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EXPONENTIAL LAMINATIONS

by

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A THESIS

Submitted to the graduate faculty of The University of Alabama at Birmingham,
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EXPONENTIAL LAMINATIONS

PATRICK BARRY HARTLEY

MATHEMATICS

ABSTRACT

Laminations have proved useful in proving facts about polynomial Julia sets. In this thesis, a process by which to construct laminations for certain exponential Julia sets will be presented. This thesis will consider those exponential Julia sets where hairs are identified at a single repelling fixed point, as opposed to multiple repelling fixed points or repelling periodic points. This construction will make use of known facts about the complex exponential function and an itinerary system similar to those used in polynomial laminations.

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Abbreviations and Symbols

$\mathbb{R}$ will stand for the real line

$\mathbb{C}$ will stand for the complex plane

E_λ will stand for λe^z, the complex exponential function with complex parameter λ

$J(E_\lambda)$ will stand for the Julia set of E_λ

$F(E_\lambda)$ will stand for the Fatou set of E_λ

1 Introduction

In this paper we will step through a description of a structure of the Julia set $J(E_\lambda)$ for the complex exponential function, where $E_\lambda(z) = \lambda e^z$, from [BD00] and then use that structure to construct a combinatorial model called a lamination.

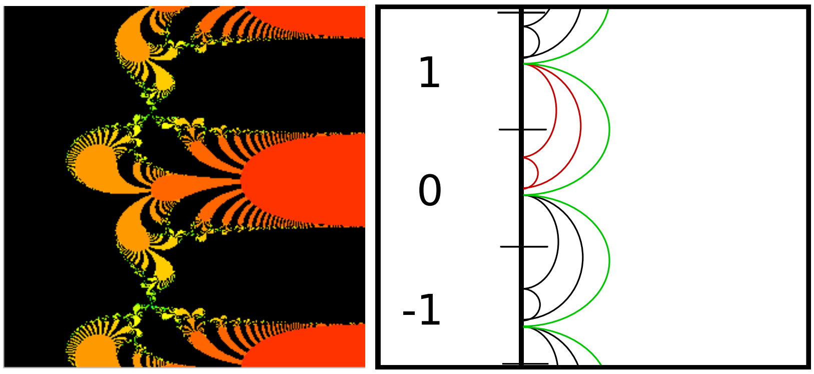

Figure 1: Example of an exponential Julia set and associated partial lamination for parameter $\lambda = 3 + \pi i$.

Laminational models have been used to study the Julia sets for complex polynomial functions for some time. At first laminations were constructed by taking a known quadratic Julia set and building a model that represented it and its dynamics. Then a set of rules was developed such that any connected, quadratic Julia set will be represented as a lamination. In addition, any lamination following those rules potentially could represent an actual polynomial Julia set. This led to studying laminations and the dynamics on them with the knowledge that theorems proved about laminations would have implications for polynomial Julia sets as well. An example of such a theorem is the central strip lemma for quadratic polynomials $P_c(z) = z^2 + c$, for complex parameter c. Proved by Thurston in [T09], the central strip lemma describes some limitations on the movements of chords in a lamination under iteration. This was then used to show that there are specific dynamical behaviors that cannot occur in Julia sets of quadratic polynomials.

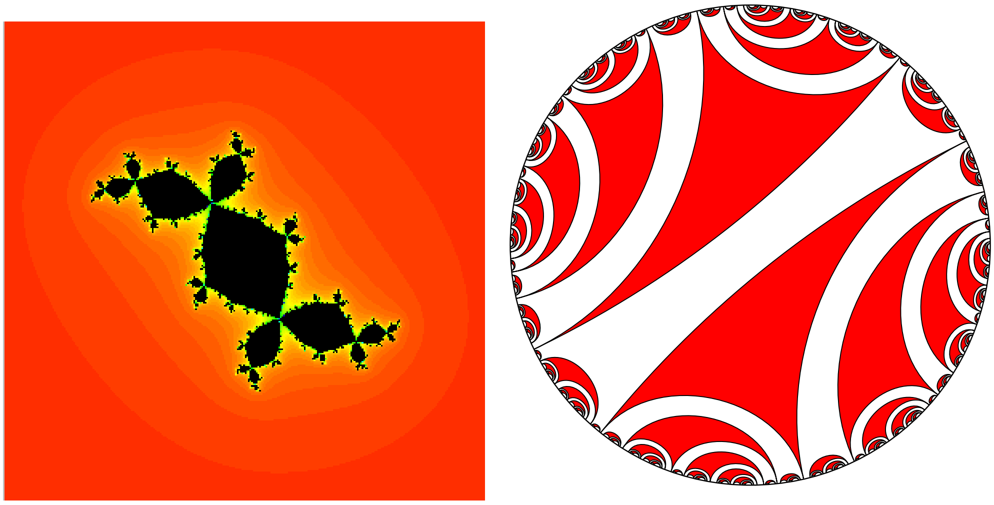

Figure 2: A Douady Rabbit quadratic Julia set, $P_c(z) = z^2 + c$ for parameter $c = -.125 + .75i$, and associated lamination.

The Julia set $J(E_\lambda)$ is defined as the closure of the set of points that escape to ∞ under iteration, and the Fatou set $F(E_\lambda)$ as the complement of the Julia set. Analogously in the polynomial case, the Julia set is the boundary of the set of points that escape to ∞ [M06]. The exponential function is strongly analogous to the quadratic function because they both have just one singular point. Quadratic functions have one critical point and the exponential function has one omitted value. To get an idea of the behavior of E_λ, we take a look at it's dynamics under only a few iterations with $\lambda = 1$. E_λ maps vertical lines to circles around the origin, the half plane to the left of that line inside of the circle, and the other half plane to the outside of the circle. This shows that every horizontal strip of width 2π maps over $\mathbb{C} \setminus \{0\}$. Because of the 2π periodicity of E_λ up and down the imaginary axis, we get infinitely many horizontal strips of width 2π that map over $\mathbb{C} \setminus \{0\}$. If we choose our strips so that the horizontal border lines (height odd multiples of π) map to the negative real axis, under two iterations a π wide strip around each of them will map into the interior of a circle about the origin. It is known that in the case of an attracting fixed point, points near enough to the origin will be attracted to said fixed point. This implies, if we have an attracting fixed point, that far enough to the right our strips that map into a circle about the origin in two steps will map into a small enough circle that they will ultimately be attracted to the attracting fixed point. This shows that the strips in question will not be in the Julia set and thus will fall in the Fatou set. Changing λ results in a few changes. Changing the argument of λ results in shifting horizontally the strips that map in two iterations into a circle about the origin, and changing the modulus of λ changes the radius of the aforementioned circle about the origin. This ultimately results in shifting up or down our strips in the far right half plane that are in the Fatou set.

