(S/E) + (r_F - r_D) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_2 = \frac{\ln(S/E) + (r_F - r_D) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}.$$

4. Currency Exchange

The foreign exchange market (Forex, FX, or currency market) is a global decentralized or over-the-counter (OTC) market for the trading of currencies. This market determines foreign exchange rates for every currency. It includes all aspects of buying, selling and exchanging currencies at current or determined prices. In terms of trading volume, it is by far the largest market in the world, followed by the credit market. The foreign exchange market has a huge trading volume, that represents the largest asset class in the world, and leads to high liquidity. According to the Bank for International Settlements, the preliminary global results from the 2019 Triennial Central Bank Survey of Foreign Exchange and OTC Derivatives Markets Activity shows that trading in foreign exchange markets averaged $6.6 trillion per day in April 2019 (which is the largest volume financial market, followed by the bond market with $700 billion and the stock market with $500 billion per day). This is up from $5.1 trillion in April 2016. Measured by value, foreign exchange swaps were traded more than any other instrument in April 2019, at $3.2 trillion per day, followed by spot trading at $2 trillion.

An exchange rate is the value of one currency for the purpose of conversion to another. It is also regarded as the value of one country's currency in relation to another currency.

Currency	Currency Name	Exchange Rate = 1 EUR
USD	US dollar	**1.0906**
JPY	Japanese yen	**117.06**
BGN	Bulgarian lev	**1.9558**
CZK	Czech koruna	**27.553**
DKK	Danish krone	**7.4661**
GBP	Pound sterling	**0.87738**
HUF	Hungarian forint	**363.73**
PLN	Polish zloty	**4.5697**
RON	Romanian leu	**4.8320**
SEK	Swedish krona	**10.9265**
CHF	Swiss franc	**1.0551**

FIGURE 1.3. Exchange rates of the Euro to other currencies (Source: FRED) 04/02/2020

The most traded currencies pairs are, in order, Euro/US Dollar (EUR/USD), US Dollar/Japanese Yen (USD/JPY), British Pound Sterling/US Dollar (GBP/USD), Australian Dollar/US Dollar (AUD/USD), US Dollar/Swiss Frac (USD/CHF), New Zealand Dollar/US Dollar (NZD/USD), US Dollar/Canadian Dollar (USD/CAD). The graph given below illustrates the exchange rates of the Euro to other currencies with their trends and volatilities.

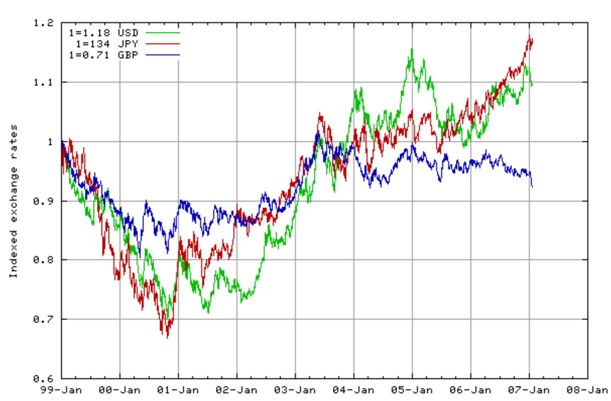


FIGURE 1.4. Graph of Exchange rates of the Euro to other currencies (Source: Wikipedia)

5. The Currency Pair EUR/USD

Among the major currencies pairs EUR/USD, USD/JPY, GBP/USD, AUD/USD, USD/CHF, NZD/USD, USD/CAD, figure 1.3 shows that the pair GBP/JPY is the most volatile and the pair EUR/USD is the least volatile. The currency pair EUR/USD is also the most actively traded currency pair in the world. We choose to study EUR/USD for our project mostly because it is the most influential of the major pairs, and would then be used as a template for the study of the others.

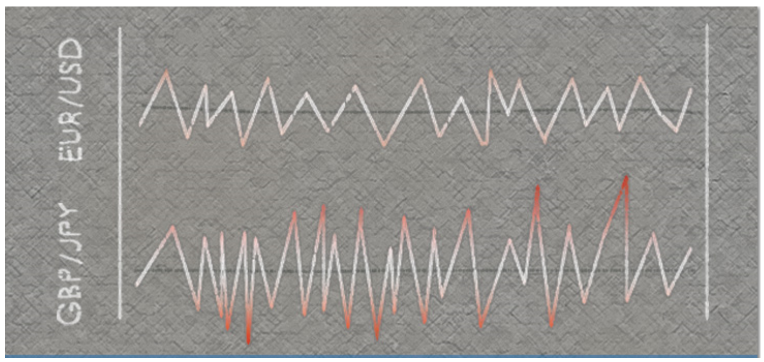


FIGURE 1.5. The most and least volatile currency pairs (Source: FXSSI)

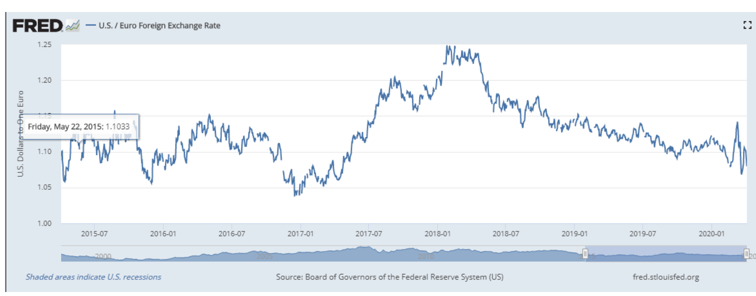


FIGURE 1.6. Exchange rates of EUR/USD (Source: FRED)

6. Overview of Thesis

This thesis contains six chapters, which are explained briefly below:

Chapter 1: Our first chapter consists of basic definitions of financial markets and factors affecting currency exchange rates, currency options and some special currency pairs.

Chapter 2: Factors that affect the currency exchange rates of the currency pair EUR-USD are explained in chapter 2, in which we also discuss some important models that could provide the precise trend of FX rates.

Chapter 3: This chapter considers the volatility of currency exchange rates, along with important models that provides promising results for the recovery of volatility, and for forecasting future exchange rates.

Chapter 4: In this chapter, we discuss FX trend prediction from the ODE system MatLab simulation and volatility recovery from C code inverse algorithms..

Chapter 5: The results that we obtained may not be always accurate. So, after obtaining results, some statistical tools/formulae are used to assist in measuring the accuracy of the models.

Chapter 6: The final chapter contains concluding remarks on open questions and possible improvements. We are interested in using machine learning and artificial neural networks concepts to predict the exchange rates. Possible future work related with this project is also discussed.

CHAPTER 2

The Trend of Currency Exchange Rates

Effective prediction of the exchange rate for any currency pair, requires that one predict both the trend and the future volatility of the exchange rate. These are then combined in the currency price model (financial asset model) given by the following stochastic differential equation:

$$\frac{dS}{S} = \mu dt + \sigma dB_t$$

where, for each time t, $S = S_t(\omega)$ is a random variable representing the price of a financial asset for the trial ω, $\mu = \mu(S,t)$ is the drift, $\sigma = \sigma(S,t)$ is volatility and $B_t(\omega)$ is the Brownian motion stochastic process used to model the randomness. Here, we consider S as the exchange rate, μ as the trend and σ as the volatility of the exchange rate of the currency pair EUR/USD.

1. Factors affecting FX rates of EUR and USD

To predict the trend, we study the factors, called macroeconomic variables, affecting currency exchange rates. In our case, for the prediction of EUR/USD, we determine the factors that impact the Euro and the US dollar.

Factors impacting the Euro to US dollar:

i) Economic growth (GDP) of the Eurozone countries

ii) Monetary policy of the European Central Bank (ECB)

iii) Unemployment rates, or job opportunities

iv) Budget deficits and national debt levels of the Eurozone countries

v) Domestic politics and international policies

Factors impacting the US dollar to Euro:

i) US GDP

ii) Interest rates set by the Fed

iii) Money supply set by the Fed

iv) Unemployment rates

v) The balance of payments account

vi) US national debt and annual budget deficits

vii) Trade agreements, tariffs, and duties set internationally

viii) Consumer savings and household income rates

We need to investigate the macroeconomic variables affecting a currency exchange rate to predict its trend. There are many variables that affect the currency exchange rates directly or indirectly, including

- **Quantitative Theory of Money-Demand and Supply of Money**

  The most common factor affecting foreign exchange rates is the demand and supply of money. Demand-supply theory helps to predict future exchange rates. We can predict the trend or direction of the changes in exchange rates by understanding the demand-supply framework. An exchange rate represents the relative price of currency. The Euro-US dollar, for instance , determines how many euros are needed to buy one US dollar, that is the price of a dollar in euros. The exchange market consists of international banks, multinational companies, speculators and other entities who wish to trade in currencies.

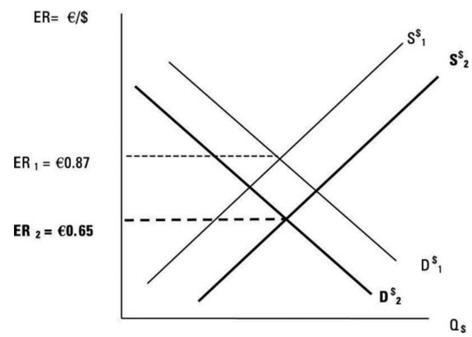


FIGURE 2.1. Demand-Supply of Euro-US dollar (Source: Dommies: A Wiley Brand)

There are certain factors that affect the demand for and supply of dollars in foreign exchange markets. Variables like inflation rates, interest rates, economic growth (GDP), and government restrictions in the US or Eurozone may effect the euro-dollar exchange rate. For example, high inflation in the US leads to the depreciation of the US dollar, and vice versa. Similarly, high interest rates appreciate exchange rates. Economic growth refers to an increase in a country's output, or real gross domestic product (real GDP). The demand-supply model predicts that the higher growth rate country's currency will depreciate. For example, if the Euro-zone's real GDP growth rate is higher than that of the U.S., this model predicts that the euro will depreciate.

The demand-supply model of exchange rate determination implies that the equilibrium exchange rate changes when the factors that affect the demand and supply conditions change.

- **Consumer Price Index (CPI)**

A Consumer Price Index (CPI) is a statistical estimate constructed using the prices of a sample of representative items whose prices are collected periodically. The CPI measures changes in the price level of a weighted average market basket of consumer goods and services purchased by households. The CPI is considered to be an important indicator for tracking price change, and this affects inflation rates. So most investors and economists consider the CPI as a crucial factor for changes in the price of a predetermined group of goods and services which are bought by households within a country. A rise in the CPI indicates a weakening in the purchasing power of the country's currency. Especially high inflation relative to inflation rates in other countries magnifies the effect of this factor.

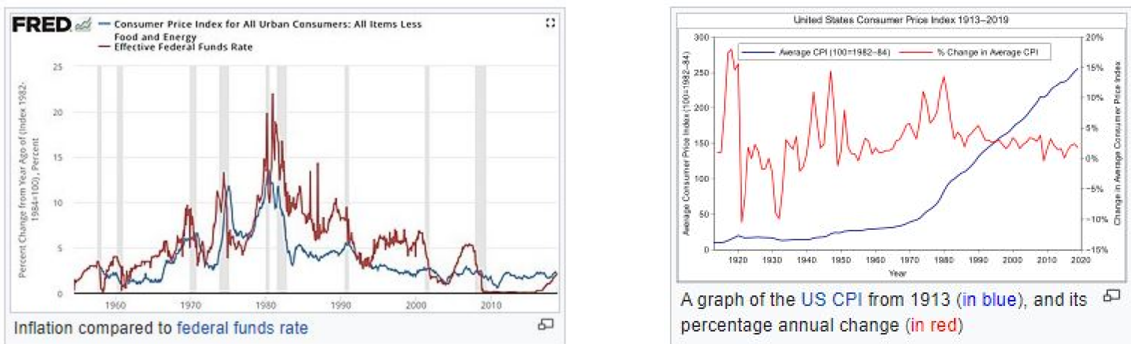


FIGURE 2.2. Consumer Price Index (Source: FRED)

- **Economic growth or Gross Domestic Product (GDP)**

  The gross domestic product (GDP) of any country is the monetary value of all goods and services which have been produced within that country in one year. The GDP strongly reflects the basic size of the country's economy. Changes in the GDP reveal changes in economic growth and can directly impact the relative value of a country's currency. A high GDP reflects larger production rates, an indication of a greater demand for that country's products. An increase in demand for a country's goods and services often translates into an increased demand for the country's currency.

- **Inflation**

  The rate of inflation in a country has a key role in determining a country's currency exchange rates. Low inflation rates support a rising currency value because the purchasing power increases relative to other currencies, whereas high inflation rates causes depreciation in currency value compared to the currencies of their trading partners. Thus, there is inverse relation between a country's inflation rates and exchange rates. It is mostly considered to be a negative factor because a very low rate of inflation does not guarantee a favorable exchange rate for a country whereas an extremely high inflation rate has big negative impact on a country's exchange rates.

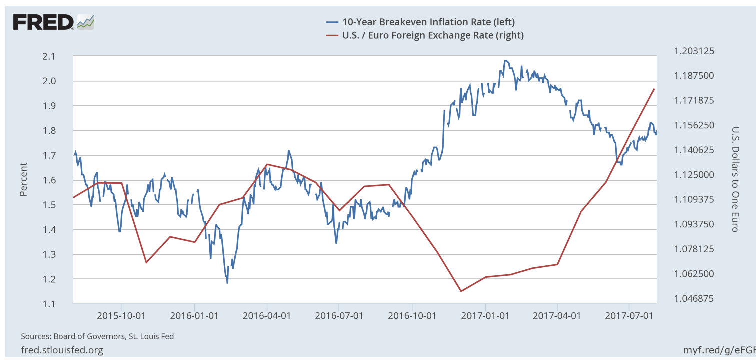


FIGURE 2.3. EUR/USD and US Inflation rate (Source: FRED)

- **Monetary Policy (Fed Funds Rate)**

  In the United States, the Federal Reserve (the country's central bank, usually just called the Fed) implements monetary policies to either strengthen or weaken the U.S. dollar. The Fed, for example, at the most basic level, implements 'easing' to lower interest rates just by purchasing bonds. Quantitative easing or simply 'easing' occurs when central banks reduce interest rates, encouraging investors to borrow money. Since the U.S. dollar is a fiat currency, meaning that it is not backed by any tangible commodity such as gold or silver, it can be created out of thin air. When more money is created, the law of supply and demand kicks in, making the existing money less valuable.

- **Interest Rates**

  The central bank interest rate is an important factor affecting a country's exchange rates. A country offering higher interest rates is usually more appealing to investors than a country offering relatively lower rates, and a decrease in interest rates lowers the cost of borrowing, which encourages businesses to increase investment spending.

- **Unemployment Rates**

  Unemployment rates are also an important indicator of a country's exchange rates. If a country has high employment rates, that indicates high demand of production of the country's goods and hence implies a high value of a country's currency. A higher demand for workers occurs if there is greater demand for products and services from a country, which surely implies that the country has more to export than import. These activities then cause more foreign currency to be exchanged in favor of the home country and causes the value of the country's currency to be higher.

- **Public (Government) Debt**

  Government debt is defined as public debt or national debt owed by the central government of the country. A high government debt shows that the country is less likely to acquire foreign capital and leads to inflation, and if inflation is high, the debt will be serviced and ultimately paid off with cheaper real dollars in the future. A large debt encourages inflation. In the worst case scenario, a government may print money to pay part of a large debt. But increasing the money supply inevitably causes inflation. The nations with large public deficits and debts are less attractive to foreign investors, and foreign investors will sell their bonds in the open market if the market predicts rising government debt within a certain country. As a result, a decrease in the value of its exchange rate will follow.

- **Balance of Trade/Payments**

  The balance of trade is the difference between exports and import of goods and the balance of payments is the difference between of the inflow and outflow of foreign exchange. A ratio comparing export prices to import prices, the terms of trade is related to current accounts and the balance of payments. If the price of a country's exports rises by a greater rate than that of its imports, its terms of trade have favorably improved. Increasing terms of trade shows greater demand for the country's exports. This, in turn, results in rising revenues from exports, which provides increased demand for the country's currency (and an increase in the currency's value). If the price of exports rises by a smaller rate than that of its imports, the currency's value will decrease in relation to its trading partners.

- **Producer Price Index (PPI)**

  The producer price index is another leading factor that affects the foreign exchange rates of a country. The PPI measures the average change in the sale price of all raw goods and services, and it examines these changes from the viewpoint of the producer and not the consumer. The PPI and CPI are obviously interrelated since increased producer costs are most often passed on to consumers. Also, a positive change in the PPI implies that costs are rising. There will also be an increase in the CPI, reflecting an increase in the level of prices.

- **Export Rate (Purchasing Power Parity)**

  Some economists regard the export rates of a country as one of the main indicators of foreign exchange rates, because when prices for a key export product fall, the currency can depreciate. For example, the Canadian dollar (known as the loonie) weakens when oil prices drop because oil is a major export product for Canada. Thus, we can see an inverse relationship between export rates and foreign exchange rates.

- **Current Account Deficit**

  The current account deficit is a measurement of a country's trade whereby the value of the goods and services it imports exceeds the value of the products it exports. The exchange rate exerts a significant influence on the trade balance, and by extension, on the current account. An overvalued currency makes imports cheaper and exports less competitive, thereby widening the current account deficit or narrowing the surplus.

2. Lotka-Volterra Models

Once we have isolated the macroeconomic variables affecting the exchange rate, we next develop a mathematical model to predict the trend of the exchange rate, which we call here the delay differential equations model. Here, we use several different concepts for formulating the relationship between the macroeconomic variables. One of the most important of these comes from the Lotka-Volterra model [**2**]. This is a system of equations, that is often used to simulate interactions between two or more populations. The first kind of interaction is competition, which refers to the possibility that an increase in one population is bad for the other populations: an example would be competition for food or habitat. The second type of interaction implies a direct relationship such as an increase in one population being good for the other: for example, owls are happy when the mouse population increases.

There are at least two cases of Lotka-Volterra models:

- Competing Species Model;
- Predator-Prey Model

Given the nature of certain macroeconomic variables, in appropriate situations we apply a predator-prey model in our case, and this model has the following assumptions:

(1) There is only one predator population x(t) and one prey population y(t), with no ecosystem limitations.

(2) There is a positive, constant predation rate β. In other words, there is no slowing of predation when prey is abundant, and no interference among predators.

(3) Predators have a positive, constant conversion rate, γ, of eaten prey into new offspring.

(4) The mortality rate of predators, $\alpha > 0$, is constant.

(5) In the absence of predators, the prey population, y(t), has a natural rate of increase, $\delta > 0$.

Using these assumptions, we can model a predator population in the absence of prey. Without a food source, it is expected that the predator population, x(t), will decrease exponentially, which can readily be described by the following equation:

$$\frac{dx}{dt} = -\alpha x,$$

where α is the predator mortality rate. However, in the presence of prey (food source), the decline is opposed by the predator birth rate, $\beta\gamma$xy, where, once again, y(t) is the prey population, β is the predation rate or searching efficiency, and γ is the predator's ability to convert food into offspring. Hence, we can describe the population dynamics of the predator by

$$\frac{dx}{dt} = -\alpha x + \beta\gamma xy.$$

Here, the product $\beta\gamma$x is known as the predator's numerical response.

In contrast, the prey population, y(t), is expected to increase exponentially without predation, which is modeled by the following equation:

$$\frac{dy}{dt} = \delta y,$$

where δ is the growth rate of the prey population. However, when predators are present, the consumption rate of the prey is βxy. Therefore, the dynamics of the prey population can be modeled by the following equation:

$$\frac{dy}{dt} = \delta y - \beta xy$$

Thus we obtain the Predator-Prey model:

$$\frac{dx}{dt} = -\alpha x + \beta\gamma xy$$

$$\frac{dy}{dt} = \delta y - \beta xy$$

Moreover, the Lotka-Volterra model predicts a cyclical relationship between the predator and prey populations, which is shown the figure 2.4 for hypothetical predator and prey population with $\alpha = -1$, $\beta = 0.01$, $\delta = 1$, and $\gamma = 2$. This can be explained by the fact that as the number of predators, x, increase so does the consumption rate, βxy, which in turn reinforces the increase in x. This increase in consumption rate, however, causes a decrease in the prey population, y, thus, forcing x, and consequently βxy, to decrease. As βxy decreases the prey population begins to recover, and y increase. As a result, x begins to increase, and the cycle restarts.

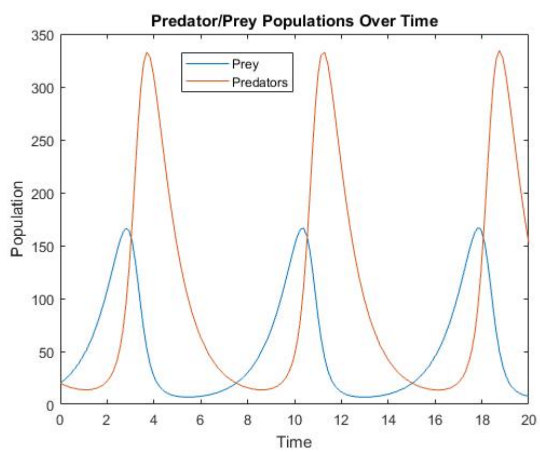


FIGURE 2.4. Cyclical relationship predicted between predators and prey

3. A Delay Differential Equation Model

In mathematics, delay differential equations (DDEs) are a type of differential equation in which the derivative of the unknown function at a certain time is given in terms of the values of the function at previous times. DDEs are also called time-delay systems. In our case, changes in the Fed funds rate cause changes that take effect after 20 months or so. Thus, under the circumstances it is reasonable that we use DDEs to predict the trend of the Forex rates.

Now, for the macroeconomic variables discussed above, we use the following notation:

$x_1(t)$ = Consumer Price Index (CPI)

$x_2(t)$ = Inflation

$x_3(t)$ = Economic Growth or Gross Domestic Product (GDP)

$x_4(t)$ = Monetary policy (Fed Funds Rates)

$x_5(t)$ = Interest Rates

$x_6(t)$ = Employment Rates

$x_7(t)$ = Public (Government) Debt

$x_8(t)$ = Balance of payment or Balance of trade

$x_9(t)$ = Producer Price Index (PPI)

$x_{10}(t)$ = Export Rate (or Purchasing Power Parity)

$x_{11}(t)$ = Currency Exchange Rates

To predict the trend of currency exchange rates, we need to formulate a system of differential equations based on the macroeconomic variables listed above.

First, using the definition of inflation as the relative price change in good and services, we see that

$$x_2(t) = a_1 \frac{x_1'(t)}{x_1(t)},$$

from which we obtain the first equation of our system (2.12):

$$x_1'(t) = a_1 x_1(t) x_2(t) \tag{2.1}$$

where x_1 is the CPI.

The inflation rate, x_2, can be represented by the following equation:

$$x_2'(t) = b_1 x_1(t) + b_2 x_2(t) x_4(t-20) + b_3 x_2(t) + b_4 x_5(t) \tag{2.2}$$

This equation consists of the definition of inflation, and the connection to the CPI, x_1. Also, we incorporate the predator-prey relationship between inflation, x_2, and the Fed funds rate with delay, $x_4(t-20)$, given that the Fed uses interest rate changes to counteract rising inflation.

The next equation models the Gross Domestic Product (GDP):

$$x_3'(t) = c_1 x_3(t) - c_2 x_5(t) + c_3 x_6(t) + c_4 x_3(t) x_5(t) \tag{2.3}$$

This equation involves the interest rate, x_5, and employment rate, x_6, and the predator-prey relationship of GDP and interest rates.

The next equation is for the Fed funds rate, which is given as

$$x_4'(t) = d_1 x_1'(t) + d_2 x_4(t) + d_3 x_6(t) + d_4 x_2(t) x_4(t-20). \tag{2.4}$$

Here, the first term in the equation stems from one known way the Fed decides whether to change interest rates. If the CPI, x_1, decreases (increases) too quickly, the Fed will lower (raise) the discount rate (and the federal funds rate, x_4, will change accordingly). The last part comes from the Lotka-Volterra (predator-prey) relationship between the Fed funds rate and inflation.

Now, we constitute the equation for the interest rate:

$$x_5'(t) = e_1 x_5(t) + e_2 x_3(t) x_5(t) + e_3 x_4(t) + e_4 x_5(t) x_6(t). \tag{2.5}$$

Since the interest rate is set by the central bank, there is a very important role played by the Fed funds rate, x_4 and the predator-prey relationship between interest rates and the Fed funds rates and between interest rates and employment rates.

The next equation models the employment rate:

$$x_6'(t) = -f_1 x_6(t) + f_2 x_3(t) + f_3 x_4(t) - f_4 x_5(t) x_6(t). \tag{2.6}$$

As the employment is determined by the GDP and sometimes by the Fed funds rate, we have the macroeconomic variables x_3 and x_4 in equation (2.6). Also the last term in the equation comes from a predator-prey relationship.

