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MATHEMATICS

ABSTRACT

Fractional calculus has existed since 1695, when L'Hôpital asked Leibniz about his notation for a derivative, $\frac{d^n f(x)}{dx^n}$. L'Hôpital asked what the result would be if n were $\frac{1}{2}$. Leibniz's response: "An apparent paradox, from which one day useful consequences will be drawn." While fractional calculus has only been intensely explored in recent decades, this prediction seems to have been true, with fractional derivatives often seeming contradictory and difficult to work with. However, our exploration has found that these definitions are far more compatible after looking at their non-local properties, in particular their ability to account for causality. We then looked into applications of fractional calculus models to some typical problems involving differential equations such as oscillation and diffusion. Exploring different cases of fractional oscillation indicated that fractional calculus can better describe the evolution of oscillatory motion and damping over time, while exploring anomalous diffusion indicated that fractional calculus is more accurate when modeling the physical process of subdiffusion. The non-local property of fractional derivatives appears to be the same property that allows for these models to capture this additional information, though there remains much research to be done both in the theory of fractional calculus and in its applications.

Keywords: fractional calculus, fractional derivative, fractional oscillation, anomalous diffusion, subdiffusion

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LIST OF NOTATIONS

${}_aJ_x^\alpha f$	the α order fractional integral of $f(x)$ from a to x
${}_aD_x^\alpha f$	the α order left hand fractional derivative of $f(x)$ from a to x
${}_xD_a^\alpha f$	the α order right hand fractional derivative of $f(x)$ from a to x
$\mathbb{Z}$	the set of integers
$\mathbb{N}$	the set of natural numbers; the set of positive integers
$\mathbb{Z}_-$	the set of non-positive integers
$\mathbb{R}$	the set of real numbers
$a_\flat$	sub-acceleration
$b_\flat$	sub-damping
$m_\flat$	sub-mass
$v_\flat$	sub-velocity
$b_\sharp$	super-damping
$x_\sharp$	super-displacement
$k_\sharp$	super-potential
$v_\sharp$	super-velocity

1 INTRODUCTION

1.1 HISTORICAL OVERVIEW

For most of the history of mathematical modeling, calculus has been used to explain the behavior of phenomena ranging from the physics of kinetics and electromagnetism, to the biology of disease spread, and various systems in between [1]. For a long time, it was believed that calculus described all that was needed in order to model these systems. However, more recent research has indicated that a more generalized branch of calculus known as fractional calculus is better at modeling these phenomena [1, 2]. The idea that fractional calculus, and not traditional calculus, offers better models is still being explored. Immediate implications of such better models include being able to describe with more detail the specifics behind systematic evolutions and anomalous behaviors in many physical problems. These behaviors often go unnoticed in the models formed under traditional calculus.

Fractional calculus itself generalizes the methods of calculus to non-integer orders of integrodifferentiation. This branch of mathematics began with a letter from L'Hopital to Leibniz in 1695 [3]. In the letter, L'Hopital commented on the notation Leibniz had chosen for differentiation; particularly, with the notation for a derivative, $\frac{d^n f(x)}{dx^n}$, L'Hopital asked what the result would be if n were $\frac{1}{2}$. Leibniz's response: "An apparent paradox, from which one day useful consequences will be drawn." However, fractional calculus had not received much attention until recent decades.

Leibniz was correct in both parts of his statement. As fractional calculus has been explored, it has certainly uncovered paradoxical results. These paradoxes are derived from generalizing many rules and formulas already present, such as discrepancies between applications of differing definitions of the fractional derivative to a constant function, even though these differing definitions agree everywhere else [3]. Additionally, properties of fractional order calculus terms must be reevaluated, because such things as the semi-group properties, chain rule, and product rule do not function the same way they do in ordinary calculus [4].

This necessarily means that a consistent framework is needed in order to better explain physical behavior of any given problem without the possibility for the results to be interpreted in a myriad ways. It has been demonstrated that such frameworks are possible to create for a given problem, as seen in research on fractional oscillators [3, 5, 6] and on anomalous heat diffusion [3, 2]. In order to establish this framework, however, an understanding of fractional calculus is required. This includes an overview of the derivation of fractional calculus identities, functions, and formulas, as well as its methods and definitions. Through understanding the fractional calculus, it is possible to explain the patterns that arise in physical problems in a way that has not yet been captured, contributing to a better understanding of the way these problems work, and potentially leading to new developments in the way they are modeled.

Current research in such physical problems has led to generalizations of well-known equations and formulas. For instance, when considering a fractional harmonic oscillator, allowing for the idea of either fractional damping [6] or fractional inertia [7], a generalized equation can be reached in either case that returns the typical results when the differentiation order is an integer. When considering a nonhomogeneous conducting material [8], the heat diffusion differential equation can be generalized to a fractional differential equation to allow for more descriptive mathematical explanations than can be reached from the typical equation, except for cases when the order of differentiation is much closer to an integer value. In general, it appears that for any type of diffusion, fractional calculus models more completely represent what is occurring [9].

These results strongly indicate that models relying on traditional calculus alone are not as accurate as models relying on the generalized fractional calculus when it comes to predicting experimental data. Certain systematic behaviors are actually described by the math of fractional calculus that traditional calculus does not have the capacity to capture.

In the case of the oscillators [6, 7], systematic evolutions that fall somewhere between the orders of position, velocity, and acceleration contribute to a case of sub- or hyperdamping, even when no standard damping term is present. This strongly indicates a process between the integer order calculus explanations that was unnoticed by design which can only be seen in such a fractional system.

Similarly, when dealing with heterogeneous materials and heat diffusion [2] or biological diffusion [9], certain anomalous diffusions can only be described by fractional order differential equations, which are able to describe a sub- and hyperdiffusion process that otherwise goes unnoticed in traditional calculus.

In this paper, we will begin the discussion of fractional calculus by defining the fractional integral and the Riemann-Liouville and Caputo fractional derivatives. Special functions necessary for proving and defining these essential pieces will be defined along the way, and can also be found in more detail in the appendix. We will then look at and prove some of the properties of these fractional integrals and derivatives, highlighting key similarities and differences between them.

Later, we will review the application of fractional derivatives to problems of anomalous diffusion and fractional oscillation. With diffusion, we will discuss cases of both heat and biological diffusion, which will also include a discussion on probability and the Einstein-Boltzman relationship. With oscillation, we will discuss cases of unusual damping only seen through the lens of fractional calculus, and take a look at the potential physical explanations of these cases.

2 PRELIMINARIES

2.1 IMPORTANT FORMULAS AND FUNCTIONS

In order to begin an exploration of fractional calculus, it is necessary to define the method by which we find fractional derivatives and integrals. To that end, it is also necessary to define some other important functions which will lead to these definitions and aid in the proofs of their properties. (For proofs of special function properties, see the appendix.)

2.1.1 GAMMA FUNCTION

The gamma function is essential in many definitions for fractional operations. It acts as a generalization of the factorial operator on the natural numbers, extended for all real numbers excluding the non-positive integers.