Knowing this, we would expect an image of an exponential Julia set to show infinitely many horizontal strips in the far right half plane lying in the Fatou set and the Julia set lying between these strips. This can be seen in Figure 3. Closer to the vertical axis is where things get more complicated.

In the case of a λ that admits an attracting fixed point, it is known that the Julia set is a Cantor Bouquet (Figure 3)[BD00][DG87][B99]. In this paper we will look at the case when there is an attracting cycle of period $n \geq 3$. (The case for an attracting fixed point $n = 1$, is covered in [DG87]. The case for $n = 2$ is trivial.)

Figure 3: An exponential Julia set with an attracting fixed point. Parameter $\lambda = -2$

Another behavior of note is what we call pinching. In the quadratic case $P_c(z) = z^2 + c$, as can be seen in Figure 4, with a parameter $c = 0$ the Julia set is the unit circle. As the parameter is moved, points on the circle begin to come

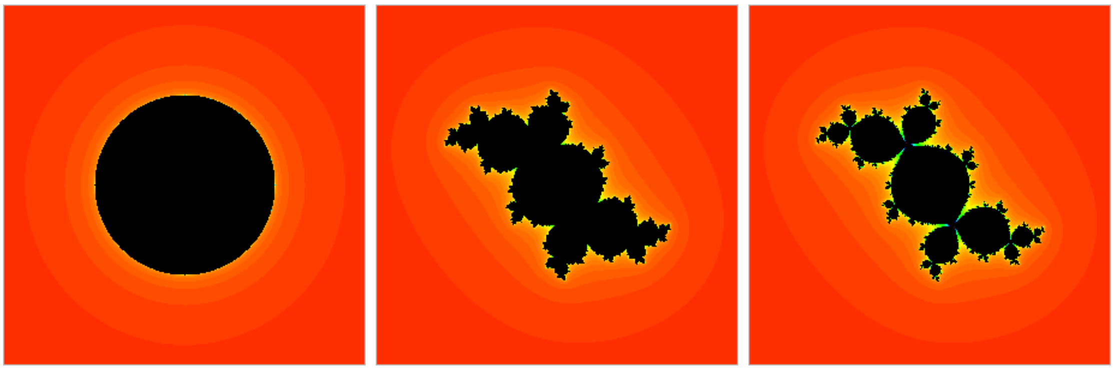

Figure 4: Example of the pinching of a quadratic Julia set. For parameters $c = 0$, $c = -.125 + .625i$, $c = -.125 + .67i$ respectively.

Figure 5: Example of the pinching of an exponential Julia set. For parameters $\lambda = -2$, $\lambda = .55 - 1.56i$, $\lambda = .8 - 1.75i$ respectively.

together. We say these points are being pinched together. Once the parameter crosses a threshold, the points that are being pinched actually come all of the way together and become identified. A similar behavior occurs with exponential Julia sets. Starting with a parameter λ that admits an attracting fixed point we see in Figure 5 that the Julia set is a Cantor Bouquet. As we move the parameter we see that some points are coming closer together in a manner reminiscent of pinching in the quadratic case. Again, crossing a threshold with λ, it appears that these points come all of the way together and are identified. It turns out that is exactly what is happening and we will elaborate on it later. The goal of this paper is to use the above behaviors to build a representative laminational model for the complex exponential function, as seen in Figure 1, that, in the future, could be used to study further the Julia sets of the complex exponential function in an manner analogous to polynomial laminations and polynomial Julia sets.

2 Kneading Sequence, Itineraries, and Hairs

In this section we will closely follow [BD00]. That paper details the structure of the attracting domain(s) of the Julia set of $E_\lambda(z) = \lambda e^z$ for those λ where there is an attractive periodic orbit in the Fatou set $F(E_\lambda)$; uses that structure to assign itineraries to points in the Julia set; constructs a kneading sequence; uses that kneading sequence to augment the previous itineraries; and uses the augmented itineraries to determine the behavior of parts of the Julia set known as hairs.

2.1 Approximating Fundamental Sets of Attracting Domains

To begin, we will describe a convenient structure for a given attracting periodic orbit. First, we need to introduce a definition from [BD00] which will be used throughout this section.

Definition 1. *An unbounded, simply connected $F \subset \mathbb{C}$ is called a* finger *of width c if*

1. *F is bounded by a simple curve $\gamma \subset \mathbb{C}$*
2. *There exists a $\nu > 0$ such that $F \cap \{z|Re(z) > \nu\}$ is simply connected, extends to infinity, and satisfies*

$$\{F \cap \{z|Re(z) > \nu\}\} \subset \{z|Im(z) \in [\xi - \frac{c}{2}, \xi + \frac{c}{2}]\}$$

for some $\xi \in \mathbb{R}$

We then let $z_0, ..., z_{n-1}$ be an attracting periodic orbit for $E_\lambda = \lambda e^z$ with $n \geq 3$. Suppose $0 \in A^*(z_1)$ where $A^*(z)$ is the immediate basin of attraction of z. Then it is known [BD00] that there exist disjoint, open, simply connected sets $C_1, ..., C_{n-1}$ such that:

1. $z_j \in C_j, C_j \subset A^*(z_j)$
2. $E_\lambda(C_j) = C_{j+1}, j = 0, ..., n-2$ and $E_\lambda(C_{n-1}) \subset C_0$
3. $C_1, ..., C_{n-1}$ are fingers of width $c_j \leq 2\pi$
4. The complement of C_0 consists of infinitely many disjoint fingers.

We note that the fingers C_j for $j = 1, ..., n-1$ do not contain any points of $J(E_\lambda)$, however the fingers in the complement of C_0 contain all of $J(E_\lambda)$ as well as all of the other C_j. A collection of such sets will become important so we will use the following

Definition 2. *A collection of open subsets $C_0, ..., C_{n-1}$ satisfying the above conditions is called a fundamental set of attracting domains for the cycle $z_0, ..., z_{n-1}$.*

These attracting domains are fingers and subsets of the $A^*(z_j)$, but beyond this we don't know anything about them. We would like to have an idea of their sizes. To that end, we have

Definition 3. *A smooth curve $\gamma(t)$ is called horizontally asymptotic to c if*

1. $\lim_{t\to\infty} Re(\gamma(t)) = +\infty$
2. $\lim_{t\to\infty} Im(\gamma(t)) = c$
3. $\lim_{t\to\infty} arg(\gamma'(t)) = 0$

In [BD00], the authors show the following

Theorem 2.1. *For a cycle $z_0, ..., z_{n-1}$ there exists a fundamental set of attracting domains, denoted B_j for $j = 0, ..., n-1$, with the following properties. There are integers k_j and a parameterization $\gamma_j(t)$ of the boundary of B_j which is horizontally asymptotic to*