The next equation is about public debt:

$$x_7'(t) = -g_1 x_7(t) + g_2 x_2(t) + g_3 x_6(t) \tag{2.7}$$

This equation shows how public debt is related to inflation, x_2, and the employment rate, x_6.

The following equation models the balance of payment:

$$x_8'(t) = h_1 x_8(t) + h_2 x_3(t) + h_3 x_{10}(t), \tag{2.8}$$

given that the balance of payment is related to the GDP, x_3 and also to the export rate, x_{10}.

The changes in Producer Price Index (PPI) is lead to the following equation

$$x_9'(t) = m_1 x_9(t) + m_2 x_1(t) + m_3 x_2(t) + m_4 x_1(t) x_9(t), \tag{2.9}$$

where, the second and third terms are the CPI, x_1, and inflation, x_2, respectively as these variables have a leading role in determining the PPI. The last term in equation (2.9) represents a predator-prey relationship.

Finally, we consider the export rate:

$$x'_{10}(t) = n_1 x_{10}(t) + n_2 x_3(t) + n_3 x_8(t), \tag{2.10}$$

which is suggested by the fact that the export rates depend on the GDP of the country and the balance of payment.

Many macroeconomic variables affect the currency exchange rate, however, relatively few of them have a direct relationship with the exchange rate, as one can see from the following equation:

$$x'_{11}(t) = n_1 x_2(t) + n_2 x_3(t) + n_3 x_{11}(t) + n_4 x_5(t) x_{11}(t) + n_5 x_7(t). \tag{2.11}$$

Here, the terms $x_5(t)$ and $x_{11}(t)$ come from a predator-prey relationship, and the other variables inflation, x_2, GDP, x_3, interests rates, x_5 and public debt, x_7 have a direct impact on the currency exchange rate. Therefore, in summary our system of delay differential equations is as follows:

$$
\begin{aligned}
x_1'(t) &= a_1x_1(t)x_2(t) \\
x_2'(t) &= b_1x_1(t) + b_2x_2(t)x_4(t-20) + b_3x_2(t) + b_4x_5(t) \\
x_3'(t) &= c_1x_3(t) - c_2x_5(t) + c_3x_6(t) + c_4x_3(t)x_5(t) \\
x_4'(t) &= d_1x_1'(t) + d_2x_4(t) + d_3x_6(t) + d_4x_2(t)x_4(t-20) \\
x_5'(t) &= e_1x_5(t) + e_2x_3(t)x_5(t) + e_3x_4(t) + e_4x_5(t)x_6(t) \\
x_6'(t) &= f_1x_6(t) + f_2x_3(t) + f_3x_4(t) + f_4x_5(t)x_6(t) \qquad (2.12) \\
x_7'(t) &= g_1x_7(t) + g_2x_2(t) + g_3x_6(t) \\
x_8'(t) &= h_1x_8(t) + h_2x_3(t) + h_3x_{10}(t) \\
x_9'(t) &= l_1x_9(t) + l_2x_1(t) + l_3x_2(t) + l_4x_1(t)x_9(t) \\
x_{10}'(t) &= m_1x_{10}(t) + m_2x_3(t) + m_3x_8(t) \\
x_{11}'(t) &= n_1x_2(t) + n_2x_3(t) + n_3x_{11}(t) + n_4x_5(t)x_{11}(t) + n_5x_7(t)
\end{aligned}
$$

4. Existence of a Unique Solution

We verify that the requirements in the following theorem 4.1 [**3**] for the existence of a unique solution are satisfied before we solve the delay differential system numerically using MATLAB.

THEOREM 4.1. *Let $F : [t_0, \beta) \times \mathscr{C}_D \to \mathbb{R}^n$ be continuous and locally Lipschitzian. Then, for each $\phi \in \mathscr{C}_D$, the initial value problem*

$$x'(t) = F(t, x_t), \quad x_{t_0} = \phi$$

has a unique solution on $[t_0 - r, t_0 + \delta]$ for some $\delta > 0$, where $r, t_0 \in \mathbb{R}$, $t_0 < \beta \leq \infty$, D is an open set in $\mathbb{R}^n$, $\mathscr{C}_D = \mathscr{C}([-r,0], D)$ for $r \geq 0$, and $\mathscr{C}([-r,0],\mathbb{R}^n))$ is the space of continuous mapping $[-r,0] \to \mathbb{R}^n$.

For this, we need only a function $F(t, x_t)$ in (2.12) that is continuous and locally Lipschitzian. Continuity is easily satisfied as each component of F is a linear combination of products and compositions of continuous functions. Consider $F(t, x_t) = f(t, x(g_1(t)), \ldots, x(g_m(t)))$, where $x(g_i(t)) = x_t(g_i(t) - t)$, and $t - r \leq g_i(t) \leq t$ for $t \geq t_0$. Now, to prove that F is locally Lipschitzian, it is sufficient to show that f is continuously differentiable that is f has continuous first partial derivatives with respect to all of its dependent arguments. Again consider that the function f defined as:

$$f(t, x_1, x_2, \ldots, x_{11}, \psi) = \begin{bmatrix} a_1 x_1 x_2 \\ b_1 x_1 + b_2 x_2 \psi + b_3 x_2 + b_4 x_5 \\ c_1 x_3 - c_2 x_5 + c_3 x_6 + c_4 x_3 x_5 \\ a_1 d_1 x_1 x_2 + d_2 x_4 + d_3 x_6 + d_4 x_2 \psi \\ e_1 x_5 + e_2 x_3 x_5 + e_3 x_4 + e_4 x_5 x_6 \\ f_1 x_6 + f_2 x_3 + f_3 x_4 + f_4 x_5 x_6 \\ g_1 x_7 + g_2 x_2 + g_3 x_6 \\ h_1 x_8 + h_2 x_3 + h_3 x_{10} \\ l_1 x_9 + l_2 x_1 + l_3 x_2 + l_4 x_1 x_9 \\ m_1 x_{10} + m_2 x_3 + m_3 x_8 \\ n_1 x_2 + n_2 x_3 + n_3 x_{11} + n_4 x_5 x_{11} + n_5 x_7 \end{bmatrix}$$

where $\psi = x_4(t - 20)$.

Now, we calculate the partial derivatives as given below:

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_{11}} \right) =$$

$$\begin{pmatrix} a_1x_2 & a_1x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_1 & b_2\psi+b_3 & 0 & 0 & b_4 & 0 & 0 & 0 & 0 & 0 & b_2x_2 \\ 0 & 0 & c_1+c_4x_5 & 0 & -c_2+c_4x_3 & c_3 & 0 & 0 & 0 & 0 & 0 \\ a_1d_1x_2 & a_1d_1x_1+d_4\psi & 0 & d_2 & 0 & d_3 & 0 & 0 & 0 & 0 & d_4x_2 \\ 0 & 0 & e_2x_5 & e_3 & e_1+e_2x_3+e_4x_6 & e_4x_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & f_2 & f_3 & f_4x_6 & f_1f_4x_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & g_2 & 0 & 0 & 0 & g_3 & g_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & h_2 & 0 & 0 & 0 & 0 & h_1 & 0 & h_3 & 0 \\ l_2+l_4x_9 & l_3 & 0 & 0 & 0 & 0 & 0 & 0 & l_1+l_4x_1 & 0 & 0 \\ 0 & 0 & m_2 & 0 & 0 & 0 & 0 & m_3 & 0 & m_1 & 0 \\ 0 & n_1 & n_2 & 0 & n_4x_{11} & 0 & n_5 & 0 & 0 & 0 & n_3+n_4x_5 \end{pmatrix}$$

which shows that the first partial derivatives of f with respect to all dependent variables are continuous because each component of each derivative of f is continuous. Hence f is continuously differentiable, and therefore F is locally Lipschitzian. This proves that there exists a unique solution of the system of delay differential equations. Next, we we determine the coefficients of the system of delay differential equations.

5. Solving for the Coefficients

We use a least squares method to recover the missing coefficients, from the construction of eleven linear systems corresponding to each equation listed in (2.12). For this, we use past economic data from the variables $x_1, x_2, ..., x_{11}$ to obtain the coefficients. Given that the factors affecting foreign exchange tend to operate on different time scales, we interpolate the data down to daily data.

In order to avoid numerical instabilities associated with computing derivative values, we integrate the each equation over the interval $[t_{k-1}, t_k]$ where $1 \le k \le N$, obtain the system:

$$
\begin{aligned}
x_1(t_k)-x_1(t_{k-1}) &= a_1\int_{t_{k-1}}^{t_k} x_1(t)x_2(t)dt\\
x_2(t_k)-x_2(t_{k-1}) &= b_1\int_{t_{k-1}}^{t_k} x_1(t)dt+b_2\int_{t_{k-1}}^{t_k} x_2(t)x_4(t-20)dt+b_3\int_{t_{k-1}}^{t_k} x_2(t)dt+b_4\int_{t_{k-1}}^{t_k} x_5(t)dt\\
x_3(t_k)-x(t_{k-1}) &= c_1\int_{t_{k-1}}^{t_k} x_3(t)dt-c_2\int_{t_{k-1}}^{t_k} x_5(t)dt+c_3\int_{t_{k-1}}^{t_k} x_6(t)dtc_4\int_{t_{k-1}}^{t_k} x_3(t)x_5(t)dt\\
x_4(t_k)-x_4(t_{k-1}) &= d_1(x_1(t_k)-x_1(t_{k-1}))+d_2\int_{t_{k-1}}^{t_k} x_4(t)dt+d_3\int_{t_{k-1}}^{t_k} x_6(t)dt+d_4\int_{t_{k-1}}^{t_k} x_2(t)x_4(t-20)dt\\
x_5(t_k)-x_5(t_{k-1}) &= e_1\int_{t_{k-1}}^{t_k} x_5(t)dt+e_2\int_{t_{k-1}}^{t_k} x_3(t)x_5(t)dt+e_3\int_{t_{k-1}}^{t_k} x_4(t)dt+e_4\int_{t_{k-1}}^{t_k} x_5(t)x_6(t)dt\\
x_6(t_k)-x_6(t_{k-1}) &= f_1\int_{t_{k-1}}^{t_k} x_6(t)dt+f_2\int_{t_{k-1}}^{t_k} x_3(t)dt+f_3\int_{t_{k-1}}^{t_k} x_4(t)dt+f_4\int_{t_{k-1}}^{t_k} x_5(t)x_6(t)dt \qquad (2.13)\\
x_7(t_k)-x_7(t_{k-1}) &= g_1\int_{t_{k-1}}^{t_k} x_7(t)dt+g_2\int_{t_{k-1}}^{t_k} x_2(t)dt+g_3\int_{t_{k-1}}^{t_k} x_6(t)dt\\
x_8(t_k)-x_8(t_{k-1}) &= h_1\int_{t_{k-1}}^{t_k} x_8(t)dt+h_2\int_{t_{k-1}}^{t_k} x_3(t)dt+h_3\int_{t_{k-1}}^{t_k} x_{10}(t)dt\\
x_9(t_k)-x_9(t_{k-1}) &= l_1\int_{t_{k-1}}^{t_k} x_9(t)dt+l_2\int_{t_{k-1}}^{t_k} x_1(t)dt+l_3\int_{t_{k-1}}^{t_k} x_2(t)dt+l_4\int_{t_{k-1}}^{t_k} x_1(t)x_9(t)dt\\
x_{10}(t_k)-x_{10}(t_{k-1}) &= m_1\int_{t_{k-1}}^{t_k} x_{10}(t)dt+m_2\int_{t_{k-1}}^{t_k} x_3(t)dt+m_3\int_{t_{k-1}}^{t_k} x_8(t)dt\\
x_{11}(t_k)-x_{11}(t_{k-1}) &= n_1\int_{t_{k-1}}^{t_k} x_2(t)dt+n_2\int_{t_{k-1}}^{t_k} x_3(t)dt+n_3\int_{t_{k-1}}^{t_k} x_{11}(t)dt+n_4\int_{t_{k-1}}^{t_k} x_5(t)x_{11}(t)dt+n_5\int_{t_{k-1}}^{t_k} x_7(t)dt
\end{aligned}
$$

Note that the constants in this model should be viewed as averages representing the system over $[t_0,t_N]$. In this model, each sub-interval $[t_{k-1},t_k]$ is one month. We chose to consider the previous seven months to determine the coefficients because it provided the best predictions and insured that we did not have any undetermined systems since each equation in (2.12) has at most five unknowns. Therefore, for each equation in (2.14) we have a least squares problem similar to the following, which solves for the coefficients:

$$
M=\begin{bmatrix}
\int_{t_0}^{t_1} x_2(t)dt & \int_{t_0}^{t_1} x_3(t)dt & \int_{t_0}^{t_1} x_{11}(t)dt & \int_{t_0}^{t_1} x_5(t)x_{11}(t)dt & \int_{t_0}^{t_1} x_7(t)dt\\
\int_{t_1}^{t_2} x_2(t)dt & \int_{t_1}^{t_2} x_3(t)dt & \int_{t_1}^{t_2} x_{11}(t)dt & \int_{t_1}^{t_2} x_5(t)x_{11}(t)dt & \int_{t_1}^{t_2} x_7(t)dt\\
\int_{t_2}^{t_3} x_2(t)dt & \int_{t_2}^{t_3} x_3(t)dt & \int_{t_2}^{t_3} x_{11}(t)dt & \int_{t_2}^{t_3} x_5(t)x_{11}(t)dt & \int_{t_2}^{t_3} x_7(t)dt\\
\int_{t_3}^{t_4} x_2(t)dt & \int_{t_3}^{t_4} x_3(t)dt & \int_{t_3}^{t_4} x_{11}(t)dt & \int_{t_3}^{t_4} x_5(t)x_{11}(t)dt & \int_{t_3}^{t_4} x_7(t)dt\\
\int_{t_4}^{t_5} x_2(t)dt & \int_{t_4}^{t_5} x_3(t)dt & \int_{t_4}^{t_5} x_{11}(t)dt & \int_{t_4}^{t_5} x_5(t)x_{11}(t)dt & \int_{t_4}^{t_5} x_7(t)dt
\end{bmatrix}
$$

Hence, we solve the following system:

$$M\begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \end{bmatrix} = \begin{bmatrix} x_{11}(t_1)-x_{11}(t_0) \\ x_{11}(t_2)-x_{11}(t_1) \\ x_{11}(t_3)-x_{11}(t_2) \\ x_{11}(t_4)-x_{11}(t_3) \\ x_{11}(t_5)-x_{11}(t_4) \end{bmatrix}$$

for coefficients n_1, n_2, n_3, n_4, and n_5. This process is completed for each equation of the system (2.12) thereby obtaining all of the coefficients. However, some of the variables only apply to a specific time period. In such a case, we define the variable representing particular factor to be zero for that a particular time, which causes one of the columns of the matrix to be zero, and results in a singular matrix. It implies the the coefficient of the variable representing the particular factor can either no longer be found or is not reliable as it causes the problem to be become ill-posed. For this reason, we assume that the coefficient of the variable itself zero and obtain another matrix, and thereby solve another system of same kind. We repeat this process for other variables as well if any of the variables mentioned above is absent in any equation of the system (2.12). Hence, we solve the remaining coefficients using the method of least squares by constructing the following system:

$$\begin{bmatrix} \int_{t_0}^{t_1} x_2(t)dt & \int_{t_0}^{t_1} x_3(t)dt & \int_{t_0}^{t_1} x_{11}(t)dt & \int_{t_0}^{t_1} x_5(t)x_{11}(t)dt & \int_{t_0}^{t_1} x_7(t)dt \\ \int_{t_1}^{t_2} x_2(t)dt & \int_{t_1}^{t_2} x_3(t)dt & \int_{t_1}^{t_2} x_{11}(t)dt & \int_{t_1}^{t_2} x_5(t)x_{11}(t)dt & \int_{t_1}^{t_2} x_7(t)dt \\ \int_{t_2}^{t_3} x_2(t)dt & \int_{t_2}^{t_3} x_3(t)dt & \int_{t_2}^{t_3} x_{11}(t)dt & \int_{t_2}^{t_3} x_5(t)x_{11}(t)dt & \int_{t_2}^{t_3} x_7(t)dt \\ \int_{t_3}^{t_4} x_2(t)dt & \int_{t_3}^{t_4} x_3(t)dt & \int_{t_3}^{t_4} x_{11}(t)dt & \int_{t_3}^{t_4} x_5(t)x_{11}(t)dt & \int_{t_3}^{t_4} x_7(t)dt \\ \int_{t_4}^{t_5} x_2(t)dt & \int_{t_4}^{t_5} x_3(t)dt & \int_{t_4}^{t_5} x_{11}(t)dt & \int_{t_4}^{t_5} x_5(t)x_{11}(t)dt & \int_{t_4}^{t_5} x_7(t)dt \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \end{bmatrix} = \begin{bmatrix} x_{11}(t_1)-x_{11}(t_0) \\ x_{11}(t_2)-x_{11}(t_1) \\ x_{11}(t_3)-x_{11}(t_2) \\ x_{11}(t_4)-x_{11}(t_3) \\ x_{11}(t_5)-x_{11}(t_4) \end{bmatrix}$$

The same process is applied for finding each coefficient of each equation of system (2.12). To determine the coefficients of the system, we collected data starting from January 31, 2000 using Bloomberg terminals provided by the Collat School of Business, University of Alabama at Birmingham. A Bloomberg terminal is a resource, from which we can get the data for any macroeconomic variable at any time. Among the eleven variables, public debt and GDP are taken from 3/31/2000, and exchange rate data is taken from 8/31/2009. So we have divided the data into two parts: old data from 2000 to 2008, and new data from 2008.

Some of the variables, such as the Fed rate and the exchange rate are reported daily, on the other hand the CPI is reported monthly and GDP is reported quarterly. In such cases, extrapolation is necessary. Since there would be no data when a month has just ended, and the new data has not yet been released, we use extrapolation to predict the value at the missing times. For example, the CPI from November 2018 was released on December 12, 2018. So predicting exchange rate for November and December required extrapolating the CPI from October. The CPI for December 2018 was predicted to be 231.105428, using the function f(x)=0.009881x+211.7, which is shown in Figure 2.5. The actual CPI for December 2018 was later reported as 252.723 which is only 0.64 percent different from our extrapolated value.

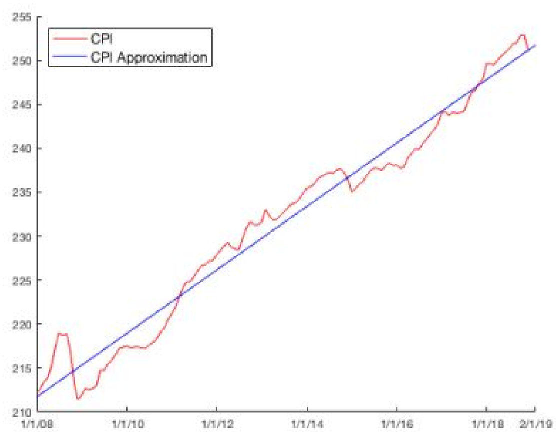


FIGURE 2.5. Extrapolating the CPI two months past the last reported value

Since some data such as GDP is only reported quarterly, we are not always able to linearly interpolate between the financial quarters when we wish to predict the future exchange rates. Therefore, if we want to predict the the exchange rate for August 2020, our last reported data would be from the financial quarter that ends on

April 30,2020. Therefore, we would not have any future data to allow us to linearly interpolate and obtain an initial condition for our system. To remedy this problem, we use several methods. First, we utilize projections that are made about Forex rates. Then we extrapolate exchange rates using the guidance/assumptions provided. Once data is available, we can compare it to our predictions. Regardless of the fact that some of the errors between our extrapolated predictions and actual values are quite large, it is important to note that investors are also basing their investment strategy on assumptions similar to ours. Therefore, we are still able to accurately forecast the trend using the extrapolated values.

CHAPTER 3

The Inverse Volatility Problem for Foreign Exchange Rate Options

We know that in transactions associated with future-oriented financial instruments, such as options and other financial derivatives, a huge amount of data is available buried inside of which is the market's best guess as to what the future holds. We consider here with the possibility of extracting future volatility information from this type of data with the aid of certain computational inverse algorithms.

It is common to model a financial entity, such as a foreign exchange (FX) rate, or a stock, via a stochastic differential equation

$$\frac{dS_t}{S_t} = \mu(S_t, t)dt + \sigma(S_t, t)dB_t, \tag{3.1}$$

where, for each time t, $S_t(\omega)$ is a random variable representing the price of the financial entity for the sample path ω, μ is the drift, which relates to the "trend" of the entity, σ is the volatility ("wobble"), and $B_t(\omega)$ is the Brownian motion stochastic process used to model this log-normal randomness.

Financial derivatives are contracts that derive their value from such an underlying entity. In particular, a *European call option* on a stock, is a derivative financial instrument that, when purchased, gives the holder the right but not the obligation to *buy* a designated number of stock shares at a pre-agreed price (the strike price) on a specified date, and a *European put option* provides the right to *sell* under the same circumstances.

In their Nobel Prize winning paper [**4**] Black and Scholes showed that, for stock options and under certain rather severe restrictions, the arbitrage-free price of a European option contract, $u(S,t)$, satisfies a deterministic PDE of diffusion type in time t and the value S of the underlying asset:

$$\frac{\partial u}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 u}{\partial S^2}+\mu s\frac{\partial u}{\partial S}-ru=0, \tag{3.2}$$

where σ is the (assumed constant) volatility, μ is the risk-neutral drift, and r is the short-term interest rate. For practical purposes the latter may be taken to be the interest rate on a 13-week US government treasury bill.

Many authors over the years [**5, 6, 7, 8, 9, 10, 11**] have noted that several of the assumptions laid down by Black and Scholes are basically incompatible with market data. Notable among these are objections to the constancy of σ. As recently as the early nineteen eighties, assertions such as "the Black-Scholes volatility is constant" seemed to hold true, at least while the market believed it so; but then, after the crash of 1987, volatility was anything but constant, and in fact it has recently become fashionable to speak of (and invest in) the volatility of the volatility! So it is common to regard the volatility as a function of S and t, $\sigma=\sigma(S,t)$.

In the case of FX options, a *European call option* on a foreign exchange rate (currency) pair, is a derivative financial instrument that, when purchased, gives the holder the right but not the obligation to *buy* a designated amount of money, denominated in one currency, in another currency at a pre-agreed exchange rate on a specified date, and a *European put option* provides the right to *sell* under the same circumstances.

Assume that S is the spot price of the deliverable currency (domestic units per foreign unit), K is the exercise price of the option (domestic units per foreign unit), t is the current time, $u(S,t)$ is the current price of an FX call option (domestic units per foreign unit), r_D is the domestic (riskless) interest rate, r_F is the foreign (riskless) interest rate, and σ is volatility of the spot currency price. The corresponding Black Scholes partial differential equation for this case was provided by Garman and

Kohlhagen [**12**] in their classic 1983 paper:

$$\frac{\partial v}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 v}{\partial S^2}+(r_D-r_F)S\frac{\partial v}{\partial S}-r_D v=0, \tag{3.3}$$

where we have replaced the time to maturity "T" in [**12**, equ. 6] with the current time t, and we note the final condition

$$v(S,T)=\begin{cases} S-K, \text{ if } S>K \\ 0, \text{ if } S\le K, \end{cases}$$

and the boundary condition

$$v(0,t)=0.$$

1. Development of the Garman Kohlhagen PDE

Before we develop Garman Kohlhagen equation (3.3), we need to derive Ito's formula:

Ito's Formula: Assume X_t is an Ito's drift-diffusion process that satisfies the stochastic differential equation

$$dX_t=\mu_t dt+\sigma_t dB_t \tag{3.4}$$

where, B_t is a Wiener-process. If f(t,x) is a twice-differentiable scalar function, its expansion in a Taylor series given as

$$df=\frac{\partial f}{\partial t}dt+\frac{\partial f}{\partial x}dx+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}dx^2+\cdots.$$

Now, we substitute X_t for x and $\mu_t dt+\sigma_t dB_t$ for dx, we get

$$df=\frac{\partial f}{\partial t}dt+\frac{\partial f}{\partial x}(\mu_t dt+\sigma_t dB_t)+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}\left(\mu_t^2 dt^2+2\mu_t\sigma_t dt\,dB_t+\sigma_t^2 dB_t^2\right)+\cdots.$$

When dt → 0, the terms dt^2 and $dtdB_t$ tend to zero faster than dB^2, which is $O(dt)$. Setting the dt^2 and $dtdB_t$ terms to zero, substituting dt for dB^2 (due to the variance of a Wiener process), and collecting the dt and dB terms, we obtain the Ito formula as

$$df = \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{\sigma_t^2}{2}\frac{\partial^2 f}{\partial x^2}\right) dt + \sigma_t \frac{\partial f}{\partial x} dB_t \tag{3.5}$$

We compare equation (3.5) with the stochastic differential equation for option price:

$$dS = \mu S dt + \sigma S dB_t$$

Set $X_t = S$, $\mu_t = \mu S$, $\sigma_t = \sigma S$, and substitute these to Ito formula above to obtain

$$df = \left(\frac{\partial f}{\partial t} + \mu S \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right) dt + \sigma S \frac{\partial f}{\partial S} dB_t, \tag{3.6}$$

which is the Ito formula for option pricing (or exchange rates in our case).