The regular definition of the factorial is:

$$n! = \prod_{k=1}^{n} k,\ \ n \in \mathbb{N}. \tag{1}$$

An important property of the factorial immediate from this definition is:

$$n! = n\,(n-1)!,\ \ n \in \mathbb{N} \setminus \{1\}.$$

In order to generalize this operator for more than the natural numbers, the resulting operator needs to have that property for its arguments. Choosing the following integral:

$$\Gamma\left(\alpha\right) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx\ ,\alpha \in \mathbb{R} \setminus \mathbb{Z}_{-}, \tag{2}$$

where $\mathbb{Z}_{-}$ denotes the non-positive integers. As a remark, the operator aligns with the factorial operator when the argument is a natural number, which gives a sense of consistency in the definition.

$$\forall\ n \in \mathbb{N} \setminus \{1\},\ \ \Gamma\left(n\right) = (n-1)!,$$

which is shown by using integration by parts and induction in (2) (see Appendix). Additionally, the operator is undefined when the argument is a non-positive integer, as this produces an integral that does not converge. As seen in the Appendix, a limiting case appears with the Gamma Function that essentially defines $0! = 1$.

2.1.2 BETA FUNCTION

The beta function, also known as Euler's Integral of the First Kind, is another important tool in fractional calculus. It is useful for calculating many complex integrals, and has an immediate relationship to the gamma function:

$$\int_0^1 (1-u)^{\alpha-1} u^{\beta-1} du = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{3}$$

2.1.3 LAPLACE TRANSFORMATION AND CONVOLUTION

The Laplace Transform is often used as a function transformation that makes complicated differential equations easier to solve. The transformation simplifies the calculus into a more algebraic problem. The Laplace transform is defined (with $\mathcal{L}$ being the transform operator) as:

$$\mathcal{L}\{f(x)\} = \int_0^\infty e^{-sx} f(x) dx = \widetilde{f}(x) \tag{4}$$

which is valid so long as this integral is convergent, meaning $f(x)$ doesn't grow faster than the exponential term decreases.

The Laplace convolution is also useful, as will be more evident later. The convolution of $f(x)$ and $g(x)$ is defined as:

$$f(x) * g(x) = \int_0^x f(x - \tau)g(\tau)d\tau = g(x) * f(x).$$

The Laplace transform of a convolution results in a much easier multiplication:

$$\mathcal{L}\{f(x) * g(x)\} = \widetilde{f}(x)\widetilde{g}(x).$$

The Laplace transform is also useful for beginning an exploration of fractional calculus due to its immediate relationship to derivatives:

$$\mathcal{L}\{f^{(n)}(x)\} = s^n \widetilde{f}(x) - \sum_{k=0}^{n-1} s^k f^{(n-k-1)}(0).$$

2.1.4 MITTAG-LEFFLER FUNCTION

The Mittag-Leffler function is another important function for our exploration of fractional calculus. The Mittag-Leffler function acts as a generalization of the exponential function which has its own ubiquitous presence in calculus. The function is given as:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma\left(\alpha k + \beta\right)},\ \ \alpha, \beta > 0. \tag{5}$$

Often, β is taken to be 1, which gives the more common form of the function. When both $\alpha = \beta = 1$, it reduces to the series expansion of the standard exponential function. In any case, the function is as connected with fractional calculus as the special case of the exponential function is with ordinary calculus.

2.1.5 CAUCHY REPEATED INTEGRAL

Cauchy's formula for the repeated integral is a valuable starting point for the definition of a fractional integral. It is a compact way to represent n number of antidifferentiations of a given function. Using the letter J to represent integration, the formula is defined as:

$$_aJ_x^n f\left(x\right) = \frac{1}{(n-1)!} \int_a^x \left(x-y\right)^{n-1} f\left(y\right) dy,\ \ n \in \mathbb{N},\ \ x \geq a. \tag{6}$$

In ordinary calculus, it only makes sense to think about a natural number of antidifferentiations, with the $0th$ integral of a function being defined as the identity operator, or the function itself. This definition, however, does lead us into our definition in Chapter 3, allowing us to consider - just as L'Hôpital did - what may happen if we select a non-natural value for n.

2.1.6 CAUSALITY

Causality is something that is very useful to consider when working with the fractional operators, as it can contain important information about whatever a fractional differential equation may be modeling.

When a system is causal, this means that only past and present inputs will affect the output of the system. In the setting of fractional calculus, this value determines several factors in the resulting operation on a function. The chosen causality constant for a fractional integral or derivative is the lower bound a from the repeated integral. As the definitions for fractional derivatives used in this paper will depend on the definition of the fractional integral (in the following section), these derivatives are going to be non-local, as opposed to the local versions in traditional calculus. This causality then will be a major player in the properties of the fractional derivative, considering now all behavior of the function since this starting point a. This is what is meant when referring to past and present inputs on the system. As will be seen in applications, this allows for fractional models to account for information traditional models cannot.

As an example, setting this value to 0 when evaluating a power function returns only the antiderivative of the function with no unknown constant, (a property that will be very useful when exploring the contradictory nature of fractional derivatives). In functions like the exponential, this value must approach negative infinity for the same effect, already hinting at a complicated relationship between functions and their causality constant.

3 FRACTIONAL CALCULUS

Now that we have seen the tools necessary for defining fractional operators, we can take a look at the operators themselves, and explore the relationship between different approaches. We can also look at the properties and conditions of each definition, to compare and contrast.

3.1 THE FRACTIONAL INTEGRAL

Returning to the formula for repeated integration, we can immediately extend the definition from natural number arguments into real arguments, sans the non-positive integers, by simply using the gamma function instead of the factorial. Keeping with the notation for the iterated integral operator, we can represent the αth antiderivative of a function with the formula:

$$ {}_aJ_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y)\,dy,\ \alpha \in \mathbb{R} \setminus \mathbb{Z}_-,\ x \geq a. \tag{7} $$

3.1.1 CONDITIONS

As mentioned above, one major condition is that α must not be in $\mathbb{Z}_-$, as the gamma function for those values would not be defined. We can, however, define ${}_aJ_x^0 f(x) = f(x)$ as the identity operator, leaving the function unchanged.

For the function to have an αth antiderivative, the weighted function

$$ (x-y)^{\alpha-1} f(y) $$

must be integrable on the interval a to x. In this case, we will say that $f(x)$ is fractionally integrable. In addition, depending on the problem being analyzed, the causality of the formula, a, will need to be set appropriately such that the proper evolution of the system is captured. Later, we discuss what different values for a can represent for different types of functions.