1. $2\pi k_{n-1} - arg(\lambda) \pm \frac{\pi}{2}$ *if* $j = n-1$

2. $2\pi k_j - arg(\lambda)$ *if* $j = 0, ..., n-2$

where $k_j \in \mathbb{Z}$

From here on we will assume the fundamental set of attracting domains is chosen to satisfy the above conditions. We note that Figure 6, and others like it in this

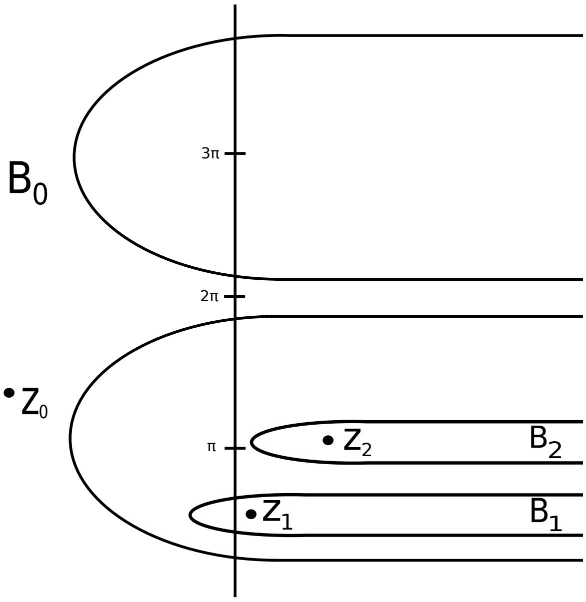

Figure 6: Fundamental domains of attraction for $\lambda = 3 + \pi i$.

paper, is not to scale, because as seen in Theorem 2.1, B_{n-1} will have width π and the other B_j for $j = 1, ..., n-2$ can have arbitrarily small width.

2.2 Symbol Space

We now turn our attention to defining a symbol space that we will use to describe the dynamics of E_λ on its Julia set. We start by recalling that the complement of B_0 is the union of infinitely many closed, unbounded fingers in the right half-plane. We label these fingers by $\mathcal{H}_k$ where $k \in \mathbb{Z}$ and we index them so that $0 \in \mathcal{H}_0$ and k increases with increasing imaginary parts. Note that because $J(E_\lambda)$ is contained in the complement of B_0 and $\cup_{z\in\mathbb{Z}}\mathcal{H}_k$ is the complement of B_0, $J(E_\lambda) \subset \cup_{z\in\mathbb{Z}}\mathcal{H}_k$.

Consider the set $\Sigma = \{(s) = (s_0 s_1 s_2 ...) | s_j \in \mathbb{Z} \text{ for each } j\}$. Σ is known as the *sequence space*. There is a map on Σ called the *shift map*, denoted σ and given by $\sigma(s_0 s_1 s_2 ...) = (s_1 s_2 s_3 ...)$. We now have the tools to formulate the following

Definition 4. *For $z \in J(E_\lambda)$ we call $S(z)$ the* itinerary *of z when $S(z) = (s_0 s_1 s_2 ...)$ where $s_j = k$ iff ${E_\lambda}^j(z) \in \mathcal{H}_k$.*

Note that $S(E_\lambda(z)) = \sigma(S(z))$.

A version of this construction appears in [BD00],[DG87],[D91], and [B99]. We follow [BD00].

Our primary concern will be with itineraries with bounded entries. With that in mind, we add the following

Definition 5. *We call $\Sigma_N = \{s \in \Sigma | \; |s_j| \leq N$ for each $j\}$ the set of itineraries with entries bounded by N.*

It is at this point that we will take a brief side trip to establish a few facts that will prove useful later on.

We start by noticing that $E_\lambda(B_0) = B_1 - \{0\}$. From there it follows that $E_\lambda(\mathcal{H}_k) = \mathbb{C} - B_1$ for each k. Using this we can define a function $L_{\lambda,k}$ as $L_{\lambda,k} : \mathbb{C} - B_1 \to \mathcal{H}_k : z \mapsto {E_\lambda}^{-1}(z)$. That is, $L_{\lambda,k}$ is the branch of the inverse of E_λ on $\mathbb{C} - B_1$ which takes values in $\mathcal{H}_k$. Another way of thinking of $L_{\lambda,k}$ is as essentially a branch of the logarithm corresponding to E_λ which takes value in $\mathcal{H}_k$. This function will be useful later on, but more immediately it is used in the proof of

Theorem 2.2. *For each $N > 0$ there is an invariant subset Γ_N of $J(E_\lambda)$ that is homeomorphic to Σ_N and on which E_λ is conjugate to the shift map.*

The proof of this theorem is sketched in [BD00] and detailed in [B99]. We will, however, provide a brief summary of the construction. We first define sets V_k as the set of all points with real part less than or equal to a real number $\tau >> 0$ that lie in a subset of the complement of B_0 that lies in $\mathcal{H}_k$ but not in B_n for $n = 1, ..., n-1$ (see Figure 7). We note that E_λ maps each V_k over all of the V_k. Then $L_{\lambda,k}$ is well defined and maps $\cup V_j$ into V_k. Given $s = (s_0 s_1 s_2 ...) \in \Sigma_N$ we can use the s_n as parameters of L_{λ,s_n} and see that an appropriate composition L^n_λ of L_{λ,s_n} maps V_k properly into itself [BD00]. For this reason, L^n_λ is a contraction in the Poincaré metric on V_k. It follows that $\gamma_s = \lim_{n\to\infty} L_\lambda^n$ exists and is independent of $z \in V_k$. Let Γ_N denote the union of the γ_s for $s \in \Sigma_N$. It can be seen that the map $s \to \gamma_s$ is a homeomorphism between Γ_N and Σ_N. It is well-known that Σ_N is a Cantor set and thus so is Γ_N [BD00][B99]. It can also be seen that the action of E_λ on Γ_N is conjugate to the shift map σ on Σ_N. The Cantor set Γ_N in $J(E_\lambda)$ consists of endpoints of "hairs" in $J(E_\lambda)$, to be defined later. This completes our construction.

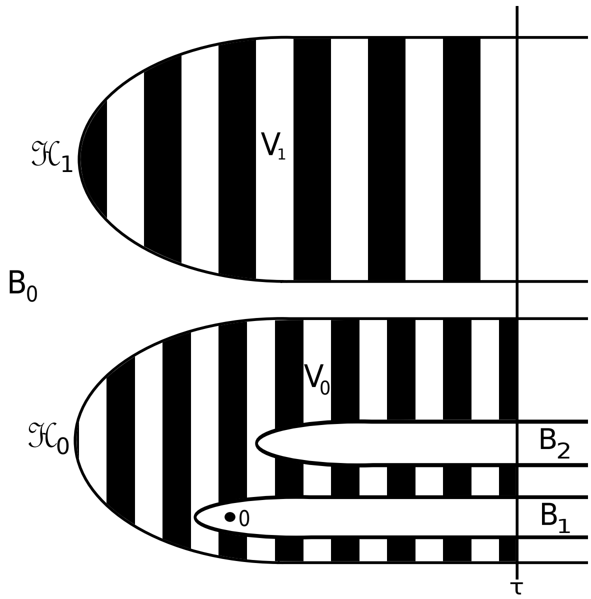

Figure 7: Construction of the $\mathcal{H}_k$ and V_k for $\lambda = 3 + \pi i$.