Here, we now derive the Garman Kohlhagen PDE for currency options and apply it to the recovery of exchange rates, volatility recovery. For this, we compare the Ito formula and the stochastic differential equations with option prices as:

$$dv = \left(\frac{\partial v}{\partial t} + \mu S \frac{\partial v}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 v}{\partial S^2}\right) dt + \sigma S \frac{\partial v}{\partial S} dB_t$$

and

$$dv = \alpha v dt + \theta v dB_t$$

where α is the expected rate of return on a security and θ is the standard deviation of the security rate of return. We obtain,

$$\alpha = \frac{1}{v}\left(\frac{\partial v}{\partial t} + \mu S \frac{\partial v}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 v}{\partial S^2}\right)$$

and

$$\theta = \frac{1}{v}\sigma S \frac{\partial v}{\partial S}.$$

Next, since the risk-adjusted expected excess returns of securities governed by our assumptions must be identical in an arbitrage-free continuous-time economy, that is, letting $C(S,t)$ be the price of European call option at time t

$$\frac{\alpha - r_D}{\theta} = \lambda$$

where λ does not depend on the security considered. We apply this fact to the ownership of foreign currency, we get,

$$\frac{(\mu + r_F) - r_D}{\sigma} = \lambda$$

By comparison of above two models, we get,

$$\frac{(\mu + r_F) - r_D}{\sigma} = \frac{\alpha - r_D}{\theta}$$

After substituting the values of α and θ, we get the Garman-Kohlhagen PDE for the FX option C:

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 v}{\partial S^2} + (r_D - r_F) S \frac{\partial v}{\partial S} - r_D v = 0. \tag{3.7}$$

Now, financial data specifying the market price of an option is readily available in quantity at various strike values K around the current price of the underlying (the "spot" price), and for values of the maturity T up to around twelve months or so into the future. Given that the computer projections currently used by most stock analysts are only valid for a week or so into the future, one is led quite naturally to the so-called *inverse volatility problem*: determine a market-inspired estimate of the future volatility function $\sigma(S,t)$ from a knowledge of current market prices of options with different strikes and future maturities.

2. Derivation of Dupire's Equation for Foreign Exchange Rates

For FX options, we now derive a Dupire-type equation for the price $v(S,t;K,T)$ of a European FX call option. There are different approaches for deriving the Dupire equation, however we adapt the approach devised by Isakov and Bouchouev [**13**]. Let $v = v(S,t;K,T)$ be the solution of the Garman-Kohlhagen PDE (3.7) satisfying the final condition

$$v(S,T)=\begin{cases} S-K, \text{ if } S>K \\ 0, \text{ if } S\le K, \end{cases}$$

and the boundary condition

$$v(0,t)=0.$$

Now, we set

$$v_{1,\epsilon}(S,t;K,T)=\frac{1}{\epsilon}[v(S,t;K+\epsilon,T)-v(S,t;K,T)] \tag{3.8}$$

Then, $v_{1,\epsilon}$ also satisfies the Black-Scholes PDE (3) in the variables S and t for any $\epsilon\neq 0$. Also,

$$\begin{aligned} v_{1,\epsilon}(S,T;K,T) &= \frac{1}{\epsilon}[(S-(K+\epsilon))^+-(S-K)^+] \\ &= \begin{cases} 0, \text{ if S} \in \text{(0,K)} \\ \frac{1}{\epsilon}(K-S), \text{ if } S\in[K,K+\epsilon] \\ -1, \text{if S } S\in(K+\epsilon,\infty) \end{cases} \end{aligned}$$

Using the maximum principle, for K and T fixed

$$-1<v_{1,\epsilon}(S,t;K,T)<0 \text{ on } \mathbb{R}^+\times(t,T)$$

Thus $v_{1,\epsilon}(S,t;K,T)$ is uniformly bounded and equicontinuous in ϵ. By the Arzela-Ascoli theorem there exists a limit of $v_{1,\epsilon}(S,t;K,T)$ when $\epsilon\to 0$ that is the first derivative of v with respect to K, v_K, exists and

$$v_K(S,T;K,T)=-H(S-K) \tag{3.9}$$

where H(S) is the Heaviside step function. In similar fashion, set

$$v_{2,\epsilon}(S,t;K,T)=\frac{1}{\epsilon^2}[v(S,t;K-\epsilon,T)-2v(S,t;K,T)+v(S,t;K+\epsilon,T)] \tag{3.10}$$

Again, $v_{2,\epsilon}$ also satisfies the Garman-Kohlhagen PDE (3.7) in the variables S and t for any $\epsilon \neq 0$. If t=T in $v_{2,\epsilon}(S,t;K,T)$ then

$$v_{2,\epsilon}(S,T;K,T) = \frac{1}{\epsilon^2}[(S-(K-\epsilon))^+ - 2(S-K)^+ + (S-(K+\epsilon)^+]$$

$$= \begin{cases} 0, \text{ if } S \in (0, K-\epsilon) \\ \frac{1}{\epsilon^2}(\epsilon + S - K), \text{ if } S \in [K-\epsilon, K] \\ \frac{1}{\epsilon^2}(\epsilon - S + K), \text{ if } S \in (K, K+\epsilon] \\ 0, \text{if S } S \in (K+\epsilon, \infty) \end{cases}$$

where the right hand side is a 'hat' function. Again, using the the maximum principle, $v_{2,\epsilon}(S,t;K,T)$ is uniformly bounded and equicontinuous in ϵ for $t < T$. By the Arzela-Ascoli theorem there exists a limit of $v_{2,\epsilon}(S,t;K,T)$ when $\epsilon \to 0$ i.e. the second derivative of v with respect to K, v_{KK} exists. Define the function $G(S,t;K,T)$ by

$$G(S,t;K,T) = v_{KK}(S,t;K,T) \tag{3.11}$$

Thus, $G(S,t;K,T)$ also satisfies the Garman-Kohlhagen PDE (3.7).

Proposition: The function $G(S,T;K,T)$ is the Dirac Delta function concentrated at K i.e.

$$G(S,T;K,T) = \delta(S-K)$$

Proof: Let $\phi(p)$ be a continuously differentiable function on $[K-\epsilon, K+\epsilon]$. Now,

$$\begin{aligned} & \int_{-\infty}^{\infty} G(p,T;K,T)\phi(p)dp \\ = & \int_{K-\epsilon}^{K} \lim_{\varepsilon\to 0}\left(\frac{p}{\epsilon^2} - \frac{K}{\epsilon^2} + \frac{1}{\epsilon}\right)\phi(p-\epsilon)dp + \int_{K}^{K+\epsilon} \lim_{\varepsilon\to 0}\left(-\frac{p}{\epsilon^2} + \frac{K}{\epsilon^2} + \frac{1}{\epsilon}\right)\phi(p)dp \end{aligned}$$

by lebesgue dominated convergence theorem and change of variables, we get,

$$
\begin{aligned}
&= \lim_{\epsilon\to 0}\int_{K}^{K+\epsilon}\left(\frac{p}{\epsilon^2}-\frac{K}{\epsilon^2}\right)\phi(p-\epsilon)dp+\lim_{\epsilon\to 0}\int_{K}^{K+\epsilon}\left(-\frac{p}{\epsilon^2}+\frac{K}{\epsilon^2}+\frac{1}{\epsilon}\right)\phi(p)dp \\
&= \lim_{\epsilon\to 0}\int_{K}^{K+\epsilon}(K-p)\frac{\phi(p)-\phi(p-\epsilon)}{\epsilon^2}dp+\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{K}^{K+\epsilon}\phi(p)dp
\end{aligned}
$$

Using mean value theorem and boundedness of $\phi'(p)$, one can show that the first limit goes to zero. So,

$$
\int_{-\infty}^{\infty} G(p,T;K,T)\phi(p)dp = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{K}^{K+\epsilon}\phi(p)dp
$$

By the extreme value theorem, there exists a minimum m and a maximum M such that

$$
m \le \phi(p) \le M \text{ for all } p \in [K, K+\epsilon].
$$

Now, by the sandwich theorem,

$$
\int_{-\infty}^{\infty} G(p,T;K,T)\phi(p)dp = \phi(K).
$$

Thus we obtain,

$$
G(S,T;K,T) = \delta(S-K)
$$

THEOREM 2.1. *The function $G(S,t;K,T)$ defined above is the fundamental solution of the Garman-Kohlhagen PDE.*

PROOF.

$$
\begin{aligned}
& G(S,t;K,T) \\
&= \lim_{\varepsilon\to 0} v_{2,\epsilon}(S,t;K,T) \\
&= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^+}\Gamma(S,t;p,T)\phi_\epsilon(p)dp \quad \text{(by theorem 12 in [\textbf{14}])}
\end{aligned}
$$

where $\Gamma(S,t;p,T)$ is the fundamental solution of the Garman-Kohlhagen PDE

$$
= \lim_{\varepsilon\to 0}\int_{K-\epsilon}^{K}\Gamma(S,t;p,T)\left(\frac{p}{\epsilon^2}-\frac{K}{\epsilon^2}+\frac{1}{\epsilon}\right)dp
$$

$$+\lim_{\varepsilon\to 0}\int_K^{K+\epsilon}\Gamma(S,t;p,T)\left(-\frac{p}{\epsilon^2}+\frac{K}{\epsilon^2}+\frac{1}{\epsilon}\right)dp$$

by change of variable we get,

$$\begin{aligned}
&= \lim_{\varepsilon\to 0}\int_K^{K+\epsilon}\Gamma(S,t;p-\epsilon,T)\left(\frac{p}{\epsilon^2}-\frac{K}{\epsilon^2}\right)dp\\
&\quad +\lim_{\varepsilon\to 0}\int_K^{K+\epsilon}\Gamma(S,t;p,T)\left(-\frac{p}{\epsilon^2}+\frac{K}{\epsilon^2}+\frac{1}{\epsilon}\right)dp\\
&= \lim_{\varepsilon\to 0}\int_K^{K+\epsilon}\frac{\Gamma(S,t;p-\epsilon,T)-\Gamma(S,t;p,T)}{\epsilon^2}(p-K)\,dp\\
&\quad +\lim_{\varepsilon\to 0}\frac{1}{\epsilon}\int_K^{K+\epsilon}\Gamma(S,t;p,T)\,dp
\end{aligned}$$

If we can show that the first limit goes to zero then we have shown that $G(S,t;K,T)$ is the fundamental solution to the Garman-Kohlhagen PDE.

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_K^{K+\epsilon}\frac{\Gamma(S,t;p-\epsilon,T)-\Gamma(S,t;p,T)}{\epsilon^2}(p-K)\,dp\\
&= \lim_{\varepsilon\to 0}\frac{1}{\epsilon}\int_K^{K+\epsilon}\frac{\Gamma(S,t;p-\epsilon,T)-\Gamma(S,t;p,T)}{\epsilon}(p-K)\,dp
\end{aligned}$$

by Mean Value Theorem $\exists$ a point $q\in(p-\epsilon,p)$ s.t.

$$\Gamma'(S,t;q,T)=\frac{\Gamma(S,t;p-\epsilon,T)-\Gamma(S,t;p,T)}{\epsilon}$$

$$\begin{aligned}
&= \lim_{\varepsilon\to 0}\frac{1}{\epsilon}\int_K^{K+\epsilon}\Gamma'(S,t;q,T)(p-K)\,dp\\
&\le \lim_{\varepsilon\to 0}\frac{M}{\epsilon}\int_K^{K+\epsilon}(p-K)\,dp \quad (\exists M \text{ that bounds } \Gamma')\\
&= \lim_{\varepsilon\to 0}\frac{M}{\epsilon}\frac{(p-k)^2}{2}\Big|_K^{K+\epsilon}\\
&= \lim_{\varepsilon\to 0}\frac{M}{\epsilon}\frac{\epsilon^2}{2}\\
&= \lim_{\varepsilon\to 0}\epsilon\frac{M}{2}\\
&= 0.
\end{aligned}$$

This completes the proof of the theorem. $\square$

Since $G(S,t;K,T)$ is the fundamental solution to the Garman-Kohlhagen PDE, we have that it is also the solution to the adjoint ([**14**], Theorem 15) of Garman-Kohlhagen PDE given below:

$$\frac{\partial G}{\partial T} = \frac{1}{2}\frac{\partial^2}{\partial K^2}(K^2\sigma^2(K,T)G) - (r_D - r_F)\frac{\partial}{\partial K}(KG) - r_D G \tag{3.12}$$

Assume that

$$v, Kv_K, K^2 v_{KK}, \text{and } K^2 v_{KKK}$$

all approach zero as $K \to \infty$. Substituting $\frac{\partial^2 v}{\partial K^2}$ for G in (3.12) and integrating twice with respect to K from K to ∞ we get,

$$\frac{\partial v}{\partial T} = \frac{1}{2}K^2\sigma^2(K,T)\frac{\partial^2 v}{\partial K^2} - (r_D - r_F)\int_K^\infty \int_\eta^\infty \frac{\partial}{\partial \zeta}(\zeta\frac{\partial^2 v}{\partial \zeta^2})\,d\zeta\, d\eta - r_D v \tag{3.13}$$

The integral in the second term on the right side can be integrated as shown below

$$\begin{aligned}
\int_K^\infty \int_\eta^\infty \frac{\partial}{\partial \zeta}\Big(\zeta\frac{\partial^2 v}{\partial \zeta^2}\Big)\,d\zeta\, d\eta &= \int_K^\infty \zeta\frac{\partial^2 v}{\partial \zeta^2}\Big|_\eta^\infty d\eta \\
&= \int_K^\infty \Big(\zeta\frac{\partial^2 v}{\partial \zeta^2}\Big|_{\zeta=\infty} - \eta\frac{\partial^2 v}{\partial \eta^2}\Big)\,d\eta \\
&= -\int_K^\infty \eta\frac{\partial^2 v}{\partial \eta^2}\,d\eta \\
&\quad \text{integrating by parts we get,} \\
&= -\eta\frac{\partial v}{\partial \eta}\Big|_K^\infty + v\Big|_K^\infty \\
&= -\eta\frac{\partial v}{\partial \eta}\Big|_{\eta=\infty} + K\frac{\partial v}{\partial K} + v\Big|_{K=\infty} - v \\
&= K\frac{\partial v}{\partial K} - v
\end{aligned}$$

Thus equation (3.13) now can be rewritten as

$$\frac{\partial v}{\partial T} = \frac{1}{2}K^2\sigma^2(K,T)\frac{\partial^2 v}{\partial K^2} - (r_D - r_F)(K\frac{\partial v}{\partial K} - v) - r_D v.$$

Simplyfying further we get the Dupire-type equation,

$$\frac{\partial v}{\partial T} = \frac{1}{2}K^2\sigma^2(K,T)\frac{\partial^2 v}{\partial K^2} - (r_D - r_F)K\frac{\partial v}{\partial K} + r_F v.$$

that is

$$\frac{\partial v}{\partial T} - \frac{1}{2}K^2\sigma^2(K,T)\frac{\partial^2 v}{\partial K^2} + (r_D - r_F)K\frac{\partial v}{\partial K} - r_F v = 0. \tag{3.14}$$

If v is known for all strikes K and maturities T then, as was noted first in [**15**], the volatility $\sigma(K,T)$ is uniquely determined in principle from the equation (3.13). But such a formula for σ is of little use in practice, as the market data for v is not only noisy (which would make the estimation of these derivatives highly ill-posed), but even worse, the data is both discrete in T and somewhat sparse in K.

We present here the "two variables" descent variational algorithm for computing, using the Dupire equation (3.13), the volatility $\sigma(K,T)$ from a knowledge of European FX option prices at various strikes and maturities. The method used is an adaption of the variational approach involving the minimization of convex functionals (with the associated distinct advantage of having unique global minima and stationary points) used in [**16**] for numerical differentiation (formulated as an inverse problem), and in [**17, 18**] for solving the inverse groundwater modeling problem.

3. Reconstruction of FX volatility

Let $T_0 < T_1 < ... < T_n$ be maturity times, and for each maturity T_i, $0 \le i \le n$, let $K_{i1},...,K_{im_i}$ be the associated strike prices. We assume that the volatility is piecewise constant in time, so that, for $1 \le i \le n$, $\sigma = \sigma_i(K)$ over the i-th time sub-interval $[T_{i-1}, T_i]$. Fixing i, set

$$w_\lambda(K) = \int_{T_{i-1}}^{T_i} e^{-\lambda T} u(K,T)\,dT, \tag{3.15}$$

where $\lambda > 0$ is a parameter. For each such fixed i, $1 \le i \le n$, we now Laplace transform the Dupire equation (3.14) over $[T_{i-1}, T_i]$ to obtain

$$0=\int_{T_{i-1}}^{T_i} e^{-\lambda T} u_T\, dT - \frac{1}{2}K^2\sigma_i^2 \underbrace{\int_{T_{i-1}}^{T_i} e^{-\lambda T} u_{KK}\, dT}_{w''_\lambda} +$$

$$+(r_D - r_F)K \underbrace{\int_{T_{i-1}}^{T_i} e^{-\lambda T} u_K\, dT}_{w'_\lambda} + r_F w_\lambda,$$

where the primes indicate differentiation with respect to K. On integrating the first term by parts we get

$$[e^{-\lambda T} u]_{T_{i-1}}^{T_i} + \lambda \underbrace{\int_{T_{i-1}}^{T_i} e^{-\lambda T} u\, dT}_{w_\lambda} - \frac{1}{2}K^2\sigma_i^2 w''_\lambda + (r_D - r_F)K w'_\lambda + r_F w_\lambda = 0,$$

and rearranging terms gives,

$$-\frac{1}{2}K^2\sigma_i^2 w''_\lambda + (r_D - r_F)K w'_\lambda + (\lambda + r_F) w_\lambda = -u(K,T_i)e^{-\lambda T_i} + u(K,T_{i-1})e^{-\lambda T_{i-1}}.$$

Next, dividing by $\frac{1}{2}K^2\sigma_i^2$ throughout, we obtain

$$-(w''_\lambda - \frac{2(r_D - r_F)}{K\sigma_i^2} w'_\lambda) + \frac{\lambda + r_F}{\frac{1}{2}K^2\sigma_i^2} w_\lambda = \frac{-u(K,T_i)e^{-\lambda T_i} + u(K,T_{i-1})e^{-\lambda T_{i-1}}}{\frac{1}{2}K^2\sigma_i^2}.$$

Finally, on multiplying by the integrating factor

$$P(K) = e^{-2(r_D - r_F)\int_{T_{i-1}}^{K} \frac{dk}{k\sigma_i^2(k)}}, \tag{3.16}$$

we now have an equation in Sturm-Liouville form:

$$-(P(K)w'_\lambda)' + (\lambda + r_F)Q(K)w_\lambda = \beta(K,\lambda)Q(K), \tag{3.17}$$

where

$$Q(K) = (\frac{2}{K^2\sigma_i^2(K)})P(K), \tag{3.18}$$

$$\beta(K,\lambda) = -u(K,T_i)e^{-\lambda T_i} + u(K,T_{i-1})e^{-\lambda T_{i-1}}. \tag{3.19}$$

If we can recover the functions $P(K)$ and $Q(K)$ for each i, $1 \le i \le n$, we can find the volatility $\sigma_i(K)$ from the formula

$$\sigma_i(K) = \sqrt{\frac{2P(K)}{K^2 Q(K)}}. \tag{3.20}$$

We now focus attention on a variational approach to the recovery of one such pair of positive coefficient functions P, Q defined on an interval $a \le K \le b$. It is assumed that we are given the functions $w_\lambda(K)$ for K in $[a,b]$ and all $\lambda > 0$. For positive functions p and q also defined on $[a,b]$, let $c = (p,q)$. Define $w_{\lambda,c}(K)$ to be the solution to the boundary value problem

$$L_{p,\lambda q} w_{\lambda,c} = -(p(K)w'_{\lambda,c})' + (\lambda + r_F)q(K)w_{\lambda,c} = \beta(K,\lambda)q(K), \tag{3.21}$$

$$w_{\lambda,c}(a) = w_\lambda(a), \quad w_{\lambda,c}(b) = w_\lambda(b). \tag{3.22}$$

Let $\mathscr{D}$ be the set of all positive function pairs $c = (p,q)$ such that boundary value problem (3.21,3.22) is *disconjugate* on [a,b], i.e. every non-trivial solution has at most one zero on [a,b]. It is known [**19**, Theorem 6.1, p. 351] that (3.21) is disconjugate if and only if the boundary value problem (3.21,3.22) can always be solved uniquely. It is also known (c.f. [**16**, Proposition 2.1]) that this set is open and convex in $\mathscr{L}[a,b] \times \mathscr{L}[a,b]$ and $\mathscr{L}^2[a,b] \times \mathscr{L}^2[a,b]$. For each $\lambda > 0$ define the functional G_λ on the convex set $\mathscr{D}$ by

$$G_\lambda(c) = \int_a^b p(K)(w'^2_\lambda - w'^2_{\lambda,c}) + (\lambda + r_F)q(K)(w^2_\lambda - w^2_{\lambda,c}) - 2\beta q(K)(w_\lambda - w_{\lambda,c})\,dK. \tag{3.23}$$

We note here that while the algorithm does require that the foreign interest rate r_F be known in advance, it is interesting to observe that we do not require that that r_D be known in advance. In fact one may recover r_D from (3.16) once P and σ_i are recovered on the interval $[T_{i-1}, T_i]$. To do this, after computing $P(K)$ one first normalizes the right side of (3.16) by selecting a K^* from among the strike values for

$[T_{i-1}, T_i]$ and computing c such that

$$P(K^*) = e^{-2(r_D - r_F)\int_c^{K^*} \frac{dk}{k\sigma^2(k)}}.$$

Then set

$$P(K) = e^{-2(r_D - r_F)\int_c^{K} \frac{dk}{k\sigma^2(k)}}$$

and solve for r_D, using the known $P(K)$ and $\sigma(K)$. So, in theory at least, one should be able to predict future (time varying) interest rates r_D, given the current value of r_F and, for example, EUR/USD call prices for known strikes and maturity times. At the same time, noting that EUR/USD calls are also USD/EUR puts one can also determine future r_F values from a current r_D value. In this way one should be able to predict market-inspired future values for both domestic and foreign interest rates from their current values using FX option data.

4. Properties of the Functional

The main properties of the functional G_λ are summarized in the following

THEOREM 4.1. *(a) For any $c = (p,q)$ in $\mathscr{D}$,*

$$G_\lambda(c) = \int_a^b p(w'_\lambda - w'_{\lambda,c})^2 + (\lambda + r_F)q(w_\lambda - w_{\lambda,c})^2. \tag{3.24}$$

(b) $G_\lambda(c) \geq 0$ for all $c = (p,q)$ in $\mathscr{D}$, and $G_\lambda(c)$=0 if and only if $w_\lambda = w_{\lambda,c}$.

(c) The first Gâteaux derivative of G_λ is given by

$$G'_\lambda(p,q)[h_1,h_2] = \int_a^b \underbrace{(w'^2_\lambda - w'^2_{\lambda,c})}_{\mathrm{L^2\ gradient\ in\ p}} h_1 + \underbrace{[(\lambda + r_F)(w^2_\lambda - w^2_{\lambda,c}) - 2\beta(w_\lambda - w_{\lambda,c})]}_{\mathrm{L^2\ gradient\ in\ q}} h_2. \tag{3.25}$$

(d) The second Gâteaux derivative of G_λ is given by

$$G''_\lambda(c)[h,k] = 2(L^{-1}_{p,\lambda q}(e(h)), e(k)), \tag{3.26}$$

where $h = (h_1, h_2)$, $k = (k_1, k_2)$,

$$e(h) = -(h_1 w'_{\lambda,c})' + (\lambda + r_F) h_2 w_{\lambda,c} - \beta h_2,$$

and $(\cdot,\cdot)$ *denotes the usual inner product in* $L^2[a,b]$.

PROOF. For convenience we set $\eta = \lambda + r_F$.