3.1.2 PROPERTIES

It is already understood that for ordinary calculus, integration obeys the semi-group property

$$ {}_aJ_x^n\, {}_aJ_x^m f(x) = {}_aJ_x^{n+m} f(x),\ n, m \in \mathbb{N}. $$

Likewise, it can be shown that the fractional integral operator also obeys this property:

Theorem 1. ${}_aJ_x^{\alpha}\,{}_aJ_x^{\beta}f\left(x\right)={}_aJ_x^{\alpha+\beta}f\left(x\right),\ \alpha,\beta\geq 0$

Proof.

$$\begin{aligned}{}_aJ_x^{\alpha}\,{}_aJ_x^{\beta}f\left(x\right)&=\frac{1}{\Gamma\left(\alpha\right)}\int_a^x\left(x-t\right)^{\alpha-1}\left[\frac{1}{\Gamma\left(\beta\right)}\int_a^t\left(t-y\right)^{\beta-1}f\left(y\right)dy\right]dt\\&=\frac{1}{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)}\int_a^x\int_a^t\left(x-t\right)^{\alpha-1}\left(t-y\right)^{\beta-1}f\left(y\right)dydt.\end{aligned}$$

By Fubini's Theorem, these limits can be exchanged,

$${}_aJ_x^{\alpha}\,{}_aJ_x^{\beta}f\left(x\right)=\frac{1}{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)}\int_a^x f\left(y\right)\int_y^x\left(x-t\right)^{\alpha-1}\left(t-y\right)^{\beta-1}dtdy.$$

For the next part of the proof, we need to show

$$\int_y^x\left(x-t\right)^{\alpha-1}\left(t-y\right)^{\beta-1}dt=\left(x-y\right)^{\alpha+\beta-1}\int_0^1\left(1-u\right)^{\alpha-1}u^{\beta-1}du.$$

If we let $t=y+\left(x-y\right)u,\ dt=\left(x-y\right)du,$

then

$\left(x-t\right)=x-y-ux+uy=\left(x-y\right)-u\left(x-y\right)=\left(1-u\right)\left(x-y\right)$

and

$\left(t-y\right)=y+\left(x-y\right)u-y=xu-yu=u\left(x-y\right).$

Combining all of the above returns

$$\begin{aligned}&\int_y^x\left(x-t\right)^{\alpha-1}\left(t-y\right)^{\beta-1}dt\\&=\int_0^1\left(x-y\right)^{\alpha-1}\left(1-u\right)^{\alpha-1}\left(x-y\right)^{\beta-1}u^{\beta-1}\left(x-y\right)du\\&=\left(x-y\right)^{\alpha+\beta-1}\int_0^1\left(1-u\right)^{\alpha-1}u^{\beta-1}du.\end{aligned}$$

Now we can return to the original proof, and have

$$\begin{aligned}&\frac{1}{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)}\int_a^x f\left(y\right)\int_y^x\left(x-t\right)^{\alpha-1}\left(t-y\right)^{\beta-1}dtdy\\=&\frac{1}{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)}\int_a^x f\left(y\right)\left(x-y\right)^{\alpha+\beta-1}\int_0^1\left(1-u\right)^{\alpha-1}u^{\beta-1}dudy.\end{aligned}$$

Using (3),

$$\begin{aligned}
{}_aJ_x^{\alpha}\,{}_aJ_x^{\beta} f(x) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x f(y)(x-y)^{\alpha+\beta-1} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} dy \\
&= \frac{1}{\Gamma(\alpha+\beta)} \int_a^x (x-y)^{\alpha+\beta-1} f(y)\, dy \\
&= {}_aJ_x^{\alpha+\beta} f(x) .
\end{aligned}$$

This is exactly the definition of the operator, so it is shown that the fractional integral obeys the usual composition rules.

□

In addition, the fractional integral can be analyzed as a convolution of two functions, giving us a quick way to see its Laplace Transformation.

3.2 THE FRACTIONAL DERIVATIVE

With the fractional integral operator defined, it is now possible to look at two potential definitions for fractional differentiation. There are different uses for both, and they are far from the only two possibilities for defining fractional differentiation. While at first, there do seem to be some inconsistencies in their properties (discussed later), there is a very direct relationship between them and causality that will be explored in the next section.

3.2.1 LEFT HAND (RIEMANN-LIOUVILLE)

The left hand derivative (LHD) is obtained by first calculating a fractional integral of a function, and then taking the integer order derivative of the resulting function. For the *ath* derivative of a function, it is necessary to find the nearest integer higher than a, which we can call n. The LHD is then defined as:

$$ {}_aD_x^{\alpha} f(x) = \frac{d^n}{dx^n} {}_aJ_x^{n-\alpha} f(x) \tag{8}$$

It can be seen that “left” refers to the position of the derivative relative to the fractional integral. In this case, it is clear that the function must first be fractionally integrable, and the fractional integral must be n times differentaible. In this case, we will say that $f(x)$ is left hand differentiable.

We can also see that a constant function would not necessarily return a fractional derivative of 0, as the fractional integral would be a function of x, namely, (by using $f(x) = c$ in (7) and solving):

$$\frac{c(x-a)^{\alpha}}{\Gamma(\alpha)}.$$

When taking the first order derivative of this function, another function is returned, which is nonzero.

3.2.2 RIGHT HAND (CAPUTO)

The right hand derivative (RHD) is obtained by first taking an integer order derivative of the starting function, and then calculating the fractional integral of the resulting function. Using the same variables as above, we represent the RHD by swapping the causal value and the function variable:

$$ {}_xD_a^{\alpha} f\left(x\right) = {}_aJ_x^{n-\alpha} f^{(n)}\left(x\right) \tag{9}$$

Here we can see how "right" refers to the placement of the derivative relative to the fractional integral. To be valid, the original function must be n times differentiable, with the resulting derivative being fractionally integrable, a slightly stronger condition than the RHD. In this case we will say that $f(x)$ is right hand differentiable. In addition to this perceived strictness, the RHD would return 0 as the fractional derivative of a constant function, as a whole order derivative is taken first, leaving a fractional integral of 0.

3.2.3 COMPARISON

There are situations where the LHD and RHD can each be useful individually, there is a relationship between the two that can illustrate exactly how the definitions are not entirely inconsistent as they first appear.

Theorem 2. ${}_xD_a^{\alpha} f\left(x\right) = {}_aD_x^{\alpha} f\left(x\right) - \sum_{k=0}^{n-1} \frac{(x-a)^{k-\alpha}}{\Gamma\left(k-\alpha+1\right)} f^{(k)}\left(a\right)$, *for any function that is both right and left hand differentiable, and fractionally integrable from a to x.*

Proof.

$$\begin{aligned}
f(x) &= f(x) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + R_{n-1} \\
&= \sum_{k=0}^{n-1} \frac{(x-a)^k}{\Gamma(k+1)} f^{(k)}(a) + R_{n-1}, where \\
R_{n-1} &= \int_a^t \frac{f^{(n)}(\tau)}{(n-1)!}(t-\tau)^{n-1} d\tau.
\end{aligned}$$

This is just the integral form of the remainder of a Taylor Series Expansion. Further,

$$\int_a^t \frac{f^{(n)}(\tau)}{(n-1)!}(t-\tau)^{n-1} d\tau = {}_aJ_x^n f^{(n)}(x)$$

so

$$\begin{aligned}
f(x) &= \sum_{k=0}^{n-1} \frac{(x-a)^k}{\Gamma(k+1)} f^{(k)}(a) + {}_aJ_x^n f^{(n)}(x) \\
{}_aD_x^\alpha f(x) &= {}_aD_x^\alpha \left[\sum_{k=0}^{n-1} \frac{(x-a)^k}{\Gamma(k+1)} f^{(k)}(a) + {}_aJ_x^n f^{(n)}(x) \right] \\
&= \sum_{k=0}^{n-1} {}_aD_x^\alpha \frac{(x-a)^k}{\Gamma(k+1)} f^{(k)}(a) + {}_aD_x^\alpha {}_aJ_x^n f^{(n)}(x) \\
&= \sum_{k=0}^{n-1} \frac{\Gamma(k+1)(x-a)^{k-\alpha}}{\Gamma(k-\alpha+1)\Gamma(k+1)} f^{(k)}(a) + {}_aJ_x^{n-\alpha} f^{(n)}(x) \\
&= \sum_{k=0}^{n-1} \frac{(x-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(a) + {}_xD_a^\alpha f(x).
\end{aligned}$$