2.3 Kneading Sequence

Now that we have established a symbol space for use in specifying itineraries, we will put it to use. It seems natural to start with the attracting cycle. We would like to create a sequence that details which fundamental attracting domains the attracting cycle visits and the order in which they are visited. However, we may not always know the points in the cycle. It is known for entire functions that the immediate basins of attraction for any attracting cycle must contain the orbit of a singular point [BD00][Bk84]. Singular points consist of critical points and omitted values. The exponential function has no critical points, but it does have an omitted value, namely 0. This is very helpful because we know that the fundamental attracting domains that 0 visits are the same ones that the attracting cycle visits [BD00]. With this in mind we define a *kneading sequence* as

Definition 6. *Let E_λ have an attracting cycle of period $n \geq 3$. The kneading sequence is the string of $n-2$ integers*

$$K(\lambda) = 0k_1k_2...k_{n-2}*$$

where $k_i = j$ iff ${E_\lambda}^i(0) \in \mathcal{H}_j$.

Note that our kneading sequences always start with 0 and end with $*$. This is because by definition 0 lies in $\mathcal{H}_0$. Also ${E_\lambda}^{n-1}(0)$ lies in B_0 which is the complement of the $\mathcal{H}_k$ so we denote this by $*$.

We now use the kneading sequence to further partition up the plane so we can more precisely determine how points of $J(E_\lambda)$ move under iteration of E_λ.

For $\tau >> 0$ as defined above, set

$$\Lambda_\tau = \{z \in \mathbb{C} \,|\, Re(z) \geq \tau\} - \bigcup_{i=0}^{n-1} B_i$$

Note that Λ_τ consists of infinitely many closed fingers extending to infinity to the right. Because B_0 is removed with all of the other B_i, each finger of Λ_τ lies in exactly one $\mathcal{H}_j$. For a given integer j, there are two cases to consider; j is not an entry in the kneading sequence; and j is an entry in the kneading sequence. In the first case there is only one finger of Λ_τ that lies in $\mathcal{H}_j$, the far right portion of $\mathcal{H}_j$ itself. This finger in Λ_τ we denote H_j. In the second case there is more than one finger of Λ_τ that lies in $\mathcal{H}_j$ because the B_j separate $\Lambda_\tau \cap \mathcal{H}_j$ into at least two fingers. For a given j in the kneading sequence we denote the fingers lying in $\Lambda_\tau \cap \mathcal{H}_j$ as H_{j_k} where j_k orders them with ascending imaginary part beginning with j_0. As a final note, because the Λ_τ lie in the half plane $Re(z) \geq \tau$, so do the H_j and H_{j_k}.

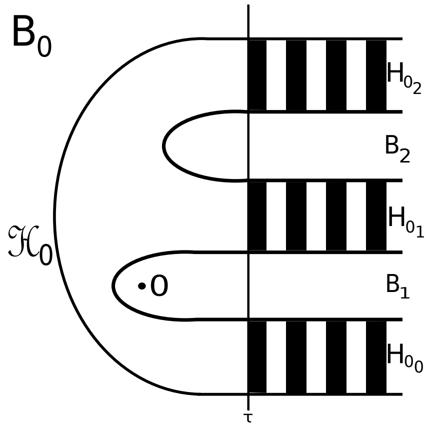

Figure 8: Construction of the H_{j_k} for $\lambda = 3 + \pi i$.

2.4 Augmented Itinerary

Now that we have further partitioned the plane, let's use this new structure. We have already given points in $J(E_\lambda)$ itineraries, but we have a potential issue. For a given point, it's itinerary keeps track of which $\mathcal{H}_k$ it visits. It is possible, however, for a point to visit a single $\mathcal{H}_k$ multiple times in a row. This would yield an itinerary like $(\overline{0})$. This provides some information, but we would like more. To that end we will use the H_{j_k} we specified in the previous section to augment our itineraries.

We note that we will define augmented itineraries for only those $z \in J(E_\lambda) \cap \{z \in \mathbb{C} \mid Re(z) \geq \tau\}$, and then relate this to our Cantor set Γ_N later. To start, let $\mathbb{Z}'$ be the set whose elements are either integers not in the kneading sequence or subscripted integers j_k corresponding to an H_{j_k} if j is in the kneading sequence. Then, following [BD00], we have

Definition 7. *The* augmented itinerary *of z is*

$$S'(z) = (s_0 s_1 s_2 ...)$$

where each $s_j \in \mathbb{Z}'$ and s_j specifies the finger in Λ_τ containing $E_\lambda(z)$.

Let Σ' denote the set of augmented itineraries. It is worth nothing that the augmented itinerary is defined only for points whose orbits always remain in Λ_τ.

It will be useful to be able to convert from augmented itineraries back to itineraries. With that in mind we have, following [BD00]

Definition 8. *The deaugmentation map is a map* $D : \Sigma' \to \Sigma$ *such that if* $s_n = j_k$ *then* $(D(s))_n = j$*. If* $s_n = j$ *then* $(D(s))_n = j$*.*

The deaugmentation map simply removes the subscript from each subscripted entry in a sequence in Σ' and does nothing to the other entries.

As it happens not all augmented itineraries correspond to an orbit in the far right half plane [BD00]. We then introduce the idea of allowable transitions to specify which augmented itineraries do correspond to points in $J(E_\lambda)$.

Definition 9. *Let* $s = (s_0 s_1 s_2 ...) \in \Sigma'$*. A* transition *is defined as any two adjacent entries* (s_i, s_{i+1}) *in* s*. The transition is called* allowable *if*

$$E_\lambda(H_{s_i}) \cap H_{s_{i+1}} \neq \emptyset.$$

In this case we say $E_\lambda(H_{s_i})$ *meets* $H_{s_{i+1}}$*. An allowable transition will be denoted as* $s_i \to s_{i+1}$*. An itinerary* $s' \in \Sigma'$ *will be called* allowable *if for all* s_j *it follows that* $s_j \to s_{j+1}$*. The set of allowable itineraries will be denoted* Σ^**.*

Now we will define some symbols we will use for the rest of the paper. From here on we will assume that N satisfies $|k_j| \leq N$ for all entries k_j in the kneading sequence. Let Σ_N^* denote the set of sequences in Σ^* whose deaugmentation is a sequence in Σ_N.

