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A local Borg -Marchenko theorem for complex potentials.

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A LOCAL BORG-MARCHENKO THEOREM FOR COMPLEX POTENTIALS

by

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ABSTRACT OF DISSERTATION
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We investigate the Sturm-Liouville problem generated by the equation

$$-y'' + qy = \lambda y$$

on $[0, b)$, where $0 < b \leq \infty$, the potential q is a complex-valued element of $L^1_{loc}([0, b))$, and a boundary condition is placed at 0 and, when necessary, at b. For such problems we define an *m-function*, which is defined in the same spirit as the Titchmarsh-Weyl m-function. The main result of the dissertation study is an extension of a result of Simon [**12**], which states that two real potentials coincide on a compact interval if and only if the m-functions for the corresponding problems are exponentially close on certain rays in **C**. We use our main result to provide a proof that, for problems of the type generated by $-y'' + qy = \lambda y$ on a compact interval, two spectra uniquely determine the potential.

Contents

CHAPTER 1

Introduction

This thesis is concerned principally with the role of the Titchmarsh-Weyl m-function in the analysis of the direct and inverse Sturm-Liouville problem generated by

$$(1.1) \qquad -y'' + qy = \lambda y$$

on $[0, b)$, where $0 < b \leq \infty$, q is complex-valued, and $q \in L^1_{loc}([0, b))$. A boundary condition is given at 0, and one is prescribed at b if necessary. Much work has been done in this direction in the case that the potential q is real-valued since (1.1) gives rise to a selfadjoint problem. The case that q is complex-valued leads to non-selfadjoint problems, which are currently the subject of active research. We now give some history, motivation, and background on direct and inverse spectral problems.

Motivated by integral equations, Weyl published a series of three papers (1908, 1909, 1910) [1] which laid some foundation for the m-function but then left this subject to return to it around 1950. Titchmarsh is given chief credit [1] for the introduction of the m-function (c. 1946). Titchmarsh and others were motivated principally by eigenfunction expansions for Sturm-Liouville problems with a real potential. These and other topics related to Sturm-Liouville problems with a real potential have been well studied and are completely understood. In the past fifty years, mathematicians have turned to investigating Sturm-Liouville problems with a complex potential. These investigations have produced results in both the direct problem (including eigenfunction expansions) and the inverse problem (recovering the potential when certain spectral data are given). Sturm-Liouville problems with a complex potential arise in physical situations involving energy dissipation and in scattering theory, which has played a central role in mathematical physics over the past century. Inverse scattering theory is a basic tool in areas such as radar, sonar, geophysical exploration, medical imaging, and nondestructive testing [**4**]. Inverse problems have become of interest recently in part because of their relation to some important nonlinear differential equations in mathematical physics. Specifically, some connections between the inverse Sturm-Liouville problem and the K-dV equation have been made [**10**].

In the 1940s the technique of transformation operators was introduced by Delsarte and Levitan. Marchenko has done extensive work in applying this technique to spectral theory and to its applications, including the Sturm-Liouville problem on a finite interval, the singular Sturm-Liouville problem on $[0, \infty)$ (where spectral functions arise), and inverse scattering theory [**11**]. Marchenko's first big application of transformation operators was for a uniqueness result in a 1952 paper, in which he proved that the spectral function of a Sturm-Liouville operator (on a compact interval or on the half-line) determines the operator uniquely [**10**].

Borg [**2**] carried out the first systematic investigation of the classical inverse Sturm-Liouville problem, where the potential q in (1.1) is real and $q \in L^1([0, \pi])$. In a 1946 paper, he showed that one spectrum does not, in general, determine the potential. In the same paper, he proved the following result.

THEOREM 1.1. *If λ_n are the eigenvalues of the problem generated by (1.1) and the boundary conditions*

$$y'(0) - hy(0) = 0, y'(\pi) - Hy(\pi) = 0,$$

and μ_n are the eigenvalues of the problem generated by (1.1) and the boundary conditions

$$y'(0) - hy(0) = 0, y'(\pi) + H_1 y(\pi) = 0,$$

and $H \neq H_1$ and $h, H, H_1 \in \mathbf{R}$, then the two sets of numbers λ_n and μ_n determine q, h, H, and H_1 uniquely.