Thus,

$${}_xD_a^\alpha f(x) = {}_xD_a^\alpha f(x) - \sum_{k=0}^{n-1} \frac{(x-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(a).$$

$\square$

This theorem highlights an important relationship between the LHD and RHD, particularly that they agree up to a term dependent on the causality of the system. Should the function and all n of its derivatives be vanishing at a, this term would be 0 and the two definitions would agree. This perfectly captures how the RHD has a slightly stronger condition than the LHD. In fact, this additional term is exactly what comes up as the constant of integration when the integral part of the RHD is performed last.

It is also visible from this theorem how the RHD and LHD composition over themselves. If $f(x)$ is both left and right hand differentiable, and the causality is chosen such that the causal term vanishes and the RHD is equal to the LHD, then the derivatives will composition as they reduce back to the composition of fractional integrals.

Theorem 3. *If* ${}_xD_a^{\alpha} f\left(x\right) = {}_xD_a^{\alpha} f\left(x\right)$, *then* ${}_aD_x^{\alpha}\,{}_aD_x^{\beta} f(x) = {}_aD_x^{\alpha+\beta} f(x)$

Proof. We begin with

$${}_aD_x^{\alpha}\,{}_aD_x^{\beta} f(x) = {}_aD_x^{\alpha}\,{}_xD_a^{\beta} f(x)$$

by assumption. By definition, the right side expands, giving

$${}_aD_x^{\alpha}\,{}_xD_a^{\beta} f(x) = \frac{d^n}{dx^n}\,{}_aJ_x^{n-\alpha}\,{}_aJ_x^{m-\beta} f^{(m)}(x), n, m \in \mathbb{N}.$$

By theorem 1, we have

$${}_aD_x^{\alpha}\,{}_xD_a^{\beta} f(x) = \frac{d^n}{dx^n}\,{}_aJ_x^{n-\alpha+m-\beta} f^{(m)}(x).$$

Again, by assumption that LHD and RHD are equal, this means

$${}_aD_x^{\alpha}\,{}_xD_a^{\beta} f(x) = \frac{d^n}{dx^n}\frac{d^m}{dx^m}\,{}_aJ_x^{n-\alpha+m-\beta} f(x).$$

Since traditional derivatives obey the semi-group property, this means

$${}_aD_x^{\alpha}\,{}_xD_a^{\beta} f(x) = \frac{d^{n+m}}{dx^{n+m}}\,{}_aJ_x^{n-\alpha+m-\beta} f(x).$$

Finally, by definition, we are left with

$${}_aD_x^{\alpha}\,{}_xD_a^{\beta} f(x) = {}_aD_x^{\alpha+\beta} f(x).$$

□

However, should the causal term not vanish, or the $f(x)$ not be both left and right hand differentiable, the derivatives will not obey the semi-group property in general.

3.2.4 DERIVATIVES OF VARIOUS FUNCTIONS

To explore some basic functions and what happens to them under the fractional differentiation and fractional integration, we can look at how they behave under traditional methods. Below is a table of the n-order derivatives and integrals of four basic functions, which have clear patterns that we would hope to generalize to fractional methods.

n order integral of $f(x)$	$f(x)$	n order derivative of $f(x)$
$\frac{c}{n!}x^n$	c	0
$\frac{m!}{(m+n)!}x^{m+n}$	x^m	$\frac{m!}{(m-n)!}x^{m-n}$
$a^{-n}e^{ax}$	e^{ax}	$a^n e^{ax}$
$a^{-n}sin(ax - \frac{n\pi}{2})$	$sin(ax)$	$a^n sin(ax + \frac{n\pi}{2})$

Table 1: Traditional integrals and derivatives of common functions.

Theorem 4. ${}_0D_x^{\alpha} x^m = \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)} x^{m-\alpha}$

Proof. We begin with

$$ {}_0{D_x}^{\alpha} x^m = \frac{d}{dx} {}_0{J_x}^{1-\alpha} x^m, $$

using the LHD. Using integration by parts, we can begin to solve the right-hand side.

$$ {}_0{J_x}^{1-\alpha} x^m = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-y)^{-\alpha} y^m dy. $$

Letting $u = y^m,\ du = my^{m-1},\ dv = (x-y)^{-\alpha},\ v = \frac{-(x-y)^{1-\alpha}}{1-\alpha}$,

$$ \begin{aligned} &\frac{1}{\Gamma(1-\alpha)} \int_0^x (x-y)^{-\alpha} y^m dy \\ =&\frac{1}{\Gamma(1-\alpha)} \left[\frac{-y^m (x-y)^{1-\alpha}}{1-\alpha} - \int_0^x \frac{-m(x-y)^{1-\alpha}}{1-\alpha} y^{m-1} dy \right]_0^x . \end{aligned} $$

It is clear that evaluating from 0 to x will cancel the first term, leaving

$$ \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-y)^{-\alpha} y^m dy = \frac{m}{\Gamma(1-\alpha)(1-\alpha)} \int_0^x (x-y)^{1-\alpha} y^{m-1} dy. $$

At this point, choosing the same terms for integration by parts each time, we can see this cascade down, with the "uv" term cancelling to 0 each time it is evaluated, while the integral term continues to accumulate more fractions, all the way until the $m - \alpha th$ iteration, when we have

$$\frac{(m)(m-1)...(1)}{\Gamma(1-\alpha)(1-\alpha)(2-\alpha)...(m-\alpha)} \int_0^x (x-y)^{m-\alpha}\, dy.$$

It is worth noting that

$$\frac{1}{(1-\alpha)(2-\alpha)...(m-\alpha)} = \frac{(-\alpha)...(1)}{(1-\alpha)...(m-\alpha)(-\alpha)...(1)} = \frac{\Gamma(1-\alpha)}{\Gamma(m-\alpha+1)},$$

which converts our integral above into

$$\frac{m!}{\Gamma(m-\alpha+1)} \int_0^x (x-y)^{m-\alpha}\, dy,$$

which can be evaluated one final time to arrive at

$$\frac{m!}{\Gamma(m-\alpha+1)} \left[\frac{-(x-y)^{m-\alpha+1}}{m-\alpha+1} \right]_0^x = \frac{m!}{\Gamma(m-\alpha+2)} x^{m-\alpha+1},$$

which now leaves us with

$${}_0J_x{}^{1-\alpha} x^m = \frac{m!}{\Gamma(m-\alpha+2)} x^{m-\alpha+1}.$$

Since

$$\begin{aligned}
{}_0D_x{}^{\alpha} x^m &= \frac{d}{dx} {}_0J_x{}^{1-\alpha} x^m, \\
{}_0D_x{}^{\alpha} x^m &= \frac{d}{dx} \frac{m!}{\Gamma(m-\alpha+2)} x^{m-\alpha+1}, \\
{}_0D_x{}^{\alpha} x^m &= \frac{m!}{\Gamma(m-\alpha+1)} x^{m-\alpha}.
\end{aligned}$$

□

Thus we have shown that the pattern established in differentiation does appear to carry over when generalized to fractional calculus. Using this, we can also see that we get the same result for the fractional derivative of a constant that we arrived at earlier. This does not at first match the pattern established by traditional calculus, though using the RHD we have already established a derivative of 0. This discrepancy in RHD and LHD can be settled using the theorem relating them, since there is no value that would make a constant function vanish unless the function is the constant 0.