(a) If $v \in W^{1,2}[a,b]$ and $\phi \in W_0^{1,2}[a,b]$ then by integration by parts we have

$$\int_a^b p(x) v' \phi' \, dx = \underbrace{p(x) v' \phi|_a^b}_{=0} - \int_a^b \phi (p(x) v')' \, dx = -\int_a^b \phi (p(x) v')' \, dx. \qquad (3.27)$$

Consequently, from (3.27) using $\phi = w_\lambda - w_{\lambda,c} \in W_0^{1,2}[a,b]$,

$$\begin{aligned}
G_\lambda(c) &= \int_a^b p(w_\lambda'^2 - w_{\lambda,c}'^2) + \eta q((w_\lambda^2 - w_{\lambda,c}^2) - 2\beta q(w_\lambda - w_{\lambda,c}) \\
&= \int_a^b p(w'_\lambda - w'_{\lambda,c})^2 + 2p w'_{\lambda,c}(w'_\lambda - w'_{\lambda,c}) \\
&\qquad + \eta q((w_\lambda^2 - w_{\lambda,c}^2) - 2\beta q(w_\lambda - w_{\lambda,c}) \\
&= \int_a^b p(w'_\lambda - w'_{\lambda,c})^2 - 2(w_\lambda - w_{\lambda,c})(p w'_{\lambda,c})' \\
&\qquad + \eta q((w_\lambda^2 - w_{\lambda,c}^2) - 2\beta q(w_\lambda - w_{\lambda,c}),
\end{aligned}$$

using (3.21) for $(p w'_{\lambda,c})'$,

$$\begin{aligned}
&= \int_a^b p(w'_\lambda - w'_{\lambda,c})^2 - 2(w_\lambda - w_{\lambda,c})(\lambda q w_{\lambda,c} - \beta q) \\
&\qquad + \eta q((w_\lambda^2 - w_{\lambda,c}^2) - 2\beta q(w_\lambda - w_{\lambda,c}) \\
&= \int_a^b p(w'_\lambda - w'_{\lambda,c})^2 + \eta q(w_\lambda - w_{\lambda,c})^2,
\end{aligned}$$

after some rearrangement.

(b) As p and q are chosen to be positive and $\eta = \lambda + r_F > 0$, from (a) we get (b).

(c) The first Gâteaux derivative of the functional G_λ is given by

$$G'_\lambda(p,q)[h_1,h_2] = \lim_{\varepsilon \to 0} \frac{G_\lambda(c + \varepsilon h) - G_\lambda(c)}{\varepsilon}$$

$$
\begin{aligned}
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_a^b (p+\varepsilon h_1)(w_\lambda'^2 - w_{\lambda,c+\varepsilon h}'^2) + \eta(q+\varepsilon h_2)(w_\lambda^2 - w_{\lambda,c+\varepsilon h}^2)\\
&\qquad -2\beta(q+\varepsilon h_2)(w_\lambda - w_{\lambda,c+\varepsilon h}) - p(w_\lambda'^2 - w_{\lambda,c}'^2)\\
&\qquad -\eta q(w_\lambda^2 - w_{\lambda,c}^2) + 2\beta q(w_\lambda - w_{\lambda,c})\\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_a^b \varepsilon(w_\lambda'^2 - w_{\lambda,c+\varepsilon h}'^2)h_1 + \varepsilon\eta(w_\lambda^2 - w_{\lambda,c+\varepsilon h}^2)h_2\\
&\qquad -2\varepsilon\beta(w_\lambda - w_{\lambda,c})h_2 + p(w_\lambda'^2 - w_{\lambda,c+\varepsilon h}'^2)\\
&\qquad +\eta q(w_\lambda^2 - w_{\lambda,c+\varepsilon h}^2) - 2q\beta(w_\lambda - w_{\lambda,c+\varepsilon h})\\
&= \lim_{\varepsilon\to 0}\int_a^b (w_\lambda'^2 - w_{\lambda,c+\varepsilon h}'^2)h_1 + [\eta(w_\lambda^2 - w_{\lambda,c+\varepsilon h}^2) - 2\beta(w_\lambda - w_{\lambda,c})]h_2\\
&\quad + \lim_{\varepsilon\to 0}\int_a^b \frac{1}{\varepsilon}p(w_{\lambda,c}'^2 - w_{\lambda,c+\varepsilon h}'^2) + \frac{1}{\varepsilon}\eta q(w_{\lambda,c}^2 - w_{\lambda,c+\varepsilon h}^2)\\
&\qquad -\frac{1}{\varepsilon}2q\beta(w_{\lambda,c} - w_{\lambda,c+\varepsilon h})
\end{aligned}
$$

If we can show the second term is zero we get (3.25). Let the integral in the second term be denoted by I. Now,

$$-(pw_{\lambda,c}')' + \eta q w_{\lambda,c} = \beta q, \tag{3.28}$$

$$-((p+\varepsilon h_1)w_{\lambda,c+\varepsilon h}')' + \eta(q+\varepsilon h_2)w_{\lambda,c+\varepsilon h} = \beta(q+\varepsilon h_2). \tag{3.29}$$

The first term in the integral I can be expanded as

$$
\begin{aligned}
& \varepsilon^{-1}\int_a^b p(w_{\lambda,c}'^2 - w_{\lambda,c+\varepsilon h}'^2)\\
= \; & \varepsilon^{-1}\int_a^b p(w_{\lambda,c}' + w_{\lambda,c+\varepsilon h}')(w_{\lambda,c}' - w_{\lambda,c+\varepsilon h}'),\\
& \text{from (3.27) using } \phi = w_{\lambda,c} - w_{\lambda,c+\varepsilon h}\ ,\\
= \; & \varepsilon^{-1}\int_a^b (w_{\lambda,c+\varepsilon h} - w_{\lambda,c})(p(w_{\lambda,c}' + w_{\lambda,c+\varepsilon h}'))'\\
= \; & \varepsilon^{-1}\int_a^b (w_{\lambda,c+\varepsilon h} - w_{\lambda,c})[(pw_{\lambda,c}')' + (pw_{\lambda,c+\varepsilon h}')'],
\end{aligned}
$$

using (3.28) and (3.29),

$$\begin{aligned}
&= \varepsilon^{-1}\int_a^b (w_{\lambda,c+\varepsilon h}-w_{\lambda,c})[\eta q w_{\lambda,c}-\beta q+\eta(q+\varepsilon h_2)w_{\lambda,c+\varepsilon h} \\
&\qquad -\beta(q+\varepsilon h_2)-\varepsilon(h_1 w'_{\lambda,c+\varepsilon h})'] \\
&= \int_a^b (w_{\lambda,c+\varepsilon h}-w_{\lambda,c})[\eta h_2 w_{\lambda,c+\varepsilon h}-\beta h_2-(h_1 w'_{\lambda,c+\varepsilon h})'] \\
&\quad +\varepsilon^{-1}\int_a^b (w_{\lambda,c+\varepsilon h}-w_{\lambda,c})[\eta q(w_{\lambda,c}+w_{\lambda,c+\varepsilon h})-2\beta q] \\
&= \int_a^b (w_{\lambda,c+\varepsilon h}-w_{\lambda,c})[\eta h_2 w_{\lambda,c+\varepsilon h}-\beta h_2-(h_1 w'_{\lambda,c+\varepsilon h})'] \\
&\quad +\varepsilon^{-1}\int_a^b \eta q(w^2_{\lambda,c+\varepsilon h}-w^2_{\lambda,c})(-2\beta q(w_{\lambda,c+\varepsilon h}-w_{\lambda,c})).
\end{aligned}$$

Substituting the above for $\varepsilon^{-1}\int_a^b p(w'^2_{\lambda,c}-w'^2_{\lambda,c+\varepsilon h})$ in I we get

$$I=\int_a^b (w_{\lambda,c+\varepsilon h}-w_{\lambda,c})[\eta h_2 w_{\lambda,c+\varepsilon h}-\beta h_2-(h_1 w'_{\lambda,c+\varepsilon h})'].$$

As the second factor in the integral is bounded, $I\to 0$ as $\varepsilon\to 0$.

(d) To find the second Gâteaux derivative of the functional G_λ we will need the following result:

$$\begin{aligned}
L_{p,\lambda q}(w_{\lambda,c+\varepsilon h}-w_{\lambda,c}) &= -(p(w_{\lambda,c+\varepsilon h}-w_{\lambda,c})')'+\lambda q(w_{\lambda,c+\varepsilon h}-w_{\lambda,c}) \\
&= -(pw'_{\lambda,c+\varepsilon h})'+\lambda q w_{\lambda,c+\varepsilon h}-[-(pw'_{\lambda,c})'+\lambda q w_{\lambda,c}], \\
&\quad \text{using (3.28) and (3.29),} \\
&= \varepsilon[(h_1 w'_{\lambda,c+\varepsilon h})'-\lambda h_2 w_{\lambda,c+\varepsilon h}+\beta h_2] \qquad (3.30)
\end{aligned}$$

The second Gâteaux derivative of the functional G_λ is given by

$$\begin{aligned}
G''_\lambda(c)[h,k] &= \lim_{\varepsilon\to 0}\frac{G'(c+\varepsilon h)[k]-G'(c)[k]}{\varepsilon} \\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_a^b (w'^2_\lambda-w'^2_{\lambda,c+\varepsilon h})k_1+[\eta(w^2_\lambda-w^2_{\lambda,c+\varepsilon h})-2\beta(w_\lambda-w_{\lambda,c+\varepsilon h})]k_2
\end{aligned}$$

$$
\begin{aligned}
& -(w_\lambda'^2 - w_{\lambda,c}'^2)k_1 - [\eta(w_\lambda^2 - w_{\lambda,c}^2) - 2\beta(w_\lambda - w_{\lambda,c})]k_2 \\
= \quad & \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_a^b (w_{\lambda,c}'^2 - w_{\lambda,c+\varepsilon h}'^2)k_1 + [\eta(w_{\lambda,c}^2 - w_{\lambda,c+\varepsilon h}^2) - 2\beta(w_{\lambda,c} - w_{\lambda,c+\varepsilon h})]k_2 \\
= \quad & \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_a^b k_1(w_{\lambda,c}' + w_{\lambda,c+\varepsilon h}')(w_{\lambda,c}' - w_{\lambda,c+\varepsilon h}') \\
& +[\eta(w_{\lambda,c}^2 - w_{\lambda,c+\varepsilon h}^2) - 2\beta(w_{\lambda,c} - w_{\lambda,c+\varepsilon h})]k_2,
\end{aligned}
$$

from (3.27) using $\phi = w_{\lambda,c} - w_{\lambda,c+\varepsilon h}$,

$$
\begin{aligned}
= \quad & \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_a^b (w_{\lambda,c} - w_{\lambda,c+\varepsilon h})(-k_1(w_{\lambda,c}' + w_{\lambda,c+\varepsilon h}'))' \\
& +[\eta(w_{\lambda,c}^2 - w_{\lambda,c+\varepsilon h}^2) - 2\beta(w_{\lambda,c} - w_{\lambda,c+\varepsilon h})]k_2,
\end{aligned}
$$

factoring $(w_{\lambda,c} - w_{\lambda,c+\varepsilon h})$,

$$
\begin{aligned}
= \quad & \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_a^b (w_{\lambda,c} - w_{\lambda,c+\varepsilon h})[(-k_1(w_{\lambda,c}' + w_{\lambda,c+\varepsilon h}'))' \\
& +(\eta(w_{\lambda,c} + w_{\lambda,c+\varepsilon h}) - 2\beta)k_2],
\end{aligned}
$$

using (3.30),

$$
\begin{aligned}
= \quad & \lim_{\varepsilon\to 0}\int_a^b L_{p,\lambda q}^{-1}[-(h_1 w_{\lambda,c+\varepsilon h}')' + \eta h_2 w_{\lambda,c+\varepsilon h} - \beta h_2] \\
& [(-k_1(w_{\lambda,c}' + w_{\lambda,c+\varepsilon h}'))' + (\eta(w_{\lambda,c} + w_{\lambda,c+\varepsilon h}) - 2\beta)k_2] \\
= \quad & \lim_{\varepsilon\to 0}\int_a^b L_{p,\lambda q}^{-1}[-(h_1(w_{\lambda,c+\varepsilon h}' - w_{\lambda,c}'))' + \eta h_2(w_{\lambda,c+\varepsilon h} - w_{\lambda,c})] \\
& [(-k_1(w_{\lambda,c}' + w_{\lambda,c+\varepsilon h}'))' + (\eta(w_{\lambda,c} + w_{\lambda,c+\varepsilon h}) - 2\beta)k_2] \\
& +\lim_{\varepsilon\to 0}\int_a^b L_{p,\lambda q}^{-1}[-(h_1 w_{\lambda,c}')' + \eta h_2 w_{\lambda,c}) - \beta h_2] \\
& [(-k_1(w_{\lambda,c}' + w_{\lambda,c+\varepsilon h}'))' + (\eta(w_{\lambda,c} + w_{\lambda,c+\varepsilon h}) - 2\beta)k_2],
\end{aligned}
$$

expanding the second integral,

$$
\begin{aligned}
= \quad & \lim_{\varepsilon\to 0}\int_a^b L_{p,\lambda q}^{-1}[-(h_1(w_{\lambda,c+\varepsilon h}' - w_{\lambda,c}'))' + \eta h_2(w_{\lambda,c+\varepsilon h} - w_{\lambda,c})] \\
& [(-k_1(w_{\lambda,c}' + w_{\lambda,c+\varepsilon h}'))' + (\eta(w_{\lambda,c} + w_{\lambda,c+\varepsilon h}) - 2\beta)k_2] \\
& +\lim_{\varepsilon\to 0}\int_a^b L_{p,\lambda q}^{-1}[-(h_1 w_{\lambda,c}')' + \lambda h_2 w_{\lambda,c}) - \beta h_2] \\
& [(-k_1(w_{\lambda,c+\varepsilon h}' - w_{\lambda,c}'))' + \eta(w_{\lambda,c+\varepsilon h} - w_{\lambda,c})k_2]
\end{aligned}
$$

$$+2\int_a^b L_{p,\lambda q}^{-1}[-(h_1 w_{\lambda,c}')' + \eta h_2 w_{\lambda,c} - \beta h_2]$$
$$[-(k_1 w_{\lambda,c}')' + \eta k_2 w_{\lambda,c} - \beta k_2].$$

The first and second terms equal zero. Thus we get

$$G_\lambda''(c)[h,k] = 2(L_{p,\lambda q}^{-1}(e(h)), e(k)),$$

where

$$e(h) = -(h_1 w_{\lambda,c}')' + \eta h_2 w_{\lambda,c} - \beta h_2,$$
$$e(k) = -(k_1 w_{\lambda,c}')' + \eta k_2 w_{\lambda,c} - \beta k_2.$$

This completes the proof of the theorem.

□

With some additional work one can show that the first and second Gâteaux derivatives of G_λ are also Fréchet derivatives. As $L_{p,\lambda q}$ is a positive operator on $W_0^1[a,b]$, we have from Theorem 4.1(d) that $G_\lambda''(c) \geq 0$ for all c in the convex set $\mathscr{D}$. By [**20**, Corollary 42.8] the functional G_λ is therefore convex on $\mathscr{D}$. We know from Theorem 4.1(b) that G_λ has a global minimum (zero) at $c = (p,q)$ if and only if $w_\lambda = w_{\lambda,c}$. Choose $N \geq 3$ positive distinct real numbers λ_j, $1 \leq j \leq N$, so that

$$0 < (\lambda_j + r_F)T < 2, \quad T \in [T_{i-1}, T_i].$$

Define a convex functional G on the domain $\mathscr{D}$ (defined above) by

$$G(c) = \sum_{j=1}^{N} G_{\lambda_j}(c). \tag{3.31}$$

From the uniqueness theorem [**21**, Theorem 3.5] we know that, under certain (computer-verifiable) conditions on the nature of the flows of certain associated vector fields (which amount here to an admissibility restriction on the data $u(K,T)$), the condition $w_\lambda = w_{\lambda,c}$ for at least three distinct values of λ implies that $c = (p,q) = (P,Q)$. By

[20, Proposition 42.6(1)] we know that if the convex functional G has a stationary point at (p,q) then it must have a global minimum there, and from the foregoing (assuming admissible data) that stationary point must uniquely occur at (P,Q). So, the desired function pair (P,Q) now appears as the unique global minimum of a convex functional with a unique stationary point. In practical numerics this is an important consideration, as many (if not most) least-square type minimization methods suffer greatly from the minimization process getting stuck in spurious local minima. That this cannot happen here is one of the significant advantages of our approach.

5. A Recovery Algorithm

$G(c)$ is a nonnegative convex functional since it is the sum of nonnegative convex functionals, and it also has a unique stationary point at $c=(P,Q)$. The idea here is that by using G rather than just one of the G_λ, in addition to gaining favourable uniqueness properties, we are blending additional time-based data into the inverse problem, and this is intended to improve the well-posedness of the problem. We note in passing from [22] that this inverse recovery is conditionally well-posed in the weak-L^2 sense, so from a theoretical standpoint, the recoveries are expected to be quite stable, which indeed is the case.

We minimize this functional for $N=20$ using the steepest descent method to recover the coefficients $P(K)$ and $Q(K)$. The L^2-direction of steepest descent for G at $c_0=(p_0,q_0)$ with respect to p is

$$-\nabla_{L^2,p}G(c_0)=\sum_{j=1}^{N}(w'^2_{\lambda_j}-w'^2_{\lambda_j,c_0}),$$

and the L^2-direction of steepest descent for G at (p_0,q_0) with respect to the variable q is given by

$$-\nabla_{L^2,q}G(c_0)=\sum_{j=1}^{N}[(\lambda_j+r_F)(w^2_{\lambda_j}-w^2_{\lambda_j,c_0})-2\beta(w_{\lambda_j}-w_{\lambda_j,c_0})].$$

Instead of using these L^2-gradients we use the corresponding Neuberger-gradients (see [**23**]) as the L^2-gradient has numerical problems that are extensively discussed in [**16**]. In particular, the L^2-gradient with respect to q is zero on the boundary of [a,b] given that w_λ and $w_{\lambda,c}$ are equal there, and thus the algorithm is unable to properly recover Q. The Neuberger-gradient smooths the L^2-gradient and preserves boundary data during the descent, an important property not shared by other descent techniques. Our Neuberger-gradient $g = \nabla_{H^1} G$ can be found from an L^2-gradient $\nabla_{L^2} G$ by solving the boundary value problem

$$\begin{aligned} -g'' + g &= \nabla_{L^2} G, \\ g(a) = g(b) &= 0. \end{aligned} \tag{3.32}$$

Below is the steepest descent algorithm used to get one descent step in p:

(i) Initialize $p(K)$ and $q(K)$ with $c_0 = (p_0, q_0)$.

(ii) Find w_{λ,c_0} and w'_{λ,c_0} by solving (3.21),(3.22).

(iii) Find the L^2 gradient of G in p, $\nabla_{L^2,p} G(c_0)$.

(iv) Find the Neuberger gradient in p, $\nabla_{H^1,p} G(c_0)$.

(v) Evaluate $p_{new}(K) = p_0(K) - \alpha \nabla_{H^1,p} G(c_0)$.

(vi) Find $G(p, q_0)$ using $p_{new}(K)$ for $p(K)$.

(vii) Find α that gives the lowest value of $G(p, q_0)$.

(viii) Set $p(K) = p_{new}(K)$.

The descent in q is similar to that of descent in p. Here we find corresponding gradients in q. The $q_{new}(K)$ is given by

$$q_{new}(K) = q(K) - \alpha \nabla_{H^1,q} G(c_0)$$

The specific order of descent is somewhat problem dependent, and different combinations of descents in p and q were tried to get the best minimization. Typically one needs more p-descent steps relative to q-descent steps as the descent progresses.

Spot Rate(S_0)	\$ 1.1342	
	Maturity(T) in days	
	30	90
Strike(K) in \$	v(K,T)	
1.1300	0.011087	0,020344
1.1310	0.011463	0.020720
1.1320	0.011849	0.021102
1.1330	0.012245	0.021490
1.1340	0.012652	0.021885
1.1350	0.013070	0.022285
1.1360	0.013498	0.022692
1.1370	0.013938	0.023105
1.1380	0.014388	0.023325
1.1390	0.014849	0.023950
1.1400	0.015321	0.024382
1.1410	0.015803	0.024821
1.1420	0.016296	0.025265
1.1430	0.016800	0.025716
1.1440	0.017314	0.026174
1.1450	0.017839	0.026638
1.1460	0.018375	0.027108
1.1470	0.018921	0.027585
1.1480	0.019477	0.028068
1.1490	0.020044	0.028557
1.1500	0.020621	0.029053
1.1510	0.021207	0.029555
1.1520	0.021804	0.030064
1.1530	0.022410	0.030579
1.1540	0.023026	0.031101
1.1550	1.023652	0.031628

TABLE 3.1. Exchange put option on 1/31/2015
Source: Bloomberg Terminals

The table 3.1 represents the exchange rate option pricing in 1 month and 3 months respectively which was recorded on 01/31/2015. At this time, the spot price is 1.1342, domestic (the US) interest rate is 0.25 percent, and foreign (the Euro) interest rate is 0.05 percent. The table includes 26 data points. The tables 3.2, 3.3, and 3.4 provide the latest data of 2020 at which both interest are very close to 0.

Spot Rate(S_0)	$ 1.1063	
	Maturity(T) in days	
	10	30
Strike(K) in $	v(K,T)	
1.1050	0.011098	0.011119
1.1051	0.011097	0.011118
1.1052	0.011097	0.011117
1.1053	0.011096	0.011117
1.1054	0.011096	0.011116
1.1055	0.011095	0.011116
1.1056	0.011095	0.011115
1.1057	0.011095	0.011115
1.1058	0.011094	0.011114
1.1059	0.011094	0.011114
1.1060	0.011093	0.011114
1.1061	0.011093	0.011113
1.1062	0.011093	0.011113
1.1063	0.011092	0.011112
1.1064	0.011092	0.011112
1.1065	0.011091	0.011111
1.1066	0.011091	0.011111
1.1067	0.011090	0.011111
1.1068	0.011090	0.011110
1.1069	0.011089	0.011110
1.1070	0.011089	0.011109
1.1071	0.011089	0.011109
1.1072	0.011088	0.011108
1.1073	0.011088	0.011108
1.1074	0.011088	0.011108
1.1075	0.011087	0.011107

TABLE 3.2. Exchange call option on 2/03/2020
Source: Bloomberg Terminals

The most widely traded of FX options is the EURUSD option. To recover the coefficient functions $P(K)$ and $Q(K)$ in (3.17) , a computer code was written in the programming language C. The volatility recovered was compared to the "implied volatility" obtained directly from the standard formula of Garman and Kohlhagen by substituting the known option price and solving for the implied volatility σ as an unknown.

Spot Rate(S_0)				$ 1.1375	
	Maturity(T) in months				
	1	2	3	6	9
Strike(K) in $	v(K,T)				
1.1150	0.0023	0.0047	0.0068	0.0065	0.0120
1.1200	0.0032	0.0060	0.0081	0.0083	0.0141
1.1250	0.0045	0.0075	0.0098	0.0106	0.0166
1.1300	0.0062	0.0093	0.0116	0.0136	0.0197
1.1350	0.0083	0.0115	0.0138	0.0172	0.0232
1.1400	0.0108	0.0139	0.0162	0.0216	0.0273
1.1450	0.0138	0.0168	0.0189	0.0269	0.0321
1.1500	0.0172	0.0199	0.0219	0.0329	0.0375
1.1550	0.0209	0.0233	0.0251	0.0397	0.0435
1.1600	0.0249	0.0270	0.0286	0.0472	0.0501

TABLE 3.3. EUR/USD put option data, June 10, 2020
Source: Investing.com

Spot Rate(S_0)				$ 1.1375	
	Maturity(T) in months				
	1	2	3	6	9
Strike(K) in $	v(K,T)				
1.1150	0.0255	0.0286	0.0315	0.0586	0.0691
1.1200	0.0214	0.0249	0.0279	0.0504	0.0612
1.1250	0.0177	0.0214	0.0245	0.0427	0.0537
1.1300	0.0144	0.0182	0.0214	0.0357	0.0467
1.1350	0.0115	0.0154	0.0185	0.0293	0.0403
1.1400	0.0091	0.0129	0.0160	0.0237	0.0344
1.1450	0.0070	0.0107	0.0137	0.0189	0.0294
1.1500	0.0054	0.0088	0.0117	0.0150	0.0246
1.1550	0.0041	0.0073	0.0100	0.0118	0.0206
1.1600	0.0031	0.0060	0.0085	0.0093	0.0172

TABLE 3.4. EUR/USD call option data, June 10, 2020
Source: Investing.com

We have option prices for discrete sets of strikes and maturities. We generated the function $v(K,T)$ by linearly interpolating the option price in both strike and maturity. The function $v(K,T)$ was mollified (c.f. [**17**, §6]) so that it could be differentiated, and the derivative $v_K(K,T)$ was found using central differences. For 20 fixed values of λ the functions $v(K,T)$ and $v'_K(K,T)$ were Laplace transformed using (3.15) to $w_\lambda(K)$ and $w'_\lambda(K)$ respectively. The functions $p(K)$ and $q(K)$ were initialized using (3.16) and

(3.18) with the initial σ_i chosen to be the implied volatility. We performed a series of descents in p using the aforementioned Neuberger steepest descent algorithm such that the functional could not be minimized any further. Then a series of descents in q were performed to the point where functional likewise could not be lowered any further. We repeated this sequence of descents in p and q. The minimization of $G(c)$ in α was done using the well known Brent minimization technique, by adapting the one-variable code in the Numerical Recipes in C function `brent()`. To avoid possible catastrophic cancellation in the Simpson rule formula used in the calculation of the integrals in the formula (3.23) for the functional G_λ, we used the alternate formula (3.24) instead. After running the code we recovered the functions $P(K)$ and $Q(K)$.