2.5 Hairs: Points with Common Itinerary

We have now described a way to give points an itinerary that represents where they move. It happens that multiple points may share the same itinerary. Points that share the same itinerary $s \in \Sigma_N^*$ actually form a continuous curve that tends to ∞ in the right half plane and limits to $\gamma_{D(s)} \in \Gamma_N$ [B99] [BD00]. We first limit our attention to points in the far right half plane ($Re(z) \geq \tau$). As stated in [BD00], we have that we may choose τ large enough so that if $j_l \to i_k$

for a sequence in Σ_N^*, then $\{\Lambda_\tau \cap L_{\lambda,s_j}(H_{i_k})\} \subset H_{j_l}$ is a closed finger that is bounded on the left by $Re(z) = \tau$ and completely contained inside some H_{j_l}. Using arguments in [B99] it can be shown that given $s' \in \Sigma_N^*$,

$$\lim_{n\to\infty} \Lambda_\tau \cap [L_{\lambda,s'_0} \circ ... \circ L_{\lambda,s'_n}(H_{s'_{n+1}})]$$

is a closed and connected set that meets ∞ and $Re(z) = \tau$, and is a continuous curve which we may parameterize by $h_{\lambda,s'} : [t_0, \infty) \to H_{s'_0}$ with $Re(h_{\lambda,s'_0}(t_0)) = \tau$. This curve we call the tail of a *hair* in $J(E_\lambda)$. From the above we now have [B99]

Proposition 2.3. *Let $s \in \Sigma_N^*$. There is a unique tail of a hair in $\Lambda_t \cap J(E_\lambda)$ that has augmented itinerary s.*

Now we have a well defined hair in the far right portion of the Julia set that has itinerary s' for each $s' \in \Sigma_N^*$. We now recall the Cantor set Γ_N we previously constructed. We will now relate these hairs to Γ_N. Given the hair $h_{\lambda,\sigma(s)}(t)$, we first note that $E_\lambda \circ h_{\lambda,s}(t)$ is properly contained in $h_{\lambda,\sigma(s)}(t)$. This means that by applying L_{λ,s_0} we may pull back $h_{\lambda,\sigma(s)}(t)$ into the region $Re(z) < \tau$. Thus, by applying

$$L_{\lambda,s_0} \circ ... \circ L_{\lambda,s_n}$$

to the hair $h_{\lambda,\sigma^{n+1}(s)}(t)$ we extend that hair back into the aforementioned region. We can use the proof that Γ_N is a Cantor set conjugate to Σ_N to see that a hair extended in this way will tend to a unique point in Γ_N [BD00]. This shows that there is only one point in Γ_N that has the same non-augmented itinerary as our hair, specifically the point whose deaugmented itinerary is given by $D(s)$.
Now we let $h^\tau_{\lambda,s}$ be the set of points on the tail of the hair $h_{\lambda,s}(t)$ where $t \in [\tau, \infty)$. We can how define a full hair as

Definition 10. *The full hair corresponding to the sequence* $s \in \Sigma_N^*$ *is given by*

$$\lim_{n \to \infty} L_{\lambda,s_0} \circ \ldots \circ L_{\lambda,s_n}(h^{\tau}_{\lambda,\sigma^{n+1}(s)}).$$

Putting this all together we have now shown, following [BD00] and [B99]

Theorem 2.4. *Let* $s \in \Sigma_N^*$. *The full hair corresponding to* s *is a curve in the Julia set that tends to* ∞ *in the right half plane and limits on* $\gamma_{D(s)} \in \Gamma_N$.

If we have two hairs that correspond to different sequences in Σ_N^* that have the same deaugmentation then it follows from Theorem 2.4 that they must limit on the same point in Γ_N. When this happens we say the hairs are attached to the same point. This is the pinching that we observed in the introduction.

3 Laminations

We have now constructed an itinerary system for continuous curves of points, called hairs, that approach ∞ in the right half plane and limit on points of a Cantor set Γ_N on the left. We've seen that it is possible for multiple hairs to limit on the same point on the left. The coordinates used in our itineraries are based on the domain(s) of attraction of our attracting point(s). We started with an attracting cycle $z_0, \ldots, z_{n-1}$, described its domains of attraction, then approximated the attracting domains with domains $B_0, \ldots, B_{n+1}$ bounded by smooth curves that we called a fundamental set of attracting domains. We noted that the complement of B_0 is the union of infinitely many closed, unbounded fingers in the right half-plane. If we look at the numerically generated picture of our example Julia set (see Figure 1) we see that there are apparent fingers in the right half-plane; we call these *natural fingers*. It would be natural to suspect that the natural fingers are the same as those in the complement of B_0. It turns out that these only coincide when the attracting cycle is made up of a single point, also known as an attracting fixed point, and B_0 is the only attracting domain. The basic case we are interested in is when we have an attracting cycle

$z_0, ..., z_{n-1}$ with $n \geq 3$ and with a kneading sequence of the form 00...0∗. In this case the fingers in the complement of B_0 do not coincide with the natural fingers.

3.1 Natural Coordinates

We would like to have an itinerary system, as exists for polynomial Julia sets, that does not change with the attracting cycle. To begin, we note that we can see from Theorem 2.1 that the vertical width of B_{n-1} is π. E_λ maps an unbounded finger of vertical width π to a left half-plane. This means that B_{n-1} corresponds to the large gaps between the natural fingers. We will make use of this fact to help define our natural itinerary system. We first recall the H_{j_k} from section 2.3. We are only interested in the H_{j_k} with j in the kneading sequence. That is, those H_{j_k} that are double subscripted. From their definition it is clear that for $z \in H_{j_k}$, and $z' \in B_{n-1}$, there will be some $j, k \in \mathbb{Z}$ such that $Im(z) > Im(z')$ and other j and k such that $Im(z) < Im(z')$. As noted in section 2.1, B_{n-1} is a preimage of part of B_0. Due to the periodic nature of E_λ there are infinitely many preimages of B_0 up and down the imaginary axis, 2π apart, and of vertical width π, like B_{n-1}. We label these preimages G_l where $G_l \subset \mathcal{H}_l$. We now use these facts to get

Definition 11. *Let G_l be preimages of B_0 as described above and $\tau >> 0$. We have two case to consider. Let $z \in B_1$ and $z' \in B_{n-1}$ then*

1. *When $Im(z) < Im(z')$ we define*

$$N_p := \{z \in J(E_\lambda) \,|\, \forall z_p \in G_p, \forall z_{p-1} \in G_{p-1}, Re(z) > \tau, Im(z_{p-1}) < Im(z) < Im(z_p)\}$$

2. *When $Im(z) > Im(z')$*

$$N_{p-1} := \{z \in J(E_\lambda) \,|\, \forall z_p \in G_p, \forall z_{p-1} \in G_{p-1}, Re(z) > \tau, Im(z_{p-1}) < Im(z) < Im(z_p)\}$$

That is when B_{n-1} is above B_1, the N_p are all of the points in $J(E_\lambda)$ and in the far right half plane that fall vertically in between G_p and G_{p-1}, and between G_p and G_{p+1} when B_{n-1} is below B_1. In $\mathcal{H}_0$ where we have been focusing we get that the tails of all of the hairs in the H_{j_k} that are below $B_{n-1} = G_0$ are in N_0 and those tails in the H_{j_k} that fall above B_{n-1} are in N_1.