Below is a table containing the most basic of functions used in many differential equations. These functions are listed alongside their fractional integral and derivative of order α. It should be noted that these functions make use of different values for a, the causality, using the values that make the sum from theorem 2 vanish. We do not prove directly the relationship for the exponential and trigonometric functions, though they can be derived from the series expansion of the exponential function and the relationship between the exponential and sine functions.

α order integral of $f(x)$	$f(x)$	α order derivative of $f(x)$
$\frac{c}{\Gamma(\alpha+1)}x^{\alpha}$	c	$\frac{c}{\Gamma(-\alpha+1)}x^{-\alpha}$
$\frac{\Gamma(m+1)}{\Gamma(m+\alpha+1)}x^{m+\alpha}$	x^m, $m > -1$	$\frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)}x^{m-\alpha}$
$a^{-\alpha}e^{ax}$	e^{ax}	$a^{\alpha}e^{ax}$
$a^{-\alpha}sin(ax - \frac{\alpha\pi}{2})$	$sin(ax)$	$a^{\alpha}sin(ax + \frac{\alpha\pi}{2})$

Table 2: Fractional integrals and derivatives of common functions.

4 FRACTIONAL OSCILLATION

Upon analysis of the mathematics behind fractional oscillation, there were three key points of interest:

The first key point was that to consider the range of equations describing harmonic oscillation with damping with no inertial term, and oscillation with an inertial term with no damping, the original harmonic equations, both in standard and integral forms, had to be transformed via the Laplace Transformation. This transformation guaranteed the successful use of the fractional derivative by relying on definition alone. Consequentially, the causality of the system in both cases was set to be 0. As such, it was possible to then consider the two forms of harmonic oscillation under the same, unified setting.

The second key point was that it was discovered that modifying the fractional counterpart of any harmonic term had a predictable effect on the system as a whole, based solely upon the mathematics of the individual derivations. Considering the typical harmonic equation:

$$ma + bv + kx = f(x)$$

where m is the inertial term, b is the damping term, and k is the potential term, and our derivatives are indicated by a for the second (acceleration), v for the first (velocity), and x for the identity (displacement), the behavior that arises from fractional oscillation can be described. As a note: the $f(x)$ term indicates a forcing term to equate the system with, though it is safe to treat this as equal to 0 for this discussion, without loss of generality.

For any derivative between the first and the second, the symbol $a_\flat$ or $v_\sharp$ may be used, and for any derivative between the function itself and the first, the symbol $v_\flat$ or $x_\sharp$ may be used. This convention was made to allow for discussion of sub-acceleration (which would be equivalent to super-velocity), and sub-velocity (equivalent to super-displacement). In addition, as the order of the derivative changes, the unit associated with that derivative's term in the harmonic equation must change accordingly, giving rise to fractional inertia ($m_\flat$ or $b_\sharp$) and fractional damping ($b_\flat$ or $k_\sharp$), with derivatives of no order simply remaining with the potential term as is.

Beginning with the damping term, when the order of the derivative was between 0 and 1, sub-velocity was being used. This term, as the derivative increased from 0 to 1, similarly increased the amount of influence damping has on the system as a whole, with the most damping occurring when the order was 1 and regular velocity was used. Mathematically, the term moved from position and potential, through super-displacement and super-potential (equivalently called sub-velocity and sub-damping), into velocity and damping. This indicated an evolution occurring in the system between the position step and the velocity step, whereupon the system gradually reacted to the presence of a damping factor as it began its motion, a concept unexplainable without the presence of a fractional derivative.

Moving into the inertial term, when the order of the derivative is between 1 and 2, sub-acceleration is being used. This term, as the derivative increased from 1 to 2, displayed the presence of damping, even when no damping term was present with a velocity term. When the order of the inertial derivative was 2 at regular acceleration, damping disappeared if no damping term was associated with velocity. Mathematically, the term moved from velocity and damping, through super-velocity and super-damping (equivalently sub-acceleration and sub-inertia), into acceleration and inertia. This indicated that the presence of damping in a fractional system continued to exist as long as the derivative associated with the inertial term was not yet at the acceleration order, and that the system was not closed as long as the order was fractional.

The third key point was that in both cases, it could be seen that the terms of a harmonic equation associated with the different orders of derivatives, mathematically behaved as if they were on a spectrum when the order was fractional. As the order of a derivative moved between the bounds of one integer order to the next, the associated physical term also moved from the respective term to the next, shifting the behavior of the system accordingly. This indicated that the physical qualities of an ordinary harmonic system are simply reflections of the most common derivatives and the physical properties that they influence, while fractional calculus is able to uncover a deeper connection between the terms rooted in the conservation of the forces acting within the whole system. This connection allows for consideration of a systematic evolution in addition to the potential difficulty in creating a truly closed system.

Considering the paper by Achar et al., it was determined that any term in the motion equation representing a super-velocity (or sub-acceleration), led to the presence of damping, even without conventional damping terms present. This surprising result came through the derivation of energy equations which relied on fractional calculus methods.

Achar et al. began with the integral form of the equation for harmonic oscillation:

$$x(t) = x(0) + x'(0)t \; - \; \omega^2 \int_0^t (t - t')x(t')dt',$$

where x(t) is the displacement of the oscillator at time t, and $x(0)$ and $x'(0)$ are the initial displacement and velocity of the oscillator, with ω being the angular frequency. They then generalized this equation to allow for fractional order derivatives (with the above equation being the case for the typical second order inertial term):

$$x(t) = x(0) + x'(0)t \;-\; \frac{\omega^2}{\Gamma(\alpha)} \int_0^t (t - t')^{\alpha-1} x(t') dt'$$

To solve this equation, the Laplace transform was used to return

$$\widetilde{x}(s) = \frac{x(0)}{s} + \frac{x'(0)}{s^2} - \omega^2 \frac{\widetilde{x}(s)}{s^\alpha},$$

where $\widetilde{x}$ is the Laplace Transformation of x. When solving for $\widetilde{x}$, this returns

$$\widetilde{x}(s) = \frac{x(0)s^{-1}}{1 + \omega^2 s^{-\alpha}} + \frac{x'(0)s^{-1}}{1 + \omega^2 s^{-\alpha}} - \omega^2 \frac{\widetilde{x}(s)}{s^\alpha}.$$

When the inverse Laplace Transformation is taken, this gives the new equation

$$x(t) = x(0)E_{\alpha,1}(-\omega^2 t^\alpha) + x'(0)tE_{\alpha,2}(-\omega^2 t^\alpha),$$

where $E_{\alpha,\beta}$ is the Mittag-Leffler function. If the initial conditions $x(0) = x_0$ and $x'(0) = 0$ are used, then this shortens to

$$x(t) = x_0 E_{\alpha,1}(-\omega^2 t^\alpha),$$

which finally reduces, in the case of $\alpha = 2$, to

$$x(t) = x_0 cos(\omega t),$$

giving us the typical function used for harmonic oscillation. This alone was not the full extent of the exploration, as Achar et al., wanted to observe the effect of fractional oscillation on the energy of the system.