From (3.20) we calculated the volatility and compared it to the implied volatility of the option at first and second expirations, as shown in the Figure 3.1 and Figure 3.2. On taking subsequent maturity intervals a volatility surface can in principle be plotted.

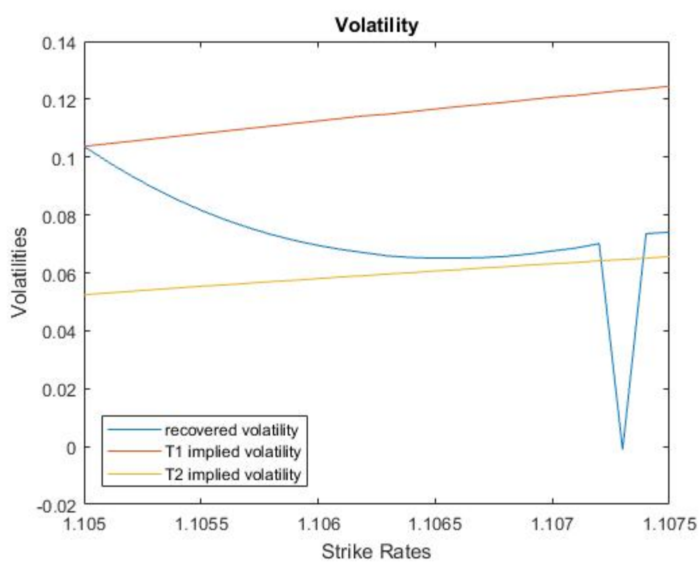


FIGURE 3.1. Volatility on 2/03/2020

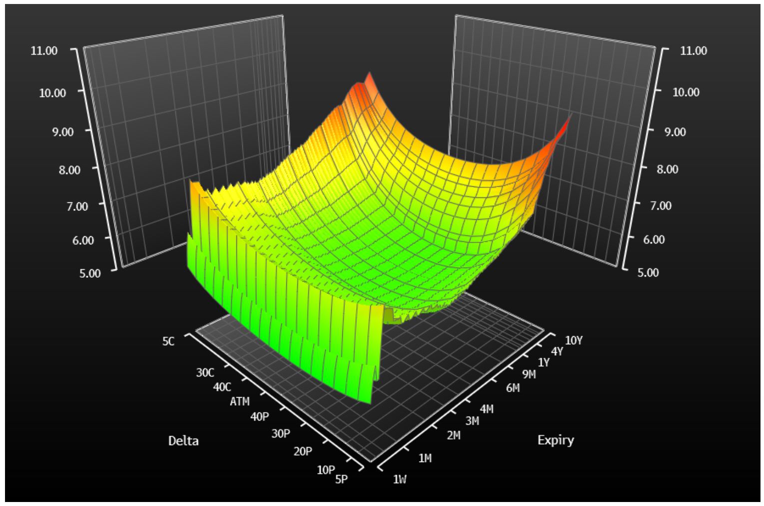


FIGURE 3.2. Volatility Surface of EURUSD (Source: Bloomberg Terminals)

Finally, from the recovered volatility we calculated the option price in MATLAB using the Binomial method and compared it to the actual price.

6. Direct Reconstruction of $\sigma(K)$

Instead of first recovering the functions $P(K)$ and $Q(K)$ and then using the formula (3.20) to determine the volatility $\sigma(K)$, there is a way to directly recover $\sigma(K)$ from the minimization of another, related, functional. This is a new variational algorithm, and we call it a "single variable" descent algorithm, by way of comparison with the two-variables algorithm above.

From (3.16) define the nonlinear integral operator A by

$$A[\sigma(K)] = P(K) = e^{-2(r_D - r_F)\int_a^K \frac{d\tau}{\tau\sigma^2(\tau)}}. \tag{3.33}$$

Then

$$Q(K) = \frac{2A[\sigma(K)]}{K^2\sigma^2(K)},$$

from (3.18).

For a generic volatility $\rho(K)$ define

$$p(K) = A[\rho(K)] = e^{-2(r_D - r_F)\int_a^K \frac{d\tau}{\tau\rho^2(\tau)}}$$

$$q(K) = \frac{2A[\rho(K)]}{K^2\rho^2(K)}.$$

For convenience below, and with an admitted abuse of notation, we denote the function $p(K)$ as $p(\rho)$, and $q(K)$ as $q(\rho)$.

Define the non-negative functional $H_\lambda(\rho)$ by $c = (p,q)$ and

$$H_\lambda(\rho) = G_\lambda(c) = \int_a^b p(w'_\lambda - w'_{\lambda,c})^2 + (\lambda + r_F)q(w_\lambda - w_{c,\lambda})^2. \tag{3.34}$$

Then we have

THEOREM 6.1. *(a) $H_\lambda(\rho) \geq 0$ for all ρ, and $H_\lambda(\rho) = 0$ if and only if*

$$w_\lambda = w_{c,\lambda},$$

where $c = (p,q)$.

(b) The first Gâteaux derivative of H_λ is given by

$$H'_\lambda(\rho)[h] = \int_a^b (w'^2_\lambda - w'^2_{c,\lambda})h_1 + [(\lambda + r_F)(w^2_\lambda - w^2_{c,\lambda}) - 2\beta(w_\lambda - w_{c,\lambda})]h_2, \tag{3.35}$$

where

$$h_1(K) = 4(r_D - r_F)e^{-2(r_D - r_F)\int_a^K \frac{dk}{k\rho^2(k)}} \cdot \int_a^K \frac{h(\tau)d\tau}{\tau\rho^3(\tau)},$$

and

$$h_2 = -\frac{4h}{K^2\rho^3}e^{-2(r_D - r_F)\int_a^K \frac{d\tau}{\tau\rho^2(\tau)}} + \\ + \frac{8}{K^2\rho^2}e^{-2(r_D - r_F)\int_a^K \frac{d\tau}{\tau\rho^2(\tau)}}(r_D - r_F)\int_a^K \frac{h}{\tau\rho^3}\,d\tau.$$

(c) The second Gâteaux derivative of H_λ is given by

$$H''_\lambda(\rho)[h,k] = 2(L^{-1}_{c,\lambda}(e(h_1,h_2)), e(k_1,k_2)), \tag{3.36}$$

where

$$e(h_1,h_2) = -(h_1 w'_{\lambda,c})' + h_2[(\lambda + r_F)w_{c,\lambda} - \beta],$$

and $(\cdot,\cdot)$ *denotes the usual inner product in* $L^2[a,b]$ *and* h_1 *and* h_2 *are calculated above from* h *and* ρ.

PROOF. (a) This follows from Theorem 4.1(a).

(b) We have

$$\begin{aligned}
H'_\lambda(\rho)[h] &= \lim_{\epsilon\to 0} \frac{H_\lambda(\rho+\epsilon h) - H_\lambda(\rho)}{\epsilon} \\
&= \lim_{\epsilon\to 0} \frac{G_\lambda(p(\rho+\epsilon h), q(\rho+\epsilon h)) - G_\lambda(p(\rho), q(\rho))}{\epsilon} \\
&= \lim_{\epsilon\to 0} \frac{G_\lambda(p(\rho)+\epsilon h_1^*, q(\rho)+\epsilon h_2^*) - G_\lambda(p(\rho), q(\rho))}{\epsilon} \\
&= G'_\lambda(p,q)[h_1,h_2].
\end{aligned}$$

where we define h_1^* and h_2^* by $p(\rho+\epsilon h) = p(\rho) + \epsilon h_1^*$ and $q(\rho+\epsilon h) = q(\rho) + \epsilon h_2^*$ and note that,

$$\begin{aligned}
h_1 &= \lim_{\epsilon\to 0} h_1^* \\
&= \lim_{\epsilon\to 0} \frac{1}{\epsilon}[p(\rho+\epsilon) - p(\rho)] \\
&= \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left[e^{-2(r_D-r_F)\int_a^K \frac{d\tau}{\tau(\rho(\tau)+\epsilon h(\tau)^2}} - e^{-2(r_D-r_F)\int_a^K \frac{d\tau}{\tau\rho^2(\tau)}}\right] \\
&= \lim_{\epsilon\to 0} \frac{1}{\epsilon} e^{-2(r_D-r_F)\int_a^K \frac{d\tau}{\tau\rho^2(\tau)}} \left\{ e^{-2(r_D-r_F)\int_a^K \frac{1}{\tau}\left[\frac{1}{(\rho+\epsilon h)^2} - \frac{1}{\rho^2}\right]d\tau} - 1 \right\} \\
&= \lim_{\epsilon\to 0} \frac{1}{\epsilon} e^{-2(r_D-r_F)\int_a^K \frac{d\tau}{\tau\rho^2(\tau)}} \left\{ e^{2(r_D-r_F)\epsilon\int_a^K \frac{1}{\tau}\frac{2\rho h+\epsilon h^2}{\rho^2(\rho+\epsilon h)^2}d\tau} - 1 \right\} \\
&= \lim_{\epsilon\to 0}\; e^{-2(r_D-r_F)\int_a^K \frac{d\tau}{\tau\rho^2(\tau)}} 2(r_D-r_F)\int_a^K \frac{1}{\tau}\frac{2\rho h+\epsilon h^2}{\rho^2(\rho+\epsilon h)^2}\,d\tau \times \\
&\qquad\qquad \times \left\{ \frac{e^{2(r_D-r_F)\epsilon\int_a^K \frac{1}{\tau}\frac{2\rho h+\epsilon h^2}{\rho^2(\rho+\epsilon)^2}d\tau} - 1}{2(r_D-r_F)\epsilon\int_a^K \frac{1}{\tau}\frac{2\rho h+\epsilon h^2}{\rho^2(\rho+\epsilon h)^2}\,d\tau} \right\} \\
&= 4e^{-2(r_D-r_F)\int_a^K \frac{d\tau}{\tau\rho^2(\tau)}}(r_D-r_F)\int_a^K \frac{h}{\tau\rho^3}\,d\tau,
\end{aligned}$$

using the fact that

$$\lim_{z\to 0}\frac{e^z-1}{z}=1;$$

and in similar fashion

$$\begin{aligned}
h_2 &= \lim_{\epsilon\to 0} h_2^* \\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}[q(\rho+\epsilon h)-q(\rho)] \\
&= \lim_{\epsilon\to 0}\frac{2}{\epsilon K^2}\left[p(\rho+\epsilon h)\left\{\frac{1}{(\rho+\epsilon h)^2}-\frac{1}{\rho^2}\right\}+\frac{1}{\rho^2}\left\{p(\rho+\epsilon h)-p(\rho)\right\}\right] \\
&= -\frac{4h}{K^2\rho^3}p(\rho)+\frac{2}{K^2\rho^2}\lim_{\epsilon\to 0}\frac{1}{\epsilon}[p(\rho+\epsilon)-p(\rho)] \\
&= -\frac{4h}{K^2\rho^3}e^{-2(r_D-r_F)\int_a^K\frac{d\tau}{\tau\rho^2(\tau)}}+ \\
&\qquad +\frac{8}{K^2\rho^2}e^{-2(r_D-r_F)\int_a^K\frac{d\tau}{\tau\rho^2(\tau)}}(r_D-r_F)\int_a^K\frac{h}{\tau\rho^3}\,d\tau.
\end{aligned}$$

(c) This follows from Theorem 4.1(d).

□

We next calculate the $L^2[a,b]$-gradient of the functional H_λ.

THEOREM 6.2. *Further to (3.35) we have*

$$H'_\lambda(\rho)[h]=\int_a^b\left[\frac{1}{K\rho^3(K)}\int_K^b m(\tau)\,d\tau-n(K)\right]h(K)dK, \tag{3.37}$$

where

$$\begin{aligned}
m(K) &= 4(r_D-r_F)e^{-2(r_D-r_F)\int_a^K\frac{d\tau}{\tau\rho^2(\tau)}}\times \\
&\qquad\times\left[[(\lambda+r_F)(w_\lambda^2-w_{c,\lambda}^2)-2\beta(w_\lambda-w_{c,\lambda})]\frac{2}{K^2\rho^2}+(w_\lambda'^2-w_{c,\lambda}'^2)\right],
\end{aligned} \tag{3.38}$$

and

$$n(K)=\left[(\lambda+r_F)(w_\lambda^2-w_{c,\lambda}^2)-2\beta(w_\lambda-w_{c,\lambda})\right]\frac{4}{K^2\rho^3}e^{-2(r_D-r_F)\int_a^K\frac{d\tau}{\tau\rho^2(\tau)}}, \tag{3.39}$$

and finally,

$$g(K) = \frac{1}{K\rho^3(K)} \int_K^b m(\tau)\,d\tau - n(K)$$

is the $L^2[a,b]$-gradient of $H_\lambda(\rho)$.

PROOF. From Theorem 6.1(b) we have

$$\begin{aligned}
&H'_\lambda(\rho)[h] \\
&= \int_a^b \Big\{ (w'^2_\lambda - w'^2_{c,\lambda}) 4(r_D - r_F) e^{-2(r_D-r_F)\int_a^K \frac{d\tau}{\tau\rho^2(\tau)}} . \int_a^K \frac{h(\tau)d\tau}{\tau\rho^3(\tau)} \\
&- h(K)[(\lambda + r_F)(w^2_\lambda - w^2_{c,\lambda}) - 2\beta(w_\lambda - w_{c,\lambda})] \frac{4}{K^2\rho^3} e^{-2(r_D-r_F)\int_a^K \frac{d\tau}{\tau\rho^2(\tau)}} \\
&+ [(\lambda + r_F)(w^2_\lambda - w^2_{c,\lambda}) - 2\beta(w_\lambda - w_{c,\lambda})] \frac{8(r_D - r_F)}{K^2\rho^2} e^{-2(r_D-r_F)\int_a^K \frac{d\tau}{\tau\rho^2(\tau)}} \int_a^K \frac{h}{\tau\rho^3}\,d\tau, \Big\}\, dK \\
&= \int_a^b \Big\{ 4(r_D - r_F) e^{-2(r_D-r_F)\int_a^K \frac{dk}{k\rho^2(k)}} . \int_a^K \frac{h(\tau)d\tau}{\tau\rho^3(\tau)} \times \\
&\qquad \times \left[[(\lambda + r_F)(w^2_\lambda - w^2_{c,\lambda}) - 2\beta(w_\lambda - w_{c,\lambda})] \frac{2}{K^2\rho^2} + (w'^2_\lambda - w'^2_{c,\lambda}) \right] \\
&\qquad - h[(\lambda + r_F)(w^2_\lambda - w^2_{c,\lambda}) - 2\beta(w_\lambda - w_{c,\lambda})] \frac{4}{K^2\rho^3} e^{-2(r_D-r_F)\int_a^K \frac{d\tau}{\tau\rho^2(\tau)}} \Big\}\, dK,
\end{aligned}$$

after collecting the terms containing $\int_a^K \frac{h(\tau)d\tau}{\tau\rho^3(\tau)}$,

$$= \int_a^b \left\{ m(K) \int_a^K \frac{h(\tau)d\tau}{\tau\rho^3(\tau)} - h(K)n(K) \right\} dK \tag{3.40}$$

where

$$\begin{aligned}
m(K) = 4(r_D - r_F) e^{-2(r_D-r_F)\int_a^K \frac{dk}{k\rho^2(k)}} \times \\
\times \left[[(\lambda + r_F)(w^2_\lambda - w^2_{c,\lambda}) - 2\beta(w_\lambda - w_{c,\lambda})] \frac{2}{K^2\rho^2} + (w'^2_\lambda - w'^2_{c,\lambda}) \right]
\end{aligned} \tag{3.41}$$

and

$$n(K) = \left[(\lambda + r_F)(w^2_\lambda - w^2_{c,\lambda}) - 2\beta(w_\lambda - w_{c,\lambda}) \right] \frac{4}{K^2\rho^3} e^{-2(r_D-r_F)\int_a^K \frac{d\tau}{\tau\rho^2(\tau)}}. \tag{3.42}$$

We now integrate the first term in the integrand in (3.40) by parts in order to isolate the h factor. Specifically,

$$\int_a^b m(K)\int_a^K \frac{h(\tau)d\tau}{\tau\rho^3(\tau)}\,dK = \int_a^b h(K)\frac{1}{K\rho^3(K)}\left[\int_K^b m(\tau)d\tau\right]dK,$$

and hence from (3.40)

$$H'_\lambda(\rho)[h] = \int_a^b h(K)\left\{\frac{1}{K\rho^3(K)}\int_K^b m(\tau)d\tau - n(K)\right\}dK, \tag{3.43}$$

and we are done. $\square$

Finding future volatility for currency pair is a very difficult task. We use the Garman-Kohlhagen equation which is basically same as the corresponding European stock option model. We recover the volatility by running our C code, that provides implied volatilities for different time periods, in addition to recovered (local) volatility as described in the beginning of this section. The following are some results obtained from our algorithm:

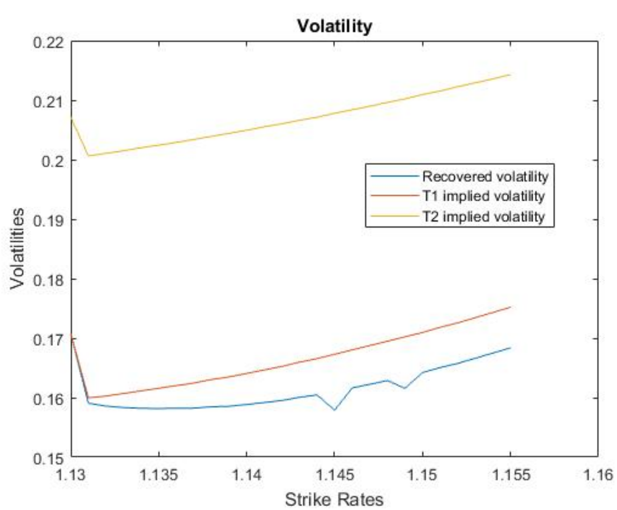


FIGURE 3.3. Graph of Table 3.5

The graphs of the figure 3.3 are plots in Matlab using the Table 3.5, in which recovered (local) volatility and $T1$ implied volatility start with the same value, and move in the same direction. The values of these volatilities are close whereas $T2$ has

Strike Rate	T1 Implied Volatility	T2 Implied Volatility	Recovered Volatility
1.1300	0.170805	0.207132	0.170805
1.1310	0.159966	0.200614	0.159056
1.1320	0.160259	0.201053	0.158573
1.1330	0.160698	0.201493	0.158337
1.1340	0.161137	0.202005	0.158204
1.1350	0.161577	0.202445	0.158168
1.1360	0.162016	0.202884	0.158217
1.1370	0.162456	0.203397	0.158238
1.1380	0.163042	0.203909	0.158458
1.1390	0.163481	0.204422	0.158550
1.1400	0.164067	0.204935	0.158842
1.1410	0.164653	0.205521	0.159170
1.1420	0.165239	0.206033	0.159529
1.1430	0.165971	0.206619	0.160060
1.1440	0.166557	0.207132	0.160455
1.1450	0.167290	0.207791	0.157884
1.1460	0.168022	0.208377	0.161631
1.1470	0.168754	0.208963	0.162239
1.1480	0.169487	0.209622	0.162875
1.1490	0.170219	0.210208	0.161563
1.1500	0.170952	0.210940	0.164216
1.1510	0.171830	0.211526	0.165061
1.1520	0.172563	0.212259	0.165758
1.1530	0.173442	0.212918	0.166618
1.1540	0.174321	0.213577	0.167496
1.1550	0.175199	0.214309	0.168357

TABLE 3.5. Volatilities from 1/31/2015 to 6/30/2015 obtained from Inverse volatility model

quite different values but same direction as $T1$. The figure 3.3 provides the behavior and connections of implied and local volatilities.

The figure 3.4 shows the volatilities corresponding to option prices for foreign currency exchange rates taken on 1/31/2015 and 1/31/2018 respectively.

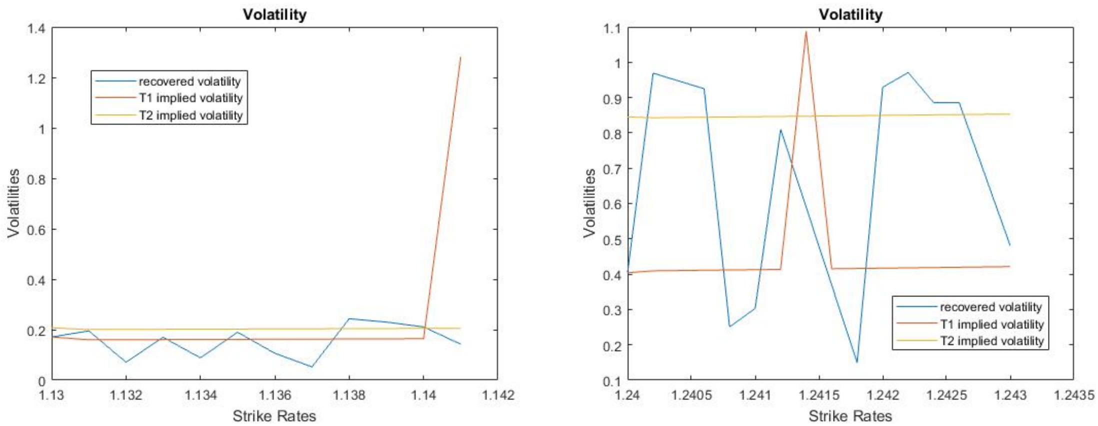


FIGURE 3.4. Volatility on 1/31/2015 and 1/31/2018 respectively

When we run the C code, that executes the table of data along with start and end strike rates, it also provides the initial value of the functional and the number of lambda values. The linesearch minimum alpha and the functional minimum value are determined. After certain number of descent steps (≥ 0), it produces the initial and final functional value and the implied and local voaltilities.

The P and Q descent algorithm in [**24**] worked well for functionals with two variables, but we have functional of one variable, so a new "single variable" variational algorithm was developed. We tried to apply old algorithm for FX options data, it sometimes fail to descend. In some cases, the code runs for couple descent steps and yields only the same initial and final functional value. In such condition, we alter the values of parameters such as lambda, delta, sigma (the start value), tolerance and linesearch minimum alpha value until we get appropriate descend and a minimum value for the functional.

CHAPTER 4

Some Results

1. Predicted Trend for Forex Options

The graphs given below are some of the predicted trends for the currency exchange rates of EUR/USD compared to how the EUR/USD rate actually performed during various time periods between 2000 and the beginning of 2020. The predicted trend is obtained by solving the system (2.12) using MATLAB's dde23 command. Since the coefficients are found using data from $[t_0, t_N]$, t_N is the start time of the DDE system. As a result, we use the vector constant functions, $x(t_N) = (x_1(t_N), \ldots, x_{13}(t_N))$, as our history function for the system.