In 1949, Levinson [9] gave a simpler proof of this result. In [8], Levin gives a proof of Theorem 1.1 when q, h, H, and H_1 are all complex. The main feature of these Borg-Levinson theorems is that two spectra determine the Sturm-Liouville operator. Since then, some uniqueness theorems for Sturm-Liouville problems have given other spectral information (such as a spectrum and norming constants) that determines the Sturm-Liouville operator. Investigating these uniqueness theorems for the case of complex potentials is the main topic of Chapter 6.

In an important piece of work in 1957 [13], Sims obtained an extension of the classical, selfadjoint two-fold classification (limit-point, limit-circle) of (1.1). His generalization allows a complex potential q and results in a three-fold classification of (1.1). In this work, Sims made a thorough study of boundary conditions and spectral properties of the corresponding operators for complex q. In [3], Brown et al. construct an analogue of the Sims result for the equation

$$-(py')' - qy = \lambda wy. \tag{1.2}$$

with certain restrictions on p, q, and w. In particular, a three-fold classification of (1.2) is obtained in [3]. That analysis is not merely a straightforward generalization of Sims' work, as problems and properties of (1.2) do not show in Sims' work, where $p = w = 1$. In [3], the m-function is defined for each of the three cases (classes) and its properties are related to the spectral properties of appropriate operators.

In Chapter 2, we obtain a two-fold classification of (1.1): we call the two classes Class I and Class II. A major difference between our work and that in [3] is the way of defining an m-function. First, our definition is based on properties of solutions of (1.1), whereas the definition in [3] is based on the nesting circle analysis. Second, the m-function in [3] for two cases is defined the same way that we define the m-function in Class I, but one of these two cases overlaps our Class II, where we have the freedom to define many more m-functions than in Class I. Thus, in this overlapping case, we define many m-functions, whereas only one is defined in [3].

In [**1**], Bennewitz gives a simplified proof of a local Borg-Marchenko Theorem, which we state presently. Let q and $\tilde{q}$ be real, locally integrable potentials on $[0, b)$ and $[0, \tilde{b})$, respectively. Let m and $\tilde{m}$ denote the m-functions for problems corresponding to q and $\tilde{q}$, respectively. Then we have the following.

THEOREM 1.2. *Let $a \in \mathbf{R}$ and $a \in (0, \min(b, \tilde{b})]$. Then $q = \tilde{q}$ on $[0, a]$ if and only if for any $\epsilon > 0$ we have*

$$m(\lambda) - \tilde{m}(\lambda) = O\left(e^{-2(a-\epsilon)\Re\sqrt{-\lambda}}\right)$$

as $\lambda \to \infty$ along some non-real ray emanating from the origin.

In Chapter 5, we state and prove a generalization of Theorem 1.2. A result similar to Theorem 1.2 is proved by Gesztesy and Simon in [**7**], although their result holds for real-valued potentials.

In a 1998 paper [**15**], Yurko established a uniqueness result involving only the m-function. Yurko shows that the eigenvalues of a Sturm-Liouville problem on a compact interval $[0, T]$ (with certain conditions on the potential) and the residues of the m-function at each eigenvalue determine the m-function uniquely. He assumes for this that the eigenvalues are algebraically simple. Yurko's uniqueness result is Theorem 1 of that paper, which states that the m-function uniquely determines the potential. In [**12**], Simon gives an improvement of this uniqueness result for real-valued potentials.

In brief, this dissertation proves a generalization of Theorem 1.2 for complex potentials and applies this result to obtain a generalization of Theorem 1.1 for complex potentials. A prominent problem left open in this dissertation is a determination of whether our generalization of Theorem 1.2 holds for all Class II problems (in particular, non-regular Class II problems). We now give a chapter-by-chapter summary of the dissertation.