Beginning this time with the total energy equation of a harmonic oscillator:

$$E_H = \frac{1}{2}kx(t)^2 + \frac{1}{2}mx'(t)^2,$$

which returns a constant energy diagram when there is no damping term, and a decreasing energy diagram when there is a damping term, Achar et al. proposed that a definition for generalized momentum be

$$p = m\frac{d^{\frac{\alpha}{2}}x(t)}{dt^{\frac{\alpha}{2}}},$$

where the total energy of the generalized oscillation equation can be written

$$E_F = \frac{1}{2}kx(t)^2 + \frac{1}{2}m\left(\frac{d^{\frac{\alpha}{2}}x(t)}{dt^{\frac{\alpha}{2}}}\right)^2.$$

Here, it is understood that m, the mass term, does not have the dimensions of the typical mass term unless α is 2. This definition ensures that $\frac{p^2}{2m}$ always has the dimensions of energy. When the displacement solution is subbed into the momentum definition, we obtain

$$p = m\frac{d^{\frac{\alpha}{2}}x_0E_{\alpha,1}(-\omega^2t^\alpha)}{dt^{\frac{\alpha}{2}}} = -mx_0\omega^2t^{\alpha/2}E_{\alpha,1+\alpha/2}(-\omega^2t^\alpha),$$

which when subbed into the energy equation returns

$$E_F = \frac{1}{2}kx_0^2(E_{\alpha,1}(-\omega^2t^\alpha))^2 + \frac{1}{2}mx_0^2\omega^4t^\alpha\left(E_{\alpha,1+\alpha/2}(-\omega^2t^\alpha)\right)^2.$$

It is with this equation that the effect of super-damping can be seen. In any case where α is less than 2 but greater than 1, this function is decreasing. However, when less than 1, this behavior disappears, as the responsible half of the sum vanishes. This immediately makes clear that damping appears to be embedded into the motion of oscillation, and that an oscillating systems moves through many stages of damping as it evolves into its motion, even when the conventional damping terms are not present.

In the paper by Rousan et al., on the other hand, the oscillation equation was approached from a different direction. The typical oscillation equation was generalized into a fractional form as

$$m\frac{d^2x}{dt^2} + \alpha b\frac{d^\alpha x}{dt^\alpha} + kx = 0,$$

where $\alpha = 1$ corresponds with simple damped oscillation and $\alpha = 0$ corresponds with simple undamped oscillation. Here, they made use of damping ratio $\eta = \frac{b}{2m\omega^{2-\alpha}}$ and the Laplace Transformation to convert the oscillation equation into

$$s^2x(s) - sx_0 + 2\alpha\eta\omega^{2-\alpha}\left\{s^\alpha x(s) - \left(\frac{d^{\alpha-1}x}{dt^{\alpha-1}}\right)_{t=0}\right\} + \omega^2x(s) = 0,$$

which when solved for $x(s)$ returned

$$x(s) = \frac{sx_0 + 2\alpha\eta\omega^{2-\alpha}\left(\frac{d^{\alpha-1}x}{dt^{\alpha-1}}\right)_{t=0}}{s^2 + 2\alpha\eta\omega^{2-\alpha} + \omega^2},$$

and when expanded algebraically and reverted back through the inverse Laplace Transform returned

$$x\left(t\right) = \sum_{p,r=0}^{\infty} \frac{\left(-1\right)^{p+r}\omega^{2(p+r)-r\alpha}\left(2\alpha\eta\right)^{r}\left(p+r\right)!t^{2(p+r)-r\alpha}x_0}{p!r!\Gamma\left[2\left(p+r\right)-r\alpha+1\right]}\left[1+\frac{2\alpha\eta\omega t}{2\left(p+r\right)-r\alpha+1}\right]$$

When α is 0 (and hence there is no conventional damping), the equation reduces to the series expansion for cosine, returning

$$x(t) = \ x_0 cos(\omega t),$$

which is again the normal function for oscillatory motion. When α is 1 (and hence there is regular damping), the equation reduces to another typical equation:

$$x(t) = x_0 e^{-\eta\omega t}\left[cos(\omega\sqrt{1-\eta^2}t) + \frac{\eta\omega}{\sqrt{1-\eta^2}} sin(\omega\sqrt{1-\eta^2}t)\right].$$

Rousan et al. analyzed this generalized solution and determined that between the damped and non-damped cases (when α is between 1 and 0), the effect of damping begins to increase, reaching its maximum effect in the regular damping case. This indicated a relationship between the evolution of oscillatory motion and an intrinsic damping effect (which was touched on in Achar et al.'s exploration).

Altogether, these separate results indicate that oscillatory motion naturally accounts for damping in some way when the damping term is varied between non-existence and a second-order derivative, even when there is no traditional damping of a first-order derivative present. Additionally, this damping increases from introduction until it reaches a maximum at the first-order, then begins to decrease but remain present until it vanishes at the second-order.

5 ANOMALOUS DIFFUSION

Analysis of anomalous diffusion revealed sub-diffusion as a very prominent feature in many real-world diffusion systems, which carried with it two key points.

The first key point was that sub-diffusion could be used to model the anomalies arising from real-world considerations of diffusive systems and the obstacles that diffusing particles would likely run into, making it a useful tool for considering general random obstacles as well as the obstacle of a non-homogenous material. The equations for both involved setting a causality once again at 0, allowing all instances to be considered within the same, unified setting.

The second key point was that the case of the heterogenous material diffusion relied upon the Laplace Transformation of the Heat Diffusion equations, while the case of the general anomalous diffusion relied upon the differential solutions to the probability distribution of a particle following Fick's Law, Einstein's relation, and the Focker-Planck equations. While these methods of derivation were substantially different, both relied upon an attempt to solve a discrepancy between original calculated predictions and actual observations. The predictions from traditional calculus were a strictly linear relationship with time, while that was not the case. In allowing for a fractional power relationship with time, the fractional form of the diffusion equations was reached.

The mathematics that the sub-diffusion equations provided allows for more complete description of the diffusive behavior in these systems. Diffusion equations are equipped to account for the normal distribution of particles in a system, but the actual diffusion of particles in a system often falls short of this normalized distribution (hence a sub-diffusion). In the general case of a sub-diffusion, the particle must encounter obstacles in a random manner, inhibiting a steady progress (or distribution), which follows more closely with the probabilistic behavior of a continuous random time walk. In the more specific case of non-homogenous diffusion, the particle must make the change between diffusing in one medium to diffusing through a new medium. The barrier provides such small obstacles as already mathematically accounted for in the general case.