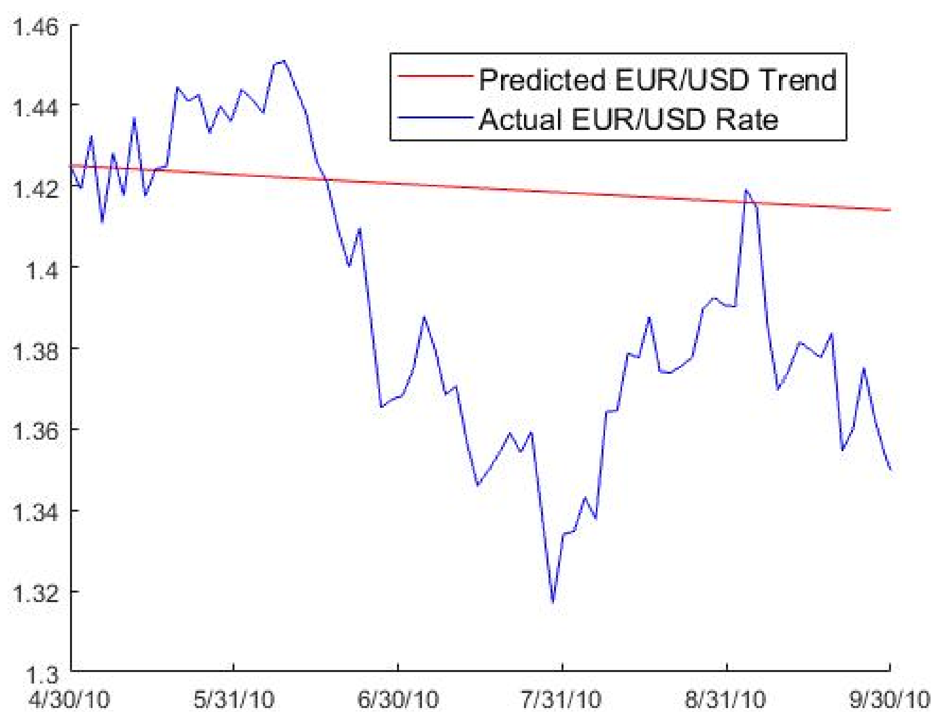


FIGURE 4.1. Comparing the predicted trend of EUR/USD rate from 4/30/2010 to 9/30/2010

The figure 4.1 shows different graph pattern for actual and predicted exchange rate, however they have the same direction at the end of the time duration mentioned in the figure. On the other hand, figure 4.2 gives a good fit over 3 months, ignoring what happens after that time.

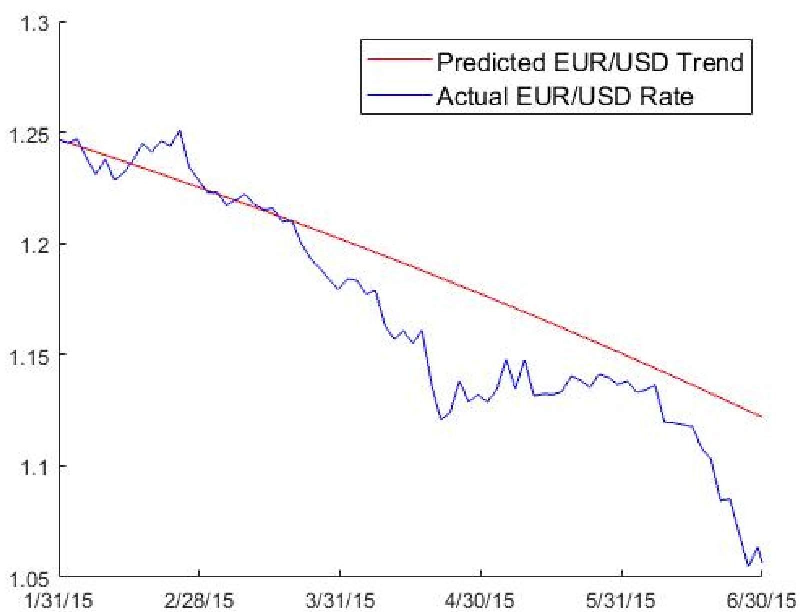


FIGURE 4.2. Comparing the predicted trend of EUR/USD rate from 11/30/2013 to 4/30/2014

For some particular time period such as from July 31, 2018 to December 31, 2018, given in figure 4.11, the predicted trend of EUR/USD is pretty close to the actual one. It implies our model is good fit to prediction of currency exchange rate.

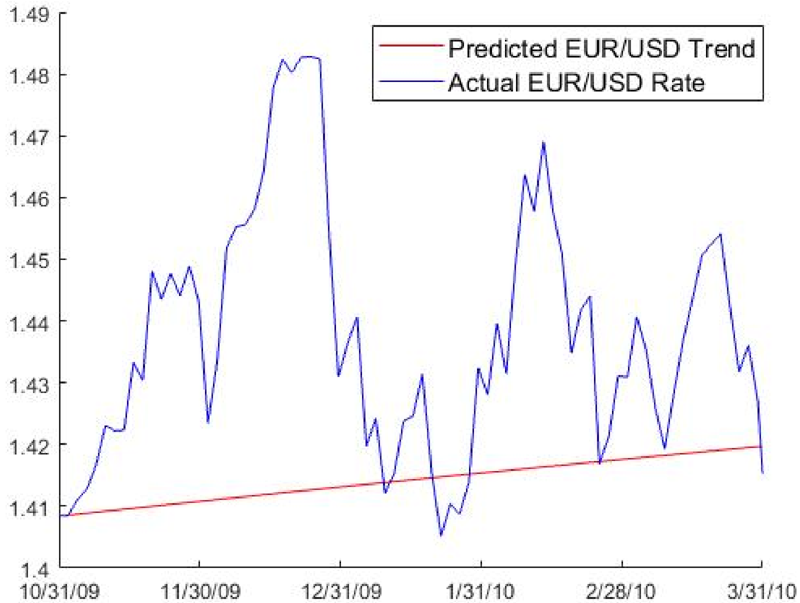


FIGURE 4.3. Comparing the predicted trend of EUR/USD rate from 7/31/2011 to 12/31/2011

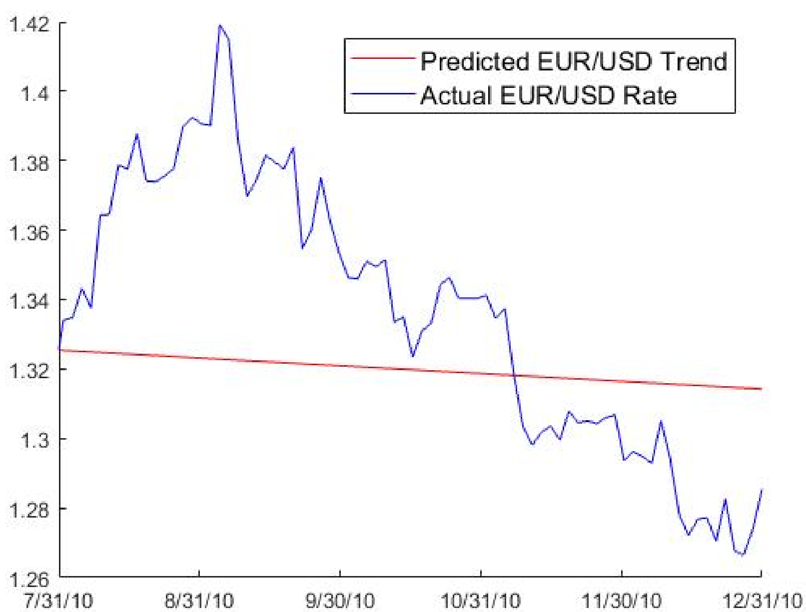


FIGURE 4.4. Comparing the predicted trend of EUR/USD rate from 10/31/2011 to 3/31/2012

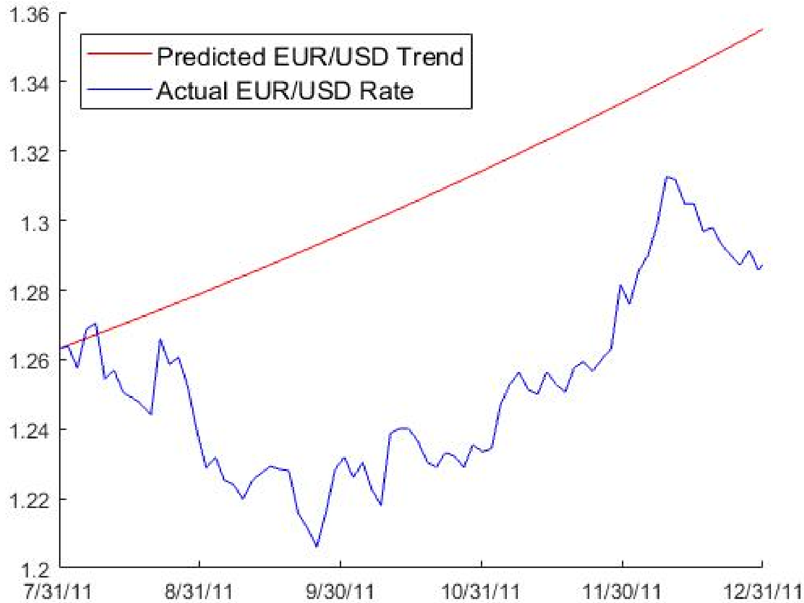


FIGURE 4.5. Comparing the predicted trend of EUR/USD rate from 4/30/2013 to 9/30/2013

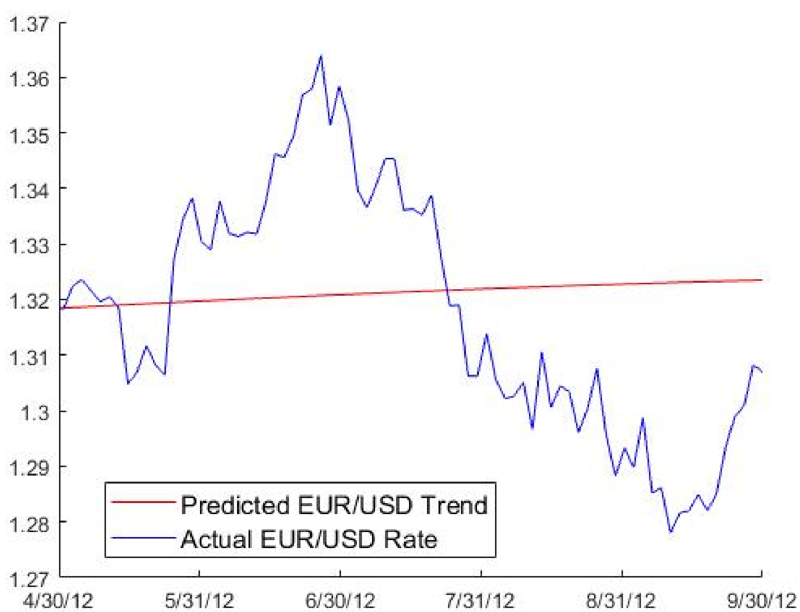


FIGURE 4.6. Comparing the predicted trend of EUR/USD rate from 10/31/2009 to 3/31/2010

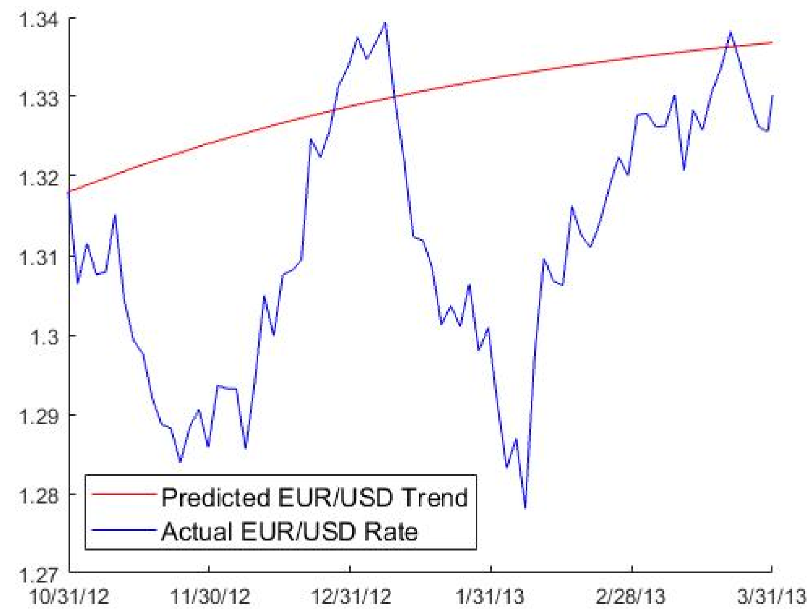


FIGURE 4.7. Comparing the predicted trend of EUR/USD rate from 11/30/2015 to 4/30/2016

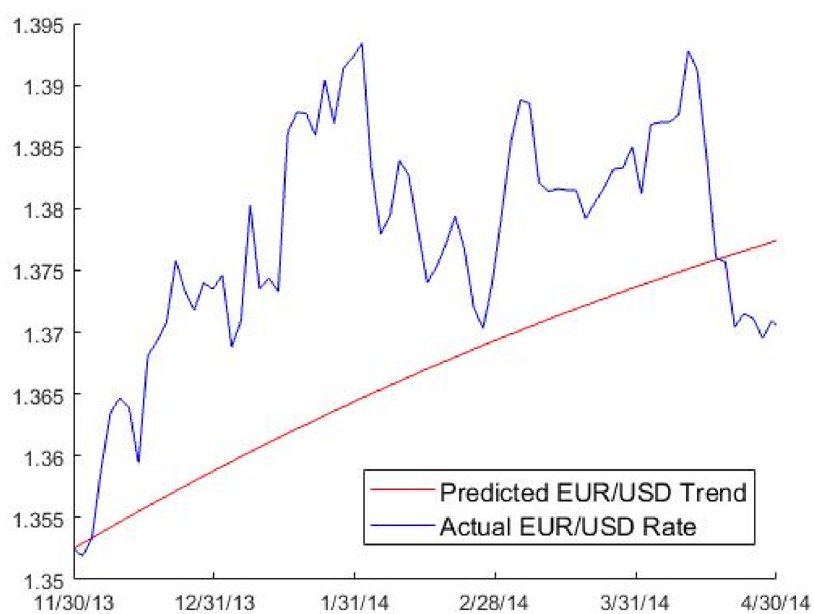


FIGURE 4.8. Comparing the predicted trend of EUR/USD rate from 7/31/2016 to 12/31/2016

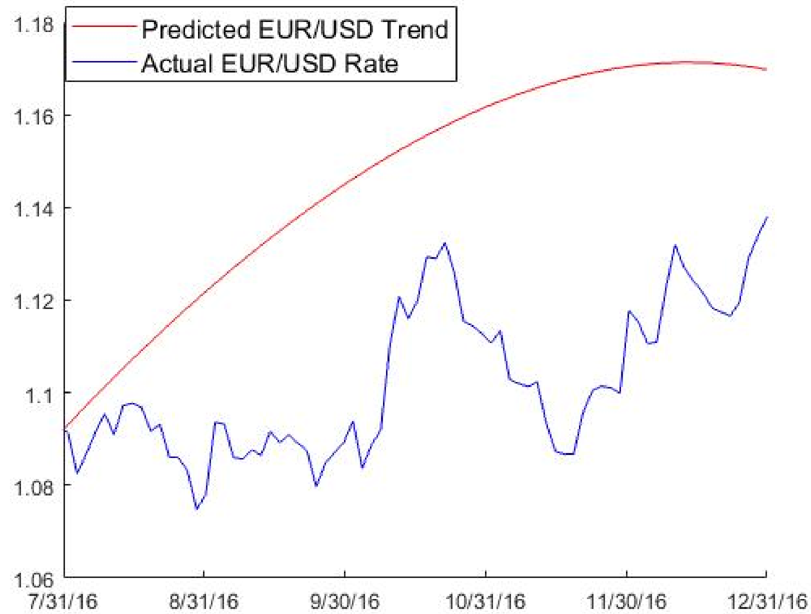


FIGURE 4.9. Comparing the predicted trend of EUR/USD rate from 1/31/2017 to 6/30/2017

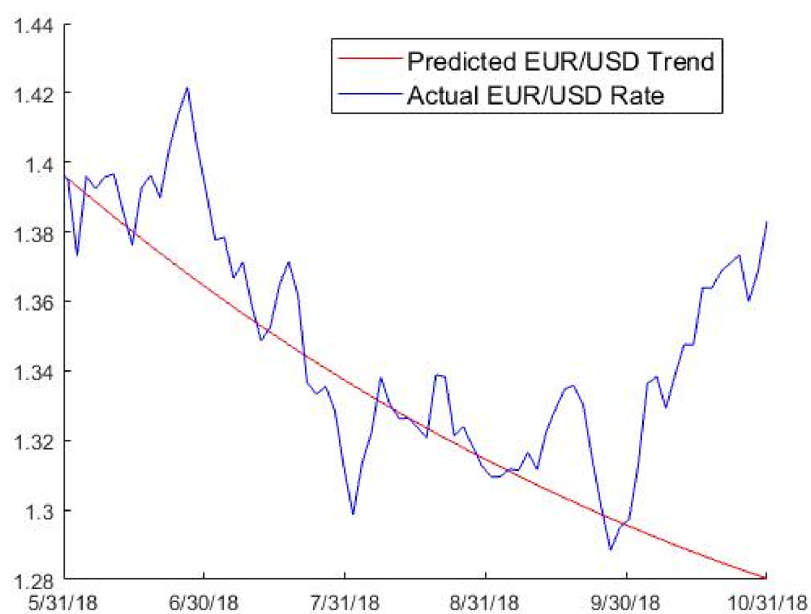


FIGURE 4.10. Comparing the predicted trend of EUR/USD rate from 5/31/2018 to 10/31/2018

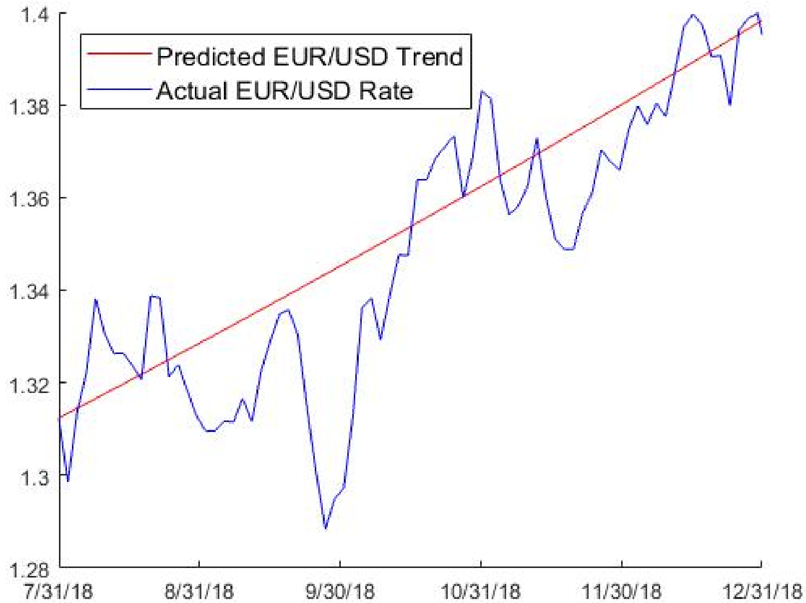


FIGURE 4.11. Comparing the predicted trend of EUR/USD rate from 7/31/2018 to 12/31/2018

Not all of these predictions are accurate, and while some of them are accurate, others are slightly off trend, and some have the correct shape only, and others are accurate for a couple of months only. Thus we have to devise a measure of the accuracy of our prediction model. For that, we define the term "successful prediction" as any prediction from which one can gain insight about the future exchange rate. Successfully predicted behavior of a Forex rate allows an investor to trade currency options accordingly. For predictions made from 2000 through 2019, there are some successful and unsuccessful trend predictions. Out of those successful predictions, we categorize them in different groups: accurate, slightly off trend, correct direction, accurate for short term or long term and so on. Based on those predictions, some are accurate for only short time like two-three months whereas some are accurate for a longer term such as for a year. All these have a very important role in predicting the future exchange rate of any currency pair. Sometimes the trend appears inaccurate for short time, but provides accurate trend over the long term. Some of the predicted trends are very unlike the actual trends. The graphs in figures from 4.12 to 4.23 are examples of unsuccessful predicted trends. The predicted trends with correct direction compared to the actual trend are considered successful whereas trends having different directions with actual trends are unsuccessful.

An alternate method to measure the accuracy of a predictive model is to look at the mean absolute percentage error (MAPE), which is defined as

$$MAPE = \frac{100}{n}\sum_{t=1}^{n} \left| \frac{A_t - F_t}{A_t} \right|, \tag{4.1}$$

where A_t is the actual exchange rate at time t, and F_t is the value of the forecasted trend at time t. Here, we choose to use a relative error instead of an absolute error, such as the mean square error, because the change in exchange rate over time make an absolute error harder to interpret since the error size depends on the scale of the data. Thus, we could define a successful prediction as one with less than a 10 percent mean absolute percentage error. Our model yields 62.86 percent successful predictions. We note in passing that Wall Street traders with a 55 percent or greater trading success rate are superstars.

Although using the mean absolute percentage error provides us with an unbiased measure of accuracy, it also has its drawbacks. Firstly, it is important to note that the MAPE is still a mean and is therefore sensitive to outliers in the data set. Also, the MAPE does not detect if the model correctly predicts the direction of the underlying exchange rate, only the degree of separation between the actual exchange rate and predicted trend.

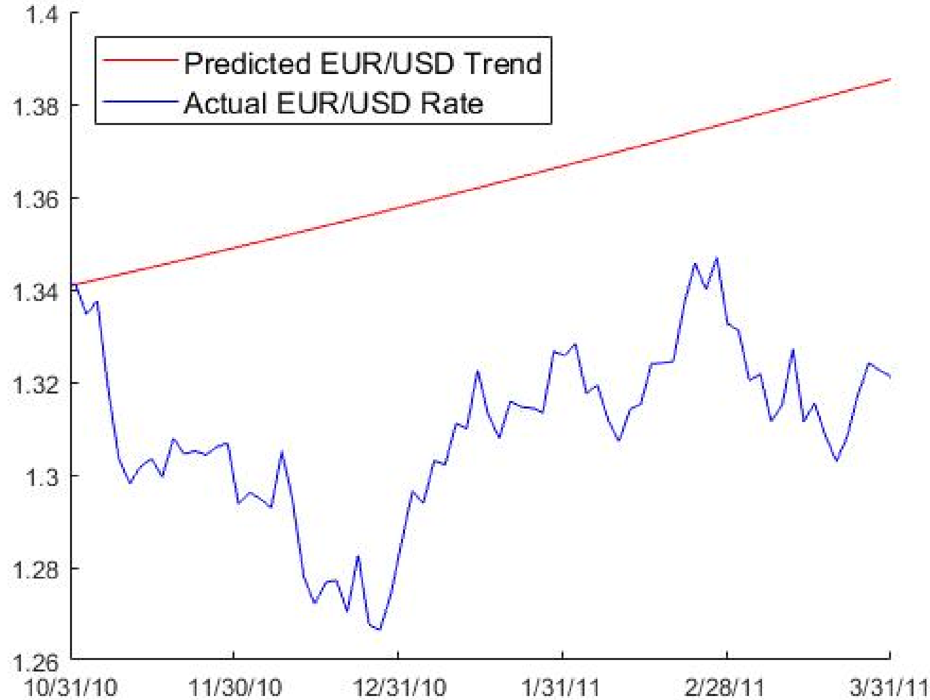


FIGURE 4.12. Comparing the predicted trend of EUR/USD rate from 11/30/2010 to 3/31/2011

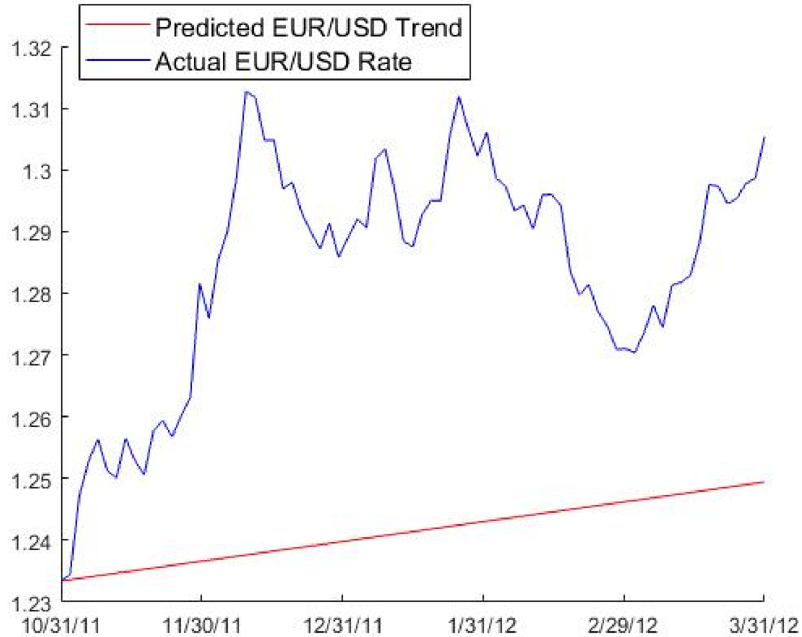


FIGURE 4.13. Comparing the predicted trend of EUR/USD rate from 11/30/2013 to 4/30/2014