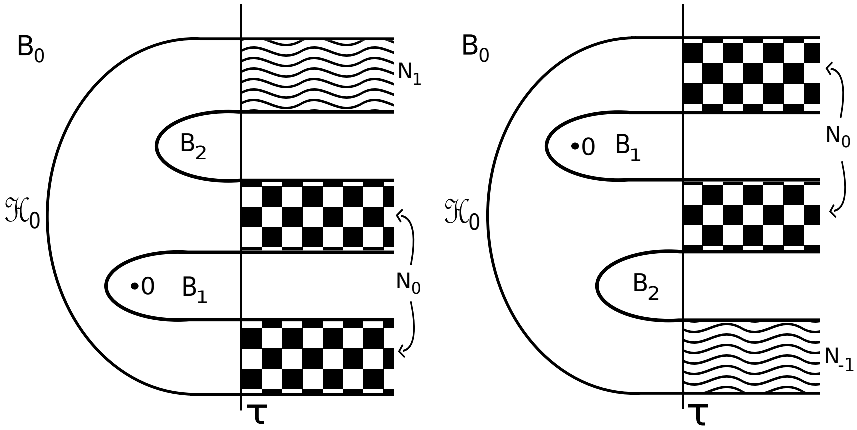

Figure 9: Construction of some of the N_p for $\lambda = 3 + \pi i$ and $\lambda = 3 - \pi i$.

We will now use the symbol space put forth in Section 2.2 to assign itineraries to the tails of hairs in the N_p.

Definition 12. *Let $T \subset J(E_\lambda)$ be the tail of a hair. We call $S_N(T)$ the* natural itinerary *of T when*

$$S_N(T) = (s_0 s_1 s_2 ...) \text{ where } s_j = k \text{ iff } {E_\lambda}^j(T) \subset N_k.$$

We now have assigned natural itineraries to the tails of hairs, but our goal is to assign natural itineraries to entire hairs. We know from arguments put forth in Section 2.5 that each tail of a hair is part of only one hair and is connected to a point of the Cantor set Γ_N. This allows us to assign entire hairs the natural itinerary associated with its tail. This can, as in our example, cause a single point of Γ_N to be assigned multiple natural itineraries. This is acceptable and a way we can use our natural itineraries to identify points in Γ_N with multiple hairs attached to them.

3.2 Leaves of the Lamination

Now that we have natural itineraries for the hairs in $J(E_\lambda)$, we can use them to construct our laminational model. We start off with a vertical axis with evenly spaced tick marks all along it. We label the intervals between tick marks with integers. We pick an arbitrary interval at which to start and label it 0. We label the interval above 0 as 1 and the interval below 0 as -1. We continue labeling all of the intervals in this manner (see Figure 10). Inside of each of these intervals is a copy of the initial set of intervals labeled as (w_0w_1) with w_0 the number of the containing interval and w_1 the subinterval numbered like the initial set of intervals. The heights of each of these subintervals won't be uniform as they need to fit inside of a single interval from our original set. Inside each subinterval is also a copy of the initial set of intervals labeled $(w_0w_1w_2)$, and so on. Each arbitrarily small interval contains a copy of the initial set of intervals inside of it (see Figure 10).

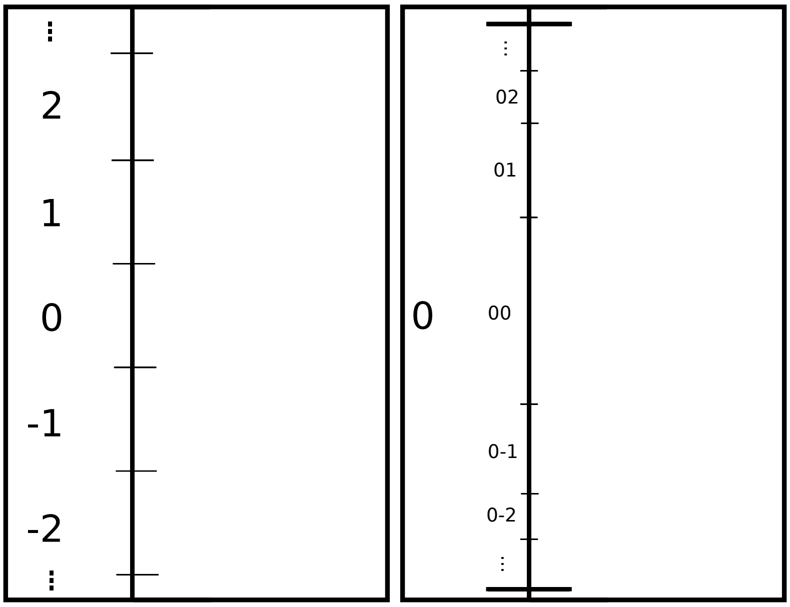

Figure 10: Labeling our vertical axis and a closer look at the 0 interval.

If we consider a label with infinitely many entries, such as $(\overline{001})$ we get an infinite intersection of ever smaller intervals. Because all of these intervals are compact sets with diameter going to 0 and each interval is contained within the previous one, it is known that their intersection will be exactly one point. From here on we let V be the set of points on our vertical axis that represent hairs. For $x, y \in V$, we define an order on V as $x > y$ iff x is above y on our vertical axis. We note the similarity between our labeling scheme for the vertical axis and our natural itineraries for hairs. This is intentional. We now relate the two by letting a hair in $J(E_\lambda)$ with natural itinerary $(s_1 s_2 s_3 ...)$ be represented by a point on our vertical axis with label $(s_1 s_2 s_3 ...)$. It is worth noting that not every point on our vertical axis will represent a hair in $J(E_\lambda)$. It turns out that both the set of points representing hairs and it's complement are dense in the vertical axis and separate each other [DG87].