Thus, the second key point was also that the different cases of diffusion, while already physically similar, underwent very different mathematical treatment and yet remained in agreement upon transformation into fractional order. This indicates that the underlying math of the fractional calculus definitions are able to remain consistent when systems are similar in their underlying physics.

Looking at the case of thermal diffusion studied in the paper by Sierociuk et al., there was experimental evidence to support the idea that heat is diffused in heterogeneous materials following a model that can be better described using a derivation of the heat-diffusion equations that rely on fractional calculus, rather than ordinary methods.

The heating process of a semi-infinite beam can be described using

$$\frac{\partial}{\partial t}T(t,\lambda) = \frac{1}{a^2}\frac{\partial^2 T(t,\lambda)}{\partial \lambda^2},$$

with initial conditions

$$T(0,\lambda) = 0, T(t,0) = u(t),$$

where $T(t,\ \lambda)$ is the temperature of the beam at time t and location λ, and $\frac{1}{a^2}$ is a beam material conductivity.

With the Laplace Transformation, this equation was converted into

$$sT(s,\lambda) - T(0,\lambda) = \frac{1}{a^2}\frac{\partial^2 T(t,\lambda)}{\partial \lambda^2},$$

which can be solved algebraically with the boundary conditions to obtain

$$\frac{\partial^2 T(t,\lambda)}{\partial \lambda^2} - a^2 sT(s,\lambda) = 0.$$

This equation has solutions of the form

$$T(s,\lambda) = C_1(s)e^{a\lambda s^{0.5}} + C_2(s)e^{-a\lambda s^{0.5}},$$

where $C_1 = 0$ due to boundary conditions, leaving

$$T(s,\lambda) = C_2(s)e^{-a\lambda s^{0.5}}.$$

From here, using $\lambda = 0$, it is clear that $C_2(s) = U(s)$, by the other boundary condition, meaning

$$T(s,\lambda) = U(s)e^{-a\lambda s^{0.5}}.$$

Next, the Heat Flux is given:

$$H(t,\lambda) = -\frac{1}{a^2}\frac{\partial^2 T(t,\lambda)}{\partial \lambda^2},$$

which is transformed via the Laplace transformation and the above equations to produce

$$H(s,\lambda) = \frac{1}{a} s^{0.5} U(s) e^{-a\lambda s^{0.5}},$$

which ultimately gives the relationship between heat flux and temperature at a point as

$$H(t,\lambda) = \frac{1}{a} \frac{\partial^{0.5}}{\partial t^{0.5}} T(t,\lambda).$$

This process was repeated with a different starting point for the heating equation, dubbed the "subdiffusion process":

$$\frac{\partial^2 T(t,\lambda)}{\partial \lambda^2} = a_\alpha^2 \frac{\partial^\alpha}{\partial t^\alpha} T(t,\lambda),$$

which uses a_α as the beam parameter, dependent on heat conductivity and material density, and $\alpha < 1$ to represent the diffusion order.

They then arrived at a similar relationship between heat flux and temperature with the equation

$$H(t,\lambda) = \frac{1}{a} \frac{\partial^{\frac{\alpha}{2}}}{\partial t^{\frac{\alpha}{2}}} T(t,\lambda).$$

Further derivations were made to better analyze data from an experimental standpoint, and Sierociuk et al. showed that real-life heat diffusion in materials that are heterogenous (and therefore do not have the same conductivity and density throughout the system) diffused in a process that was more closely predicted by the fractional diffusion equation than by the ideal diffusion equation.

In the paper on anomalous diffusion by Sokolov et al., the diffusion equations were approached from a probabilistic point of view. They begin with the discussion of Fick's Laws, and particularly his second law relating the change of concentration of a substance to the gradient of the concentration gradient:

$$\frac{\partial c\,(r,t)}{\partial t} = \kappa \nabla^2 c\,(r,t)\,,$$

where κ is the diffusion coefficient in units of length squared divided by time.

They make note, however, that Fick did not consider the more modern probabilistic point of view that came about with Einstein. Using that point of view, where concentration of particles is proportional to the probability of finding them at a specific location, this means that we can consider Fick's Second Law with probability $P(r,t)$ rather than concentration $c(r,t)$.

However, this still leads to a mean squared displacement of the particle that is linear with respect to time, which Sokolov et al., writes is not the case in a variety of physical systems. Instead, in cases deemed "subdiffusive," this scaling is proportional to a power of time that is less than one, which is analogous to the probabilistic case of a continuous-time random walk rather than in a simple random walk.

This idea leads to the fractional diffusion equation

$$\frac{\partial^\alpha P(r,t)}{\partial t^\alpha} = \kappa \nabla^2 P(r,t),$$

which can be rewritten (so long as $P(r,t)$ has the properties of a function that permit the fractional derivative to composition mentioned in chapter 3), as the equation

$$\frac{\partial P(r,t)}{\partial t} =_0 D_t^{1-\alpha} \kappa_\alpha \nabla^2 P(r,t),$$

where κ_α is still a diffusion coefficient with units of length squared divided by time to the power of α.

Considering also that some motion of the particle can be influenced by an outside deterministic force $\mathbf{f}$, the Focker-Planck equation can be invoked and used in the fractional setting to arrive at

$$\frac{\partial P(r,t)}{\partial t} =_0 D_t^{1-\alpha} \nabla(-\mu_\alpha \mathbf{f} P(r,t) + \kappa_\alpha \nabla P(r,t)),$$

where μ_α is the fractional particle mobility and obeys a generalization of the Einstein relation with the Boltzmann constant $\frac{\kappa_\alpha}{\mu_\alpha} = k_B T$.

Sokolov et al. arrive at the conclusion that this equation does indeed represent the non-linear scaling with time that real-life subdiffusive processes tend to obey.

Together, the papers from Sokolov et al. and Sierociuk et al. illustrate that anomalous diffusion processes such as subdiffusion can only be properly described with fractional calculus (if one wishes for their model to align with physical observations more closely).

6 CONCLUSION

In the case of exploring different definitions of fractional integrals and derivatives, what started out as apparently contradictory results turned out to be a result of the non-locality of the new ”differintegral” operation (combinind both into a single operation, due to the nature of fractional order). This disagreement can be put down to a single term difference in the two main definitions of the fractional derivative discussed in this paper, and is dependent on choice for the causality of the system. Should a lower bound for the fractional integral be set so that the function and its n derivatives be vanishing, then the right-hand and left-hand definitions agree, and certain properties of derivatives can be applied as usual. It should be noted, however, that these definitions do align with expected generalizations of certain function derivatives and integrals, especially in the sense of the integer order, which is comforting for an idea that was once dubbed “paradoxical”.

These equations have also had a surprising usefulness in the field of differential equations, offering models for physical problems that have either been experimentally verified to better predict results or to gain new insight into the evolution of a system. In the case of heat diffusion, “subdiffusion” on a fractional order aligns with experimental evidence that heat does not diffuse quite the way traditional models predict, with fractional models allowing more room for discussing poor insulation or the non-ideal case where the medium is not a homogenous material. This fractional order also aligns with the more probabilistic idea of particle diffusion, allowing for a model that predicts the likely evolution of a particle that scales with a fractional power of time rather than linearly, which is also confirmed by experimental evidence.