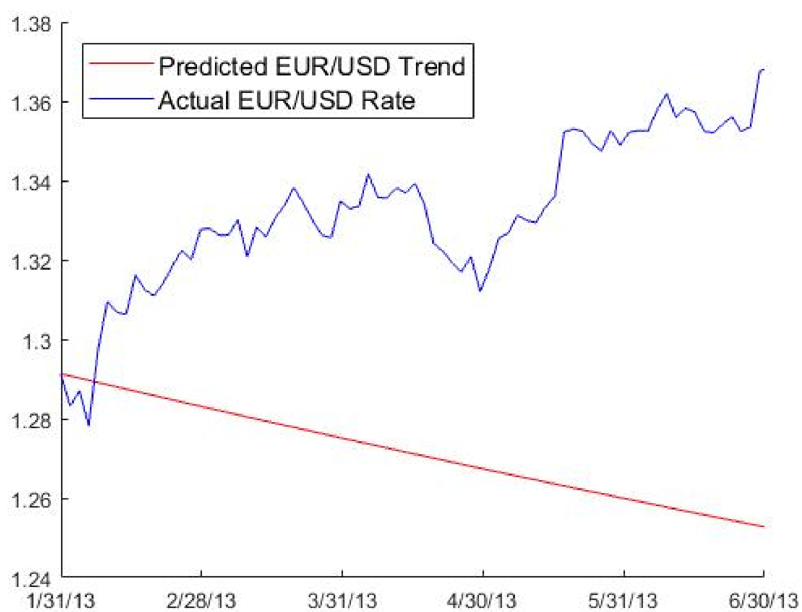


FIGURE 4.14. Comparing the predicted trend of EUR/USD rate from 7/31/2010 to 12/31/2010

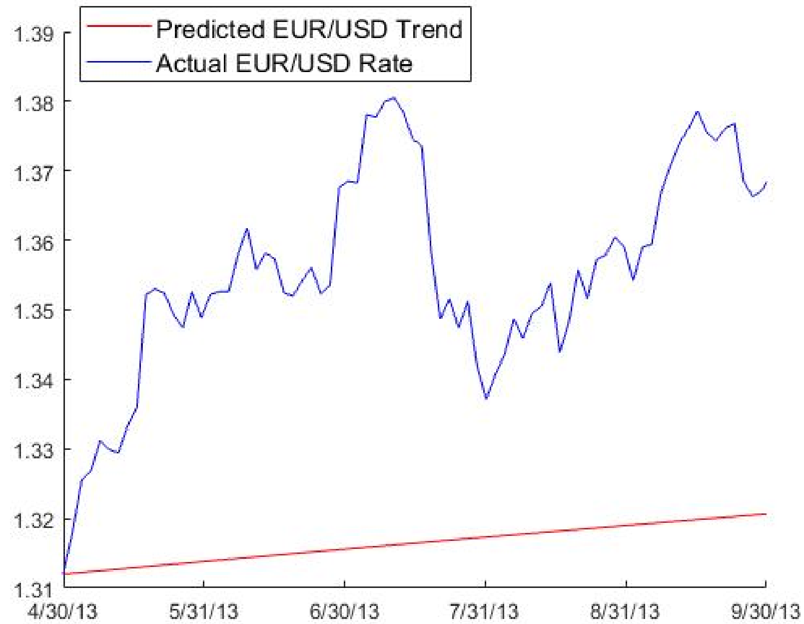


FIGURE 4.15. Comparing the predicted trend of EUR/USD rate from 4/30/2012 to 9/30/2012

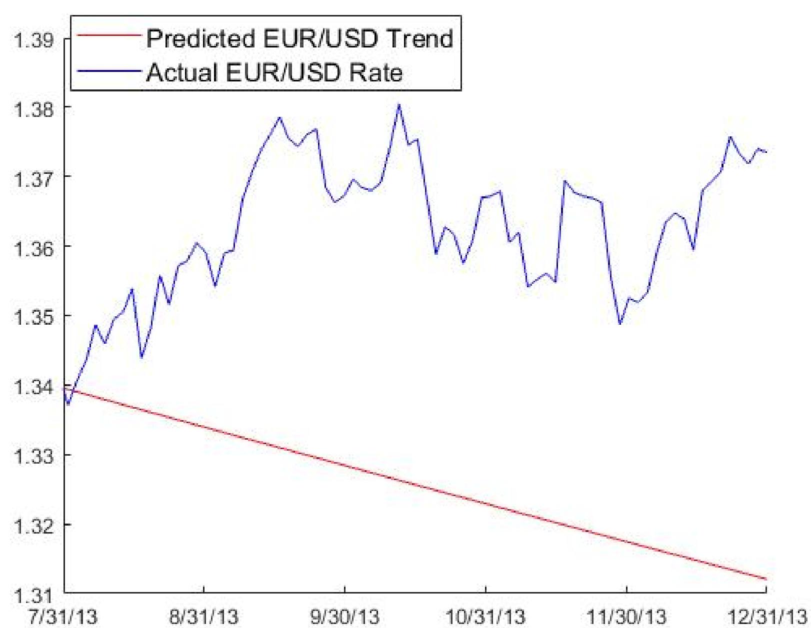


FIGURE 4.16. Comparing the predicted trend of EUR/USD rate from 10/31/2012 to 3/31/2013

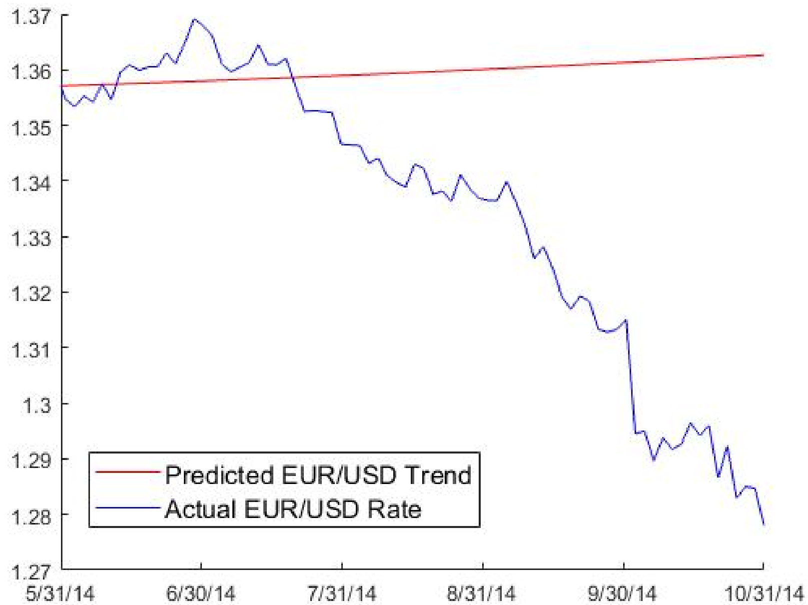


FIGURE 4.17. Comparing the predicted trend of EUR/USD rate from 7/31/2013 to 12/31/2013

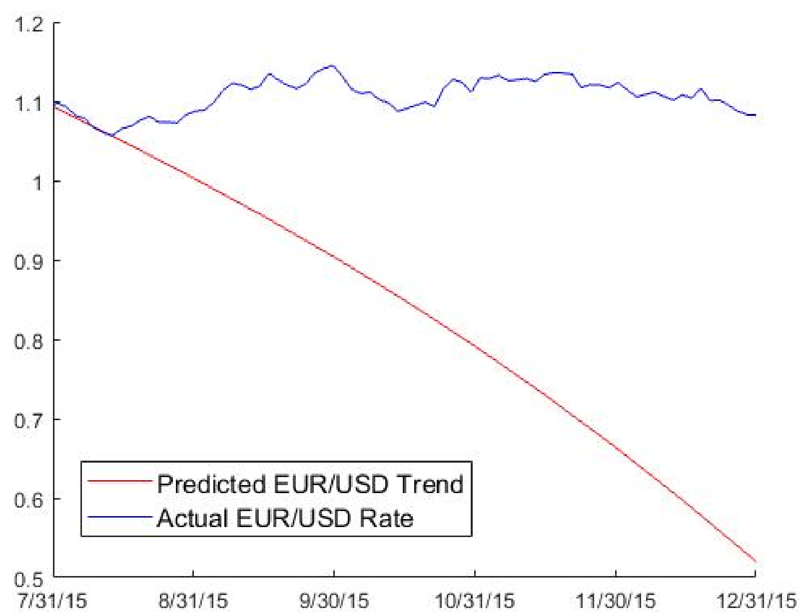


FIGURE 4.18. Comparing the predicted trend of EUR/USD rate from 5/30/2014 to 10/30/2014

Both figures 4.17 and 4.18 are showing that the predicted EUR/USD is partially accurate compared to the actual rate, that is the model provides the correct predicted rate for the short term, and after that time period the predicted and actual trend have very different directions. Some of the graphs in figures 4.20, 4.21 and 4.22 have a completely different predicted trend from the actual one.

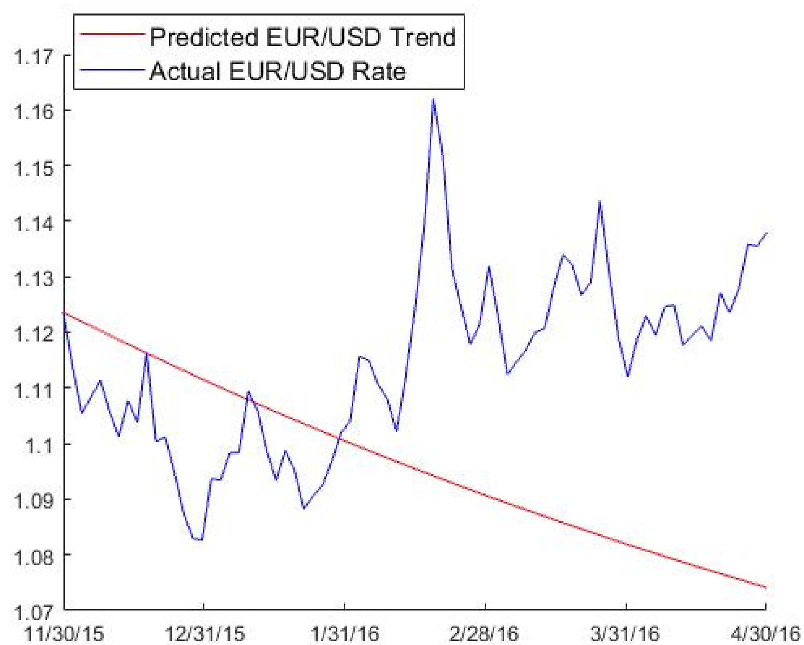


FIGURE 4.19. Comparing the predicted trend of EUR/USD rate from 1/31/2015 to 6/30/2015

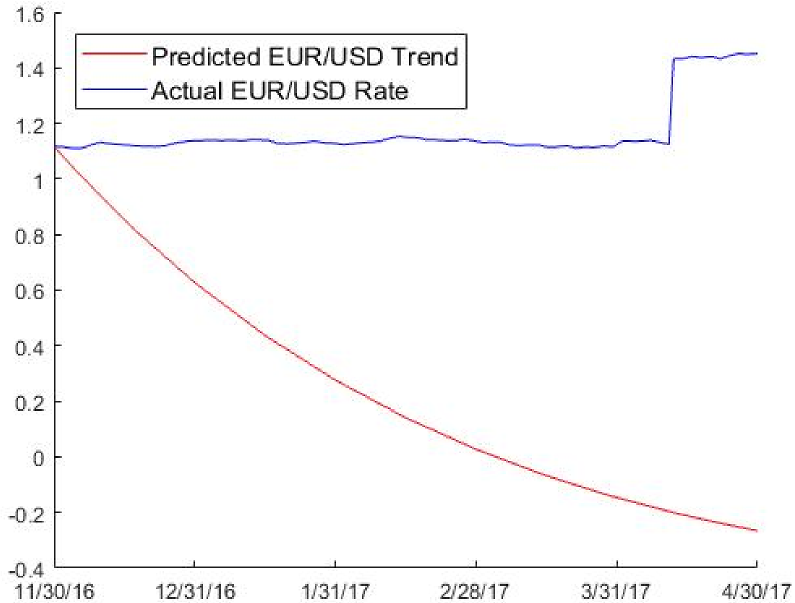


FIGURE 4.20. Comparing the predicted trend of EUR/USD rate from 7/31/2015 to 12/31/2015

FIGURE 4.21. Comparing the predicted trend of EUR/USD rate from 11/30/2016 to 4/30/2017

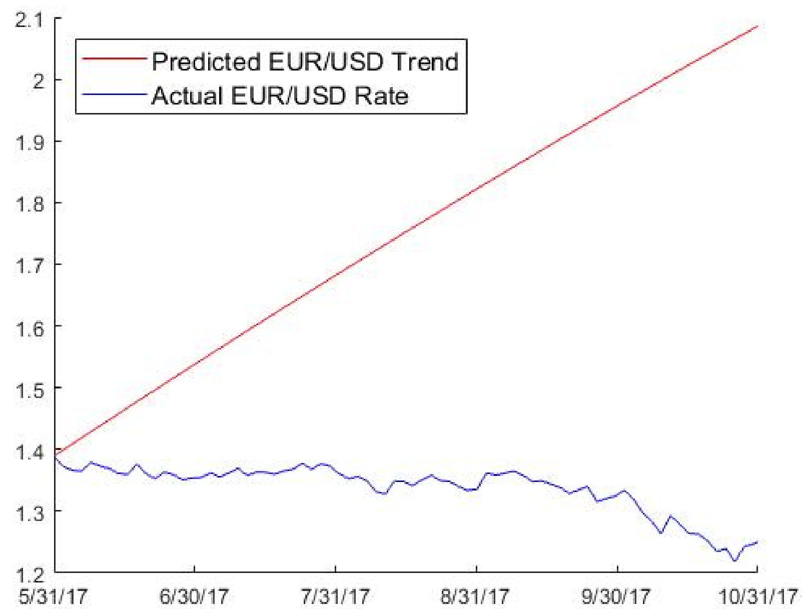


FIGURE 4.22. Comparing the predicted trend of EUR/USD rate from 5/31/2017 to 10/31/2017

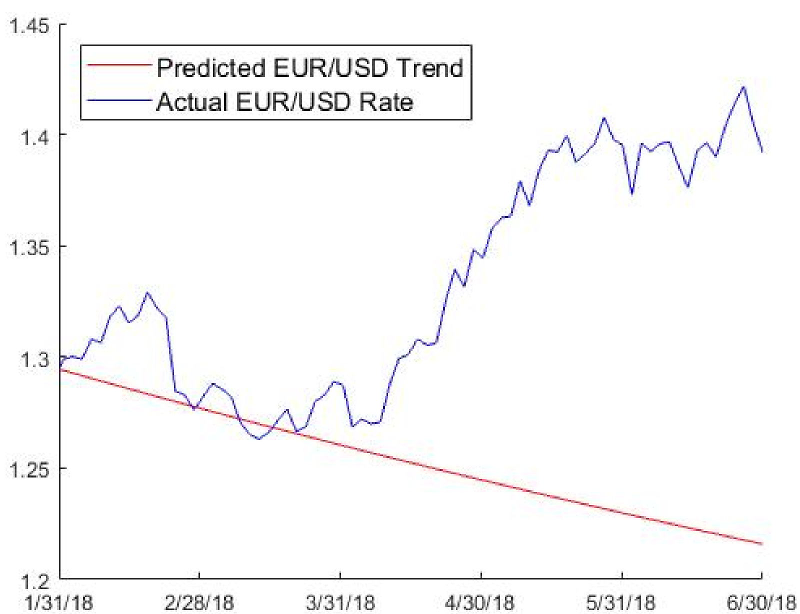
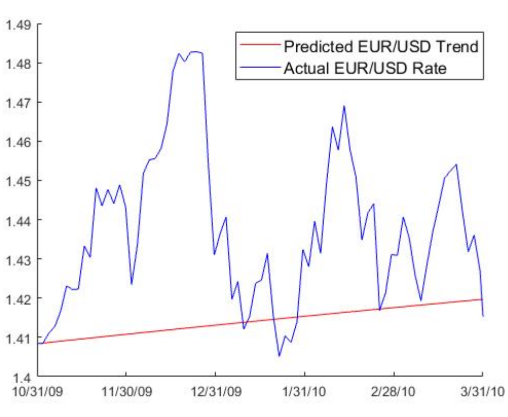


FIGURE 4.23. Comparing the predicted trend of EUR/USD rate from 1/31/2018 to 6/30/2018

2. A Forex Exchange Rate Model

In this section we combine information from our ODE trend prediction model with the predicted future volatility obtained from FX option data. The predicted the trend and recovered volatility of currency exchange rates are combined in equation (3.1) and simulated in Matlab using Matlab SDE solver dde23 with geometric Brownian motion (GBM) which produces the predicted currency exchange rates. However, not all the simulations provide the exactly same predicted rates. For example, figures 4.24 and 4.26 have predicted rate similar to actual rate whereas figures 4.27 and 4.29, predicted trend and predicted rate are which are almost same as actual rate in the starting of the duration but different at the end. They are not partially successful predicted rates. Some of the results are given below observe that quite often the prediction using the ODE trend model together with the volatility information (see the right side graphs below) is superior to the prediction using only the trend (see the left side graphs below).



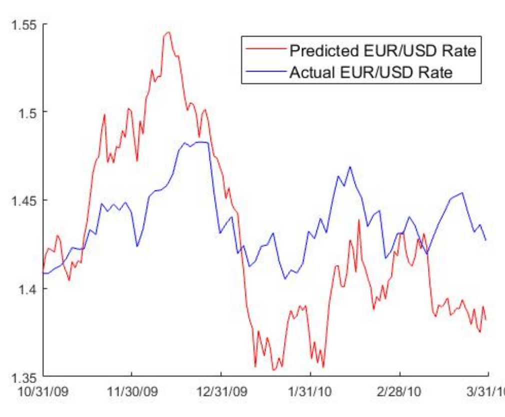


FIGURE 4.24. Predicted trend and exchange rates of from 10/31/2009 to 3/31/2009

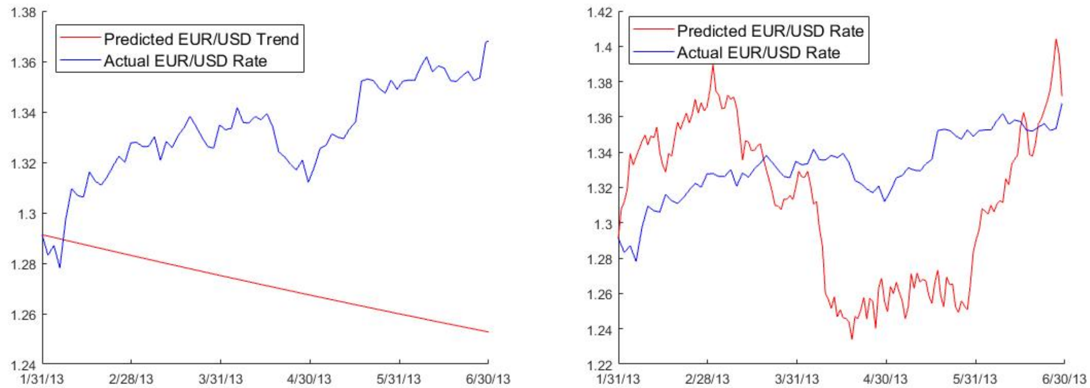


FIGURE 4.25. Predicted trend and exchange rates of from 10/31/2010 to 3/31/2011

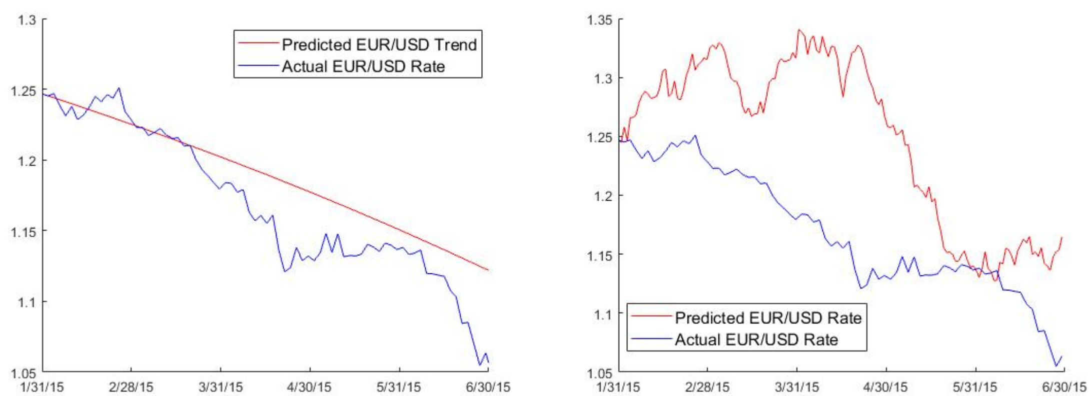


FIGURE 4.26. Predicted trend and exchange rates of from 11/30/2016 to 4/30/2017

Figure 4.30 shows the predicted exchange rate and actual rate are accurate. But in some cases such as in figures 4.27 and 4.28, the predicted trend is same as actual trend but the rate is different whereas in figure 4.25 the predicted rate is same as actual rate, however the predicted trend is not same.

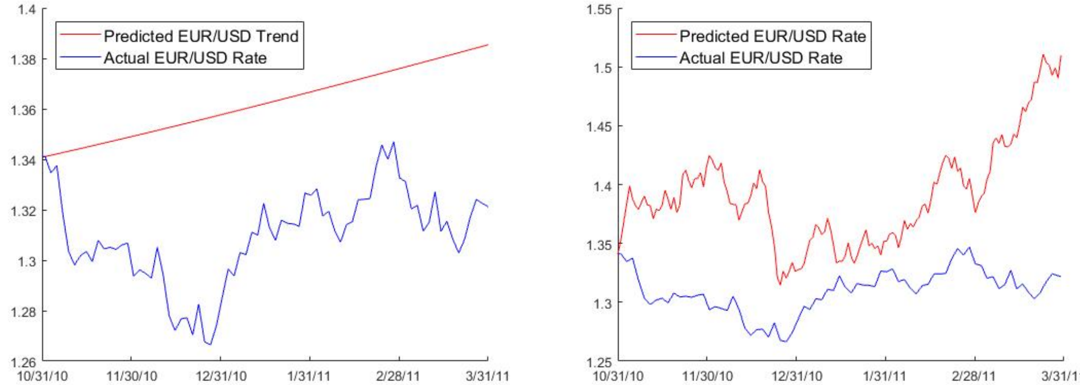


FIGURE 4.27. Predicted trend and exchange rates of from 7/31/2015 to 12/31/2015

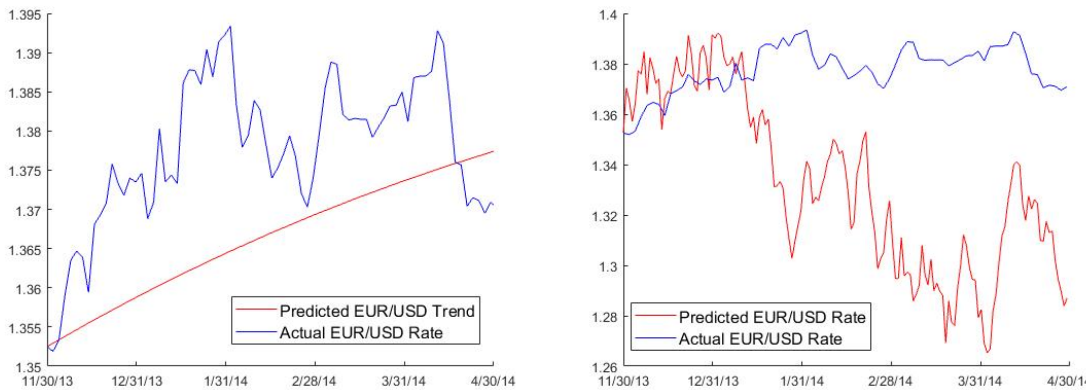


FIGURE 4.28. Predicted trend and exchange rates of from 7/31/2013 to 12/31/2013

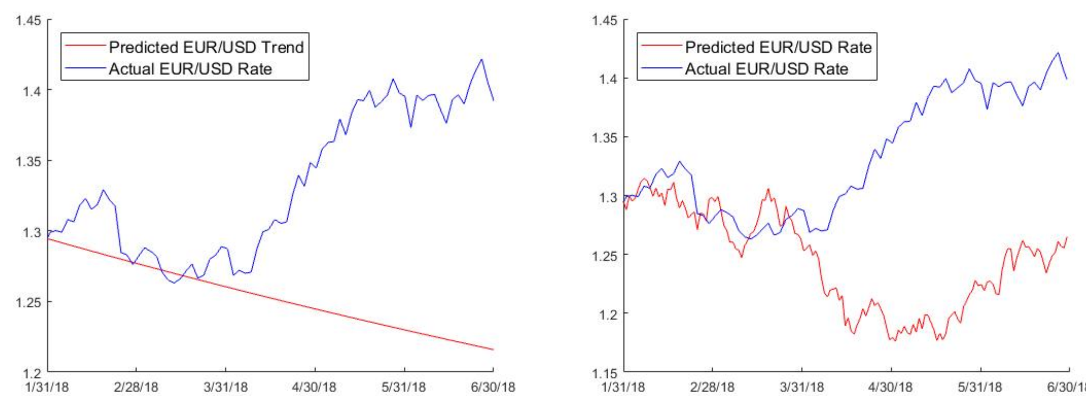


FIGURE 4.29. Predicted trend and exchange rates of from 1/31/2018 to 6/30/2018

After numbers of matlab simulations, we have both successful and unsuccessful results, some of which are given in figure 4.30:

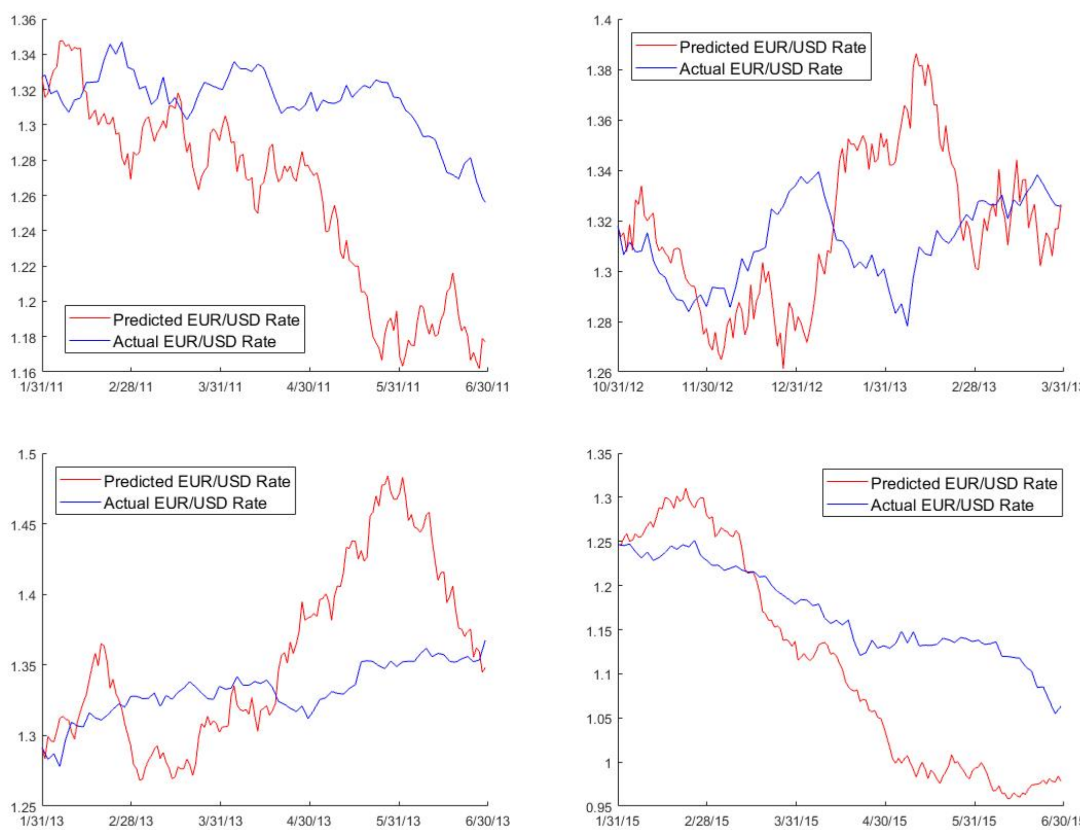


FIGURE 4.30. Some successful prediction of exchange rates.