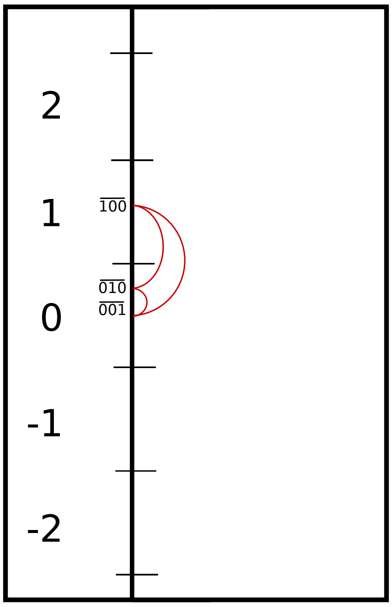

Figure 11: Our vertical axis with initial set of leaves for $\lambda = 3 + \pi i$ in red.

When we were assigning natural itineraries to hairs we noted that it is possible for an end point of a hair to be assigned multiple different natural itineraries. In that case we said that we have multiple hairs attached to the same point. We represent this in our combinatorial model as chords connecting the points that represent the hairs that are attached to the same point (see Figure 11). In the laminational model, we call these chords *leaves*. When we only have 3 hairs coming together this is a straight forward process as there is only one way to connect the points in our model. However, when we have 4 or more points there are multiple ways to connect them so we will put some constraints on how they may be connected. The leaves in our model represent the basins of attraction of our attracting cycle. Because of this, leaves connecting points may not cross. If we had two leaves crossing that would indicate two domains of attraction intersecting. In that case we would only have one domain of attraction not two.

3.3 Initial Data and Critical Chords

Now that we've established a basic version, we would like our laminational model to include information representing the dynamics of E_λ. First we note that it is easy to see that E_λ on the hairs of $J(E_\lambda)$ is conjugate to the shift map σ on our natural itineraries. Next we consider a point with n hairs tied to it. For a hair T consider the set of hairs T_i such that $E_\lambda(T_i) = T$. We call this the set of *pullbacks* of T. As these hairs are in the Julia set, they are represented by points on the vertical axis in our model. We can also get to these points by way of our model. Start with a hair T and consider its natural itinerary $S_N(T)$ of the form $(s_0 s_1 s_2 ...)$. Now consider the set of itineraries $P_T = (n s_0 s_1 s_2 ...)$ where $n \in \mathbb{Z}$. We claim that the elements of P_T represent the hairs in the pullback of T. This can be seen by the fact that E_λ is conjugate to the shift map σ and that applying σ to any element of P_T results in $S_N(T)$.

To add the idea of pullbacks to our model we need one more idea. We take two points $x, y \in V$ such that $\sigma(x) = \sigma(y)$ and a chord connecting them. We call such a chord a *critical chord* (see Figure 12).

3.4 Simplest Pullback Lamination

We now combine these ideas to complete our model. We start with a set $P = x_1, x_2, ..., x_n$ of points on our vertical axis connected by leaves. We take the point $x_k \in P$ such that $x_k \geq x_i$ for all $i = 1...n$ and the set X of points $y \in V$ such that $\sigma(x_k) = \sigma(y)$. We now connect the points in X in such a way that each point is connected to the next greater and next smaller points and no

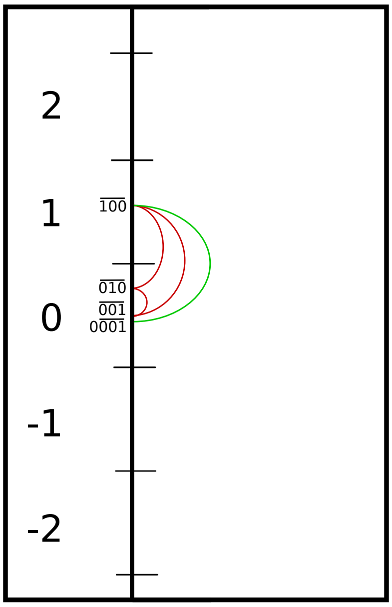

Figure 12: Our vertical axis with initial set of leaves for $\lambda = 3 + \pi i$ in red and natural critical chord in green.

others. We note that these chords are critical chords. We now consider the sets P_i of pullbacks of $x_i \in P$. We connect the points in the union of the P_i in such a way that the new leaves do not cross any existing leaves, including the critical chords, so that they form disconnected pieces where each piece contains exactly one point from each P_i, and so that the order of the points in each piece is a rotation of the order of the points in P. We call what we have now a *partial lamination* (see Figure 13). We could continue this process by considering the set of points in the pullbacks of each P_i and connecting them with leaves according to our rules. We define a *full lamination* as the limit of infinitely repeating this process with each new set of pullbacks and taking the closure. Note that because the leaves don't cross in the finite partial pullback they will not cross in the closure. Also, in the case we are considering of kneading sequence 00...0$*$, and with our natural choice of critical chords, there are no limit leaves, all limits are points on the vertical axis. We also note that given an initial set of points and leaves connecting them, there are many choices of critical chords that will work and will result in different laminations. The set we have specified is simply the most natural and simplest set of critical chords to chose.

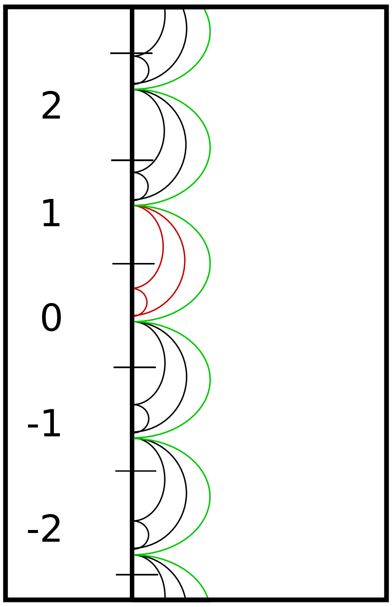

Figure 13: Partial lamination for $\lambda = 3 + \pi i$

3.5 Rotation Number In Figure 2 we see the Julia set and partial lamination for a polynomial function known as a Douady rabbit. As indicated in Figure 4, this Julia set can be viewed as a circle that has been pinched together. In the lamination in Figure 2 the points connected by chords represent the places on the Julia set where the circle has been pinched together. In the case of E_λ, we can see in Figure 5 that there is also a sort of pinching happening and the example lamination represents these pinchings in the same manner as in the Douady rabbit lamination.