In the case of oscillation, the fractional model predicts the interesting quality of harmonic motion intrinsically accounting for the idea of damping, even when no traditional damping is present. This idea was explored from two separate perspectives (focusing on either adjusting the “velocity” term or the “acceleration” term), with both giving the result that damping peaks in the traditional equations, while remaining present at fractional orders until the only derivatives are integer order and do not include the “velocity” term. This idea, and the idea of the energy and force terms adjusting as the fractional velocity alternates between displacement and acceleration, indicates a type of systemic evolution as oscillatory motion changes over time.

Altogether, this exploration has found an intricate relationship between fractional calculus and the real world, offering up ideas that seem to be more adept at describing physical systems than traditional equations (albeit in cases that do not necessarily obey ideal conditions). It seems that the very non-local property that makes fractional derivatives contradictory is the same property that allows them to capture physical information from problems that traditional calculus does not have the capacity to. Working with an operator that by nature accounts for all times since an initial time allows for an incorporation of "memory" effects error correction. The sheer fact that the building blocks of fractional calculus are paradoxical at first glance is the exact reason they are so useful upon further examination. This seems to support Leibniz's prediction to L'Hôpital, while also opening up the door for many different explorations.

6.1 FUTURE WORK

This paper did not nearly cover every case it could have, and in fact was merely a surface-level glimpse into the possibilities of fractional calculus. Though diffusion and oscillation models were explored, there remain a large number of differential equation applications to be looked at. Additionally, further exploration of the properties and definitions of fractional derivatives could lead to even more unification in contradictory material, as there are definitions other than the righ and left-hand derivatives that were not touched upon here. It is also certainly worthwhile to look further into the conditions imposed by system causality, and explore the relationship between causality and specific functions further. In the cases that were touched on, there remains the idea of expansion, such as different models of diffusion that do not necessarily pertain to heat and particles (perhaps in disease spread, population growth, and electronics, where ideal conditions are almost impossible to ensure). In the oscillation case, it would be worthwhile to explore what happens if both sub and super-velocity are employed in the same equation, perhaps while a traditional displacement, velocity, and acceleration term are all present. It would be no surprise should the equations for such an exploration become quite messy quite quickly.

Of course, there is potential for further work that may not even be predictable yet. Though fractional calculus has been explored for centuries (and seriously for a few decades), there still remain many gaps in understanding that could one day offer insight into just how useful this generalization of calculus truly is.

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SPECIAL PROPERTIES AND PROOFS

GAMMA FUNCTION

The definition of the Gamma Function arises naturally from the definition of the factorial. First we will prove that the Gamma Function shares the same product property as the factorial, then we will show that the Gamma Function naturally gives rise to the limiting case of $\Gamma(1) = 0$, effectively defining $0! = 1$.

$$\Gamma(1) = 1 \textit{ and } \Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$$

Proof.

$$\begin{aligned}
\Gamma(\alpha + 1) = \int_0^\infty x^\alpha e^{-x} dx \ &= \left[-x^\alpha e^{-x}\right]_0^\infty + \int_0^\infty \alpha x^{\alpha-1} e^{-x} dx \\
&= \lim_{t\to\infty} [-x^\alpha e^{-x} + 0] + \int_0^\infty \alpha x^{\alpha-1} e^{-x} dx \\
&= 0 + \int_0^\infty \alpha x^{\alpha-1} e^{-x} dx \\
&= \alpha \int_0^\infty x^{\alpha-1} e^{-x} dx \\
&= \alpha\Gamma(\alpha)
\end{aligned}$$

And for the limiting case:

$$\begin{aligned}
\Gamma(1) = \int_0^\infty x^0 e^{-x} dx \ &= \int_0^\infty e^{-x} dx \\
&= \lim_{t\to\infty} \left[-e^{-t} + e^0\right] = 0 + 1 \\
&= 1
\end{aligned}$$

$\square$

CAUCHY REPEATED INTEGRAL

The repeated integral can be proven by induction:

Proof.

$$ {}_aJ_x^1 f(x) = \frac{1}{(0)!}\int_a^x (x-y)^0 f(y)\,dy = \int_a^x f(y)\,dy $$

This is the first integral of $f(x)$

Assume

$$ {}_aJ_x^n f(x) = \frac{1}{(n-1)!}\int_a^x (x-y)^{n-1} f(y)\,dy $$

Show

$$ \begin{aligned} {}_aJ_x^{n+1} f(x) &= \frac{1}{n!}\int_a^x (x-y)^n f(y)\,dy \\ &= \int_a^x \frac{1}{(n-1)!}\int_a^t (t-y)^n f(y)\,dy \end{aligned} $$

This is merely taking the integral of the nth iterated integral. Then using the Leibniz-Integral Rule:

$$ \frac{d}{dx}\left[\frac{1}{n!}\int_a^x (x-y)^n f(y)\,dy\right] = \frac{1}{(n-1)!}\int_a^x (x-y)^{n-1} f(y)\,dy $$

so

$$ \begin{aligned} \frac{1}{n!}\int_a^x (x-y)^n f(y)\,dy &= \int_a^x \frac{d}{dt}\left[\frac{1}{n!}\int_a^t (t-y)^n f(y)\,dy\right] dt \\ &= \frac{1}{n!}\int_a^t (t-y)^n f(y)\,dy \end{aligned} $$

$\square$

GENERALIZED DERIVATIVE OF SINE

The observation that taking successive derivatives of sine results in shifting sine by $\frac{n\pi}{2}$ can be proven through its relation to the exponential function.

$$D_x{}^n sin(ax) = a^n sin(ax + \frac{n\pi}{2})$$

Proof. We begin by using Euler's formula to relate sine to the exponential:

$$sin(ax) = \frac{i}{2}(e^{-iax} - e^{iax}).$$

Therefore, we can take

$$\begin{aligned} D_x{}^n sin(ax) &= D_x{}^n \frac{i}{2}(e^{-iax} - e^{iax}) \\ &= \frac{i}{2} D_x{}^n (e^{-iax} - e^{iax}) \\ &= \frac{i}{2}(D_x{}^n e^{-iax} - D_x{}^n e^{iax}) \\ &= \frac{i}{2}(-i^n a^n e^{-iax} - i^n a^n e^{iax}). \end{aligned}$$

Here we stop to consider the fact that $i^n = e^{\frac{in\pi}{2}}$, leaving us with

$$\begin{aligned} D_x{}^n sin(ax) &= \frac{i}{2}(-i^n a^n e^{-iax} - i^n a^n e^{iax}) \\ &= \frac{a^n i}{2}(e^{-\frac{in\pi}{2} - iax} - e^{\frac{in\pi}{2} + iax}) \\ &= \frac{a^n i}{2}(cos(\frac{n\pi}{2} + ax) - isin(\frac{n\pi}{2} + ax) - cos(\frac{n\pi}{2} + ax) - isin(\frac{n\pi}{2} + ax)) \\ &= \frac{a^n i}{2}(0 - 2isin(\frac{n\pi}{2} + ax)) \\ &= a^n \frac{i(-2i)}{2} sin(\frac{n\pi}{2} + ax) \\ &= a^n sin(\frac{n\pi}{2} + ax) \end{aligned}$$

□