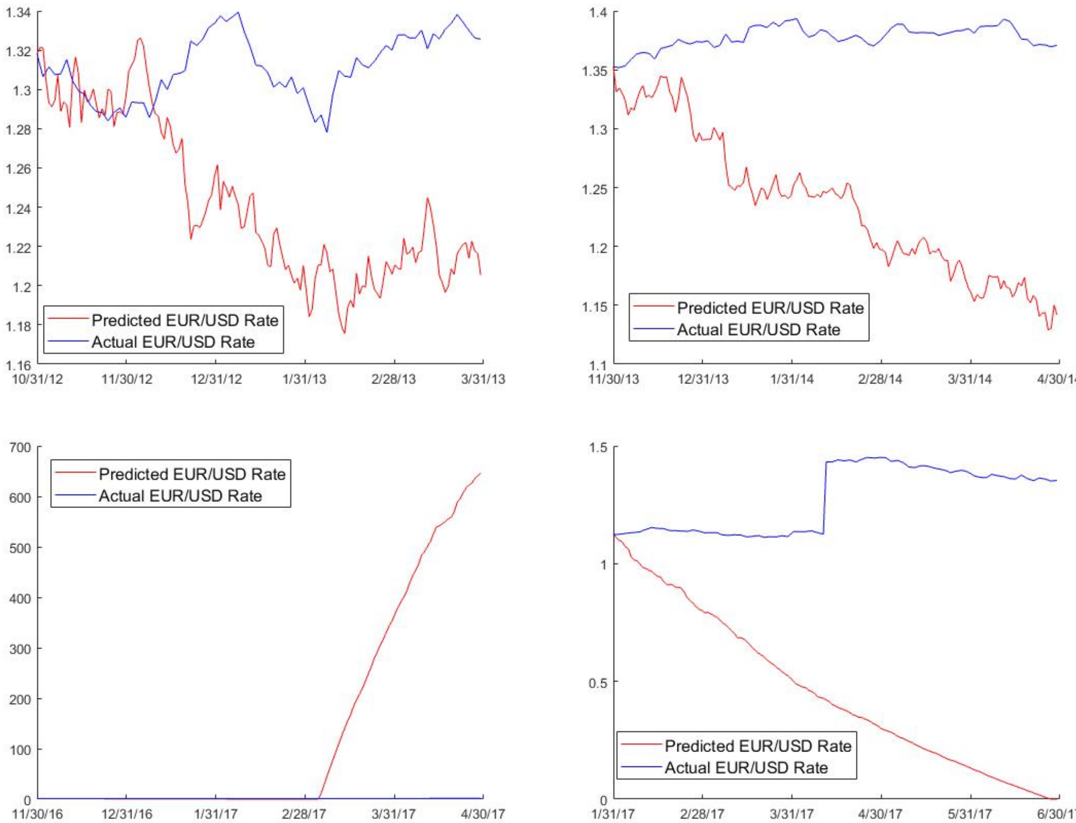


FIGURE 4.31. Some unsuccessful prediction of exchange rates.

CHAPTER 5

Statistical Inferences

We have obtained many Forex predictions using our model, however, not all of these are successful. So we now study the successful predictions of both trend and rate statistically. In particular we now perform a statistical hypothesis test on randomly selected predictions using data from January 1, 2000 to January 1, 2020.

1. Overview of Statistical Inferences

Basic terms related with statistical inferences are hypothesis and hypothesis testing. In Statistics (Elementary Statistics), a hypothesis is a claim or statement about a property of a population whereas a hypothesis test is a standard procedure for testing a claim about a property of a population. There are two types of hypothesis. The first is the null hypothesis, denoted by H_0, which is defined as a statement that the value of a population parameter (such as proportion, mean, or standard deviation) is equal to some claimed value. The second is an alternative hypothesis, denoted by H_1, and this is the statement that the parameter has a value that somehow differs from the null hypothesis. We use symbols of not equal, less than and greater than to represent the alternative hypothesis. So the following conclusions can be made:

- if the null hypothesis is rejected, the alternative hypothesis is accepted;
- if the null hypothesis is accepted, the alternative hypothesis is rejected.

Statistical inference is the process of using data analysis to deduce properties of an underlying distribution of probability. Inferential statistical analysis infers properties of a population, for example by testing hypotheses and deriving estimates. It is assumed that the observed data set is sampled from a larger population. Inferential statistics can be contrasted with descriptive statistics which is solely concerned with properties of the observed data, and it does not rest on the assumption that the data comes from a larger population.

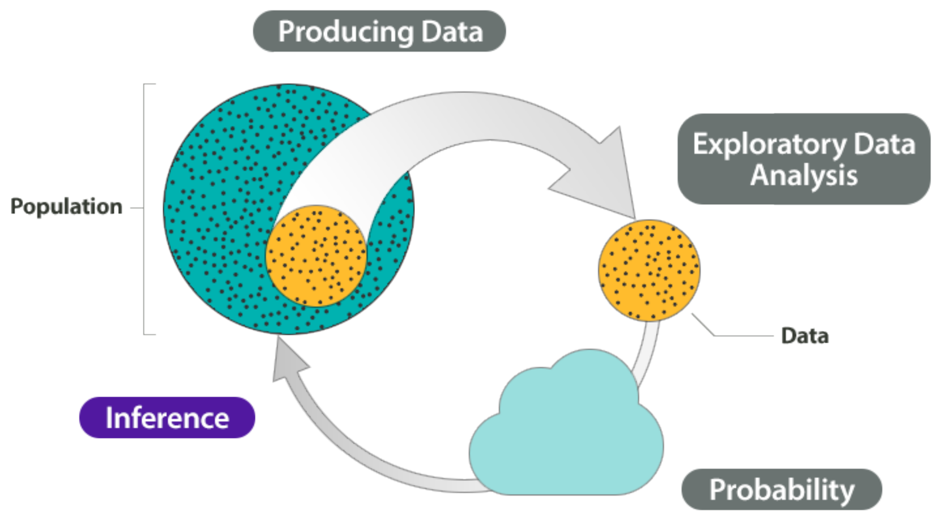


FIGURE 5.1. Statistical Inference (Source: OER Services, Concepts in Statistics)

We can formulate propositions about a population using statistical inference with the help of data drawn from the population with some form of sampling. Given a hypothesis about a population, for which we wish to draw inferences. Statistical inference consists of two keys: a) selecting a statistical model of the process that generates the data and b) deducing propositions from the model. While making propositions, the majority of the problems in statistical inference can be considered to be problems related to statistical modeling. The conclusion of a statistical inference is a statistical proposition. Some common forms of statistical proposition are the following:

- a point estimate: a particular value that best approximates some parameter of interest;
- an interval estimate: a confidence interval (or set estimate), i.e. an interval constructed using a dataset drawn from a population so that, under repeated

sampling of such datasets, such intervals would contain the true parameter value with the probability at the stated confidence level;

- a credible interval: a set of values containing, for example, 95 % of posterior belief;
- rejection of a hypothesis;
- clustering or classification of data points into groups

In our case, we use the concept of an interval estimates.

2. Model and Statistical Analysis

Our population consists of 228 months (N=228), due to the fact that we must have a minimum of 20 past months to solve for the coefficients. Therefore, we are only able to predict EUR/USD rate after August 31,2002. In order to perform a hypothesis test, we utilize the de Moivre-Laplace theorem which states that the probability mass function for a binomial distribution with n trials and probability, p, of success in each trial converges to the probability density function of the normal distribution with mean np and variance np(1-p). As always, we let q=1-p. In other words, when $n \geq 30$ we have that

$$X = bin(n,p) \sim N(np, npq),$$

then

$$\hat{p} = \frac{x}{n} \sim N\left(p, \frac{pq}{n}\right),$$

where x is an observed value of a binomial random variable, $X = bin(n,p)$. We now have that

$$\frac{\hat{p} - p}{\sqrt{\frac{pq}{n}}} \sim N(0,1).$$

We use this to test the null hypothesis that our model is successful 50 percent of the time:

$$H_0 : p = 0.5,$$

versus the alternate hypothesis that it is successful more than 50 percent:

$$H_1 : p > 0.5,$$

using a five percent significance level.

We choose to test if the model is successful more than 50 percent of the time based on the fact that if someone is randomly guessing whether a exchange rate will increase or decrease, it is expected that he will be successful 50 percent of the time. However, in the long run, if you can provide accurate predictions more than 50 percent of the time, it is expected that you will make a profit. Casinos often take advantage of this idea. For example, a casino's house edge for blackjack is only 0.5 percent. In other words, the casino is only expected to win 50.5 percent of the time.

Using a random sample of size of n=70 with 44 successful results, we obtain, $\hat{p} = \frac{44}{70} = 0.6286$, and we calculate a test statistic of

$$z = \frac{\hat{p} - p}{\sqrt{\frac{pq}{n}}} = 2.15,$$

which gives us

$$p - value = P(z \geq 2.15 | H_0) = 0.015778$$

Hence, the evidence suggests that our model provides successful predictions more than 50 percent of the time. We also construct a 90 percent confidence interval, which provides us with the following range for the accuracy of our model:

$$0.5303 < p < 0.7269$$

CHAPTER 6

Conclusion and Future Work

Some of the conclusions are made from the models and results we obtained from the models we used for forecasting currency exchange rates. The work we have done is not 100% successful and results in predictive models, so there will always be some work in progress as future work to get more successful and reliable results. Thus, we can analyze our model with the conclusion obtained from the results and a SWOT analysis.

1. Conclusion

In chapter 2, we built a model from factors that affect the currency exchange rate and this was only for predicting the trend of the exchange rate. More than 100 trials had output, with some of them quite close to the actual trend whereas others are less close and the rest of them are totally different to the actual trend of the exchange rate. Furthermore, the second part of the thesis involves finding local volatility is presented in chapter 3, which is an equally important part of this project. We use stochastic differential equations, and the Garman-Kohlhagen equation along with concept of an inverse volatility problem for exchange rates, that is able to provide implied and recovered volatility. Finally, we are able to predict the future exchange rate of the currency pair EUR/USD using the stochastic differential equation (3.1) with predicted trend and recovered volatility. After statistical analysis, currency exchange rate can be successfully predicted more than 50 percent of the time using a system of nonlinear delay differential equations consisting macroeconomic variables as factors affecting US and Euro currencies. Moreover, our model executes 62 percent successful in its predictions for the period 2010 to 2019.

2. SWOT Analysis

SWOT Analysis is the one of the best methods for analyzing the any predictive model, when discussing its strengths, weaknesses, opportunities, and threats. Most often, such analysis helps a trader to formulate decisions and strategies for successful forecasting.

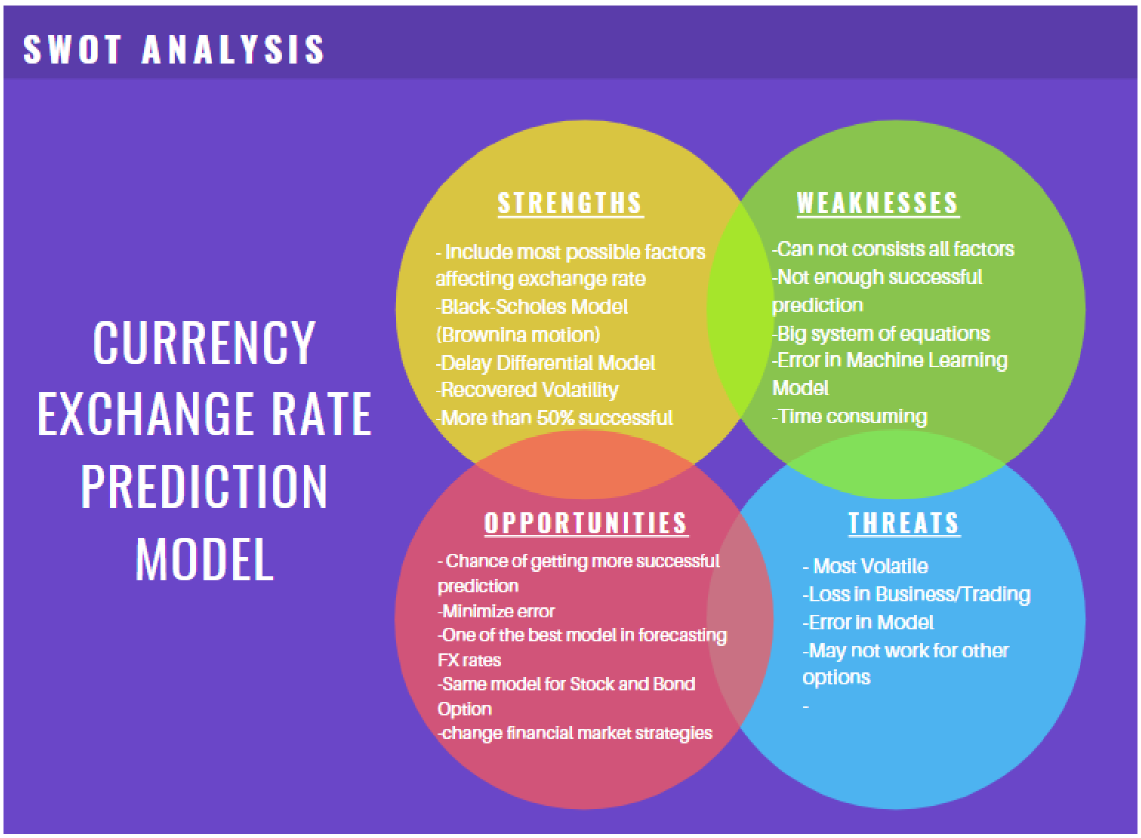


FIGURE 6.1. SWOT Analysis

- **Strengths:** There are many macroeconomic variable or factors that affect exchange rate of currency pair EU/USD, however our model consists of most possible factors such as interest rates, GDP, inflation rates of US and Euro zone that affect exchange rate of currency pair. This model applies one of the strong Model: Garmann-Kohlhagen model (similar model from Black Scholes (BS) Equation Model (Brownian Motion)) of the financial market. The Garmann-Kohlhagen model is recognized very useful and reliable model for option pricing that provides practical results with the help of Geometric Brownian Motion. Furthermore, trend is predicted by mathematical model, called delay differential equation model, which predicts the future trend, for instance exchange rate in at least 17 months. Another strength is that the model is able to recover volatility which is very unpredictable. After, nice prediction of trend and recovery of volatility, the model is more than 50% successful to predict the future exchange rate.

- **Weaknesses:** The delay differential equations involves only 11 macroeconomic variables, however there are other factors that may affect the exchange rate. Also, the system of delay differential equations contains many variable, so that we face sometimes trouble to solve the system. Because of big system and more variables, it takes time to find the solution of the system. The model provides more than 50% prediction but it is not enough successful that traders are not sure to trade option or not. Even if you build alternative method or models of forecasting currency exchange rates, they are not accurate (will have at least some percent of error).
- **Opportunities:** Traders are attracted in trading options if the predictive models provide more successful or accurate forecasting. So if we evaluate most influencing macroeconomic variables that may yield 80-90 percent of accuracy or we can modify models with variables and number of equations that can provide more successful prediction. In this way, we can minimize error or increase accuracy which helps to maximize the profits in trading. Once one can find the most effective macroeconomic variables and right numbers of equations in system delay differential equations that could possibly forecast exchange rate more accurately as establishing as one of the best models in forecasting FX rate. When we get more successful results, then we can use same model or concepts for stock market and bond market, which has second and third largest volumes in financial market. Moreover, these models not

only help traders to maximize profits but also assist economists or financial analysts to design the financial market strategies that grows the market worldwide.

- **Threats:** The big threat is the financial market is the most volatile, and some of the currency pairs are very volatile, which may cause big loss in business and put traders into financial crisis. The error of model can be minimized but can not be zero. So there is no such model that has no error. This simply implies that the model is not 100 percent accurate. the same model, sometimes, may not exactly work for other options because of their type (European and American).

3. Future Work

As mentioned in the opportunities section of the SWOT analysis, we can build better model that can yield a more accurate prediction of the FX rates, so our task would reduce to finding more macroeconomic variables, and designing models with delay differential equations consisting of more variables. We can also form more ambitious plan to predict nation's economy, provided we succeed in predicting nation's stock and bond market properly.

Next, we plan to predict the FX rate by applying machine learning techniques, such as time series analysis, logisitic regression and support vector machine/regression models. The time series analysis provides the 50 percent of accuracy as given below.

```
Testing Accuracy: 0.507246


     return  label  pred    won   pnl  equity
132 -0.0040    0.0   0.0   True  40.0     9.0
133  0.0012    1.0   1.0   True  12.0    21.0
134  0.0048    1.0   0.0  False -48.0   -27.0
135  0.0055    1.0   0.0  False -55.0   -82.0
```

```
136   0.0031    1.0   1.0   True  31.0   -51.0


Net Profit            : $-51.00
Number Winning Trades : 69
Number Losing Trades  : 68
Percent Profitable    : 50.36%
Avg Win Trade         : $38.884
Avg Los Trade         : $-40.206
Largest Win Trade     : $104.000
Largest Los Trade     : $-97.000
Profit Factor         : 0.98
```

The time series analysis reads the historical data of EUR/USD and use its model analysis, and we obtain the results with 50.72 percent accuracy of model. Along with accuracy, it provides the true/false (win/loss) prediction for FX options trade.

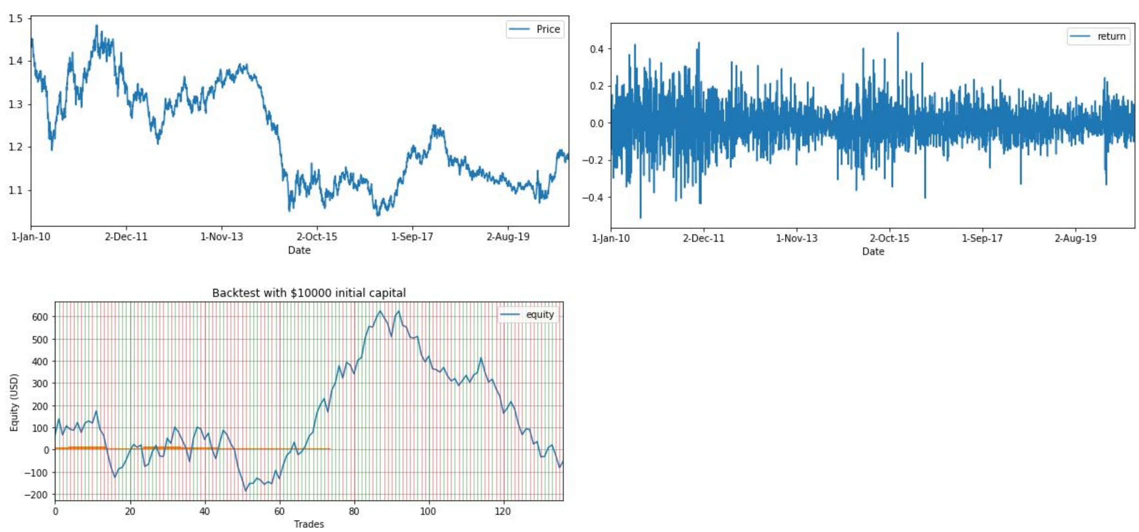
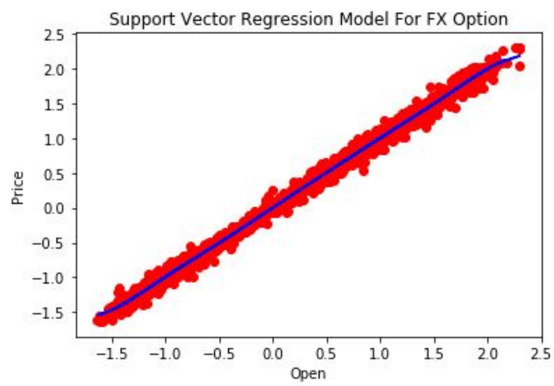


FIGURE 6.2. Exchange Rate, Volatility, and Equity obtained from Time Series Analysis

The first graph of figure shows the EUR/USE rate, the second graph is about the changes in rates, and third graph implies the FX option. In third graph, it does not provides the 100 percent accurate results for trading FX option. We have to improve this analysis, as a future work, to obtain more accurate outputs. The results and accuracy obtained from time series analysis is incomplete work. So our future work would be to find more accuracy, along with promising predicted exchange rate.

Similarly, if you apply the support vector machine/regression (SVM/SVR) approach, we can find the predicted rate but it comes with error as shown in figure 6.3



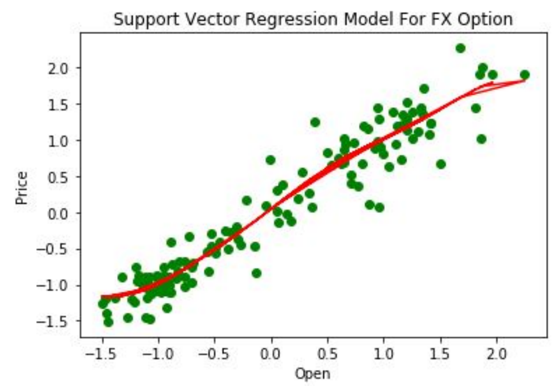


FIGURE 6.3. SVM Analysis of EURUSD on daily and monthly basis

We will seek parameters for the SVM that yields a relevant prediction with less error.

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