Another behavior to note, seen in Figure 14, is the rotation of points around the point at which the circle is pinched. This is noted by what is called a rotation number. In Figure 14 the indicated point moves, informally, one tick counter clockwise with each iteration and returns to the domain where it started in 3 iterations. We say the rotation number of the pinched point is $\frac{1}{3}$. In a similar manner, points in E_λ rotate around the pinch point and the pinch point is also assigned a rotation number. In the exponential case these rotation numbers fall between $-\frac{1}{2}$ and $\frac{1}{2}$. In Figure 15 we see examples of exponential Julia sets with rotation numbers $\frac{1}{3}$ and $-\frac{1}{3}$ respectively. In the cases we are studying, that of a

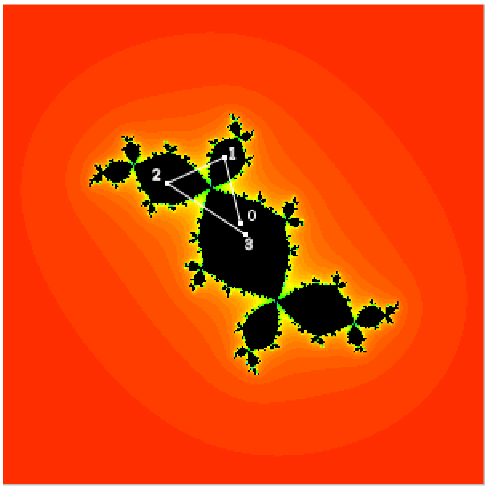

Figure 14: Quadratic Julia set with labeled orbits for $c = -.125 + .75i$

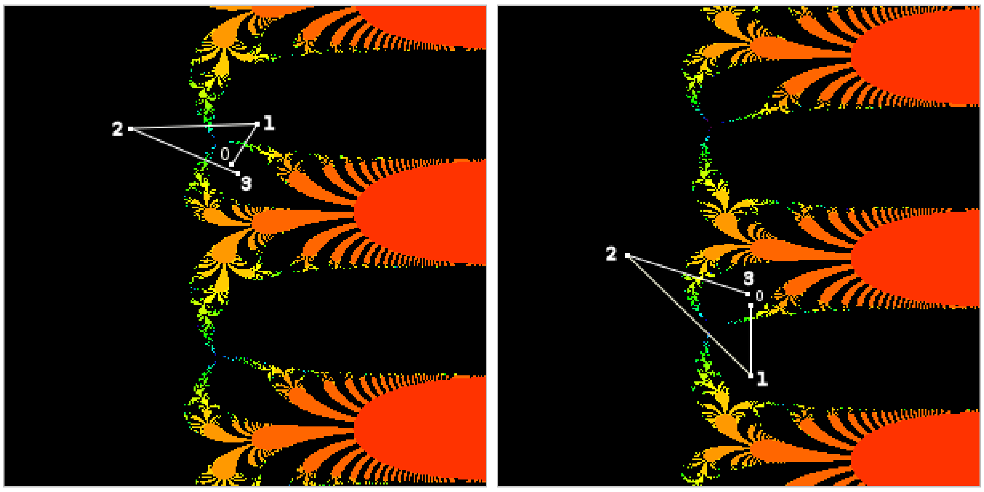

Figure 15: Exponential Julia sets with labeled orbits for $\lambda = 3 + \pi i$ and $\lambda = 3 - \pi i$ respectively

kneading sequence with a finite number of entries all zero, there are finitely many hairs coming together at a repelling fixed point. By $K\ddot{o}nig's$ $Linearization$ in [M06] the action of E_λ is conjugate to rotation with expansion about a repelling fixed point. Because in our case there are finitely many hairs connected to our repelling fixed point and the hairs are permuted in an orientation preserving manner, we get that our combinatorial rotation numbers will all be rational, even if the analytic rotation at the fixed point is irrational.

4 Conclusion and Future Work

4.1 Summary

In this paper we stepped through the construction of fundamental domains of attraction for exponential Julia sets. We used these domains to assign itineraries to points in $J(E_\lambda)$ based on where they go under iteration. We saw that these itineraries were not enough to pull out the structure we were going for. We then constructed a kneading sequence for our attracting cycle and used that to augment our itineraries. From there we showed that smooth curves of points, called hairs, share augmented itineraries and that, in the case of an attracting cycle of period more than 2, some of these hairs meet at a point.

We then used parts of that construction to construct a natural itinerary system for the hairs. We used our natural itineraries to associate hairs to points on a vertical axis and to define the notion of critical chords. We then used our natural itineraries and critical chords to define some basic rules for connecting the aforementioned points on the vertical axis with chords. We presented a system for generating a pullback given a set of points, leaves, and critical chords in our model and defined a full lamination as the limit of iterating said pullbacks and taking the closure.

4.2 Future Work

We now put forth some conjectures for future study of our laminational model. We considered only kneading sequences $00...0*$. However, we conjecture that our natural coordinates will work for all $J(E_\lambda)$ when there is an attracting periodic orbit in $F(E_\lambda)$. Given a λ, the Julia set $J(E_\lambda)$ generated by it, a full lamination $\mathcal{L}$ constructed in the manner set forth in this paper with appropriate choice of critical chords, and in analogy with the quadratic case, we conjecture

1. Every set of points connected by non-critical chords in $\mathcal{L}$ represents multiple hairs attached to a single point in $J(E_\lambda)$.

2. $J(E_\lambda)$ is homeomorphic to the quotient map on the Cantor Bouquet

represented by V that identifies points that are connected by (non-critical) chords.

3. Every identification of hairs at a single periodic point has a well-defined rational rotation number, and E_λ is transitive on attached hairs.

4. Every possible rotational number for $n \geq 3$ attached hairs that is transitive on hairs is represented by an exponential Julia set.

5. All orbits $n \geq 3$ of attached hairs are eventually periodic; that is, there are no wandering branch points for exponential Julia sets.

REFERENCES

[BD00] Bhattacharjee, Ranjit and Devaney, Robert L. Tying Hairs for Structurally Stable Exponentials. Ergodic Theory and Dynamical Systems, Vol. 20, no. 6, 2000, pp. 1603-1617.

[M06] Milnor, John. *Dynamics in One Complex Variable.* Princeton University Press, Princeton, 3rd edition, 2006.

[DG87] Devaney, Robert L. and Goldberg, Lisa R. Uniformization Of Attracting Basins For Exponential Maps. Duke Mathematical Journal, Vol. 55, no. 2, 1987, pp. 253-266.

[D91] Devaney, Robert L. e^z: Dynamics And Bifurcations. International Journal of Bifurcation and Chaos, Vol. 1, no. 2, 1991, pp. 287-308.

[Bk84] Baker, I. N. and Rippon P. J. Iteration Of Exponential Functions. Annales Academiæ Scientiarum Fennicæ. Series A. I. Mathematica. Vol. 9, 1984, pp. 49-77.

[T09] Thurston, William P. On the geometry and dynamics of iterated rational maps. Edited by Dierk Schleicher and Nikita Selinger and with an appendix by Schleicher in *Complex dynamics. Families and friends.* Edited by Dierk Schleicher. A K Peters, Ltd., Wellesley, MA, 2009.

[B99] Bodelòn, C., et.al. Hairs for the Complex Exponential Family. International Journal of Bifurcation and Chaos, Vol. 9, 1999, pp 1517-1534.