In Chapter 2, we give an account of the nesting circle analysis (limit-point, limit-circle theory) which is based on that given in [**3**]. This analysis leads to the classification of (1.1) into a problem of Class I or a problem of Class II. Our main use of the nesting circle analysis is to define the domain of the m-function for a problem of Class I.

In Chapter 3, we define the maximal and minimal operators associated with (1.1) and obtain a characterization of the domain of the closure of the minimal operator in terms of fundamental systems of solutions of (1.1). The purpose of this characterization is to provide the background needed to define the m-function for a problem of Class II.

Chapter 4 is the most important chapter. We define the Titchmarsh-Weyl m-function for problems, generated by (1.1), of Class I and Class II. After some commentary and further notes on these definitions, we prove a standard asymptotic relation for the solutions of (1.1) with λ-independent initial conditions. The focus of the remainder of the dissertation is on problems falling into one of three types: Class I and two subcases of Class II. We then prove an asymptotic relation for the Dirichlet m-function and finish Chapter 4 with an important preliminary result (Theorem 4.8), to be used in Chapter 5, and a result stating a behavior of Green's function for the types of problems we consider.

Chapter 5 is the climax of the dissertation. We prove two theorems, which, taken together, constitute a generalization of Theorem 1 of [**1**] and state roughly that the m-functions for two problems are exponentially close on certain rays in $\mathbf{C}$ if and only if the potentials coincide on some compact interval $[0, a]$.

Chapter 6 considers an application of the results in Chapter 5 to Neumann problems only. The highlight of the chapter is Theorems 6.3 and 6.4, which state equivalences of certain pieces of spectral information. To achieve these theorems, we use ideas in [**15**] to obtain an estimate of the m-function and then to characterize the m-function in terms of two pieces of spectral information. The Borg-Levinson Theorem for complex potentials is then contained in Theorems 6.3 and 6.4.

CHAPTER 2

Nesting Circle Analysis

We give here an account of the limit-point, limit-circle (nesting circle) analysis which is based on that given in [**3**]. Let $0 < b \le \infty$ and q be a complex-valued element of $L^1_{loc}([0, b))$. We associate to the differential expression

$$L(y) = -y'' - qy$$

the closed convex hull, Q, of the set $\{q(x) + r : x \in [0, b), r \in (0, \infty)\}$:

$$Q = \overline{\text{co}}\{q(x) + r : x \in [0, b), r \in (0, \infty)\}.$$

Let $K \in \partial Q$. The convexity of Q guarantees that there is a line, say L_K, which passes through K and which has the property that Q is contained in one of the two closed half-planes determined by L_K. We note here that if Q has a tangent at K, then the tangent line at K is the only such line L_K. Given such a line L_K, let m denote its slope. We now define a rotation parameter $\eta \in [-\pi/2, \pi/2]$ for L_K.

Case 1: $m \in (0, \infty]$. Then there is a number $\theta \in (0, \pi/2]$ such that $\tan(\theta) = m$. Let $\eta = \pi/2 - \theta \in [0, \pi/2)$.

Case 2: $m \in (-\infty, 0)$. Then there is a number $\theta \in (-\pi/2, 0)$ such that $\tan(\theta) = m$. Let $\eta = -\theta - \pi/2 \in (-\pi/2, 0)$.

Case 3: $m = 0$. Then let $\eta = \pi/2$ if Q lies in the lower half-plane determined by L_K. Let $\eta = -\pi/2$ if Q lies in the upper half-plane determined by L_K.

Note that there may be more than one number η associated with K because there may be more than one line such as L_K. Let S denote the set of pairs (η, K) generated by the above construction. That is,

$$S = \{(\eta, K) : K \in \partial Q \text{ and } \eta \text{ is defined from a line } L_K \text{ as above}\}.$$

For $(\eta, K) \in S$, define

$$\Lambda_{\eta,K} = \{z \in \mathbf{C} : \Re[(z - K)e^{i\eta}] < 0\}.$$

From the above three cases and from the definition of Q, we see that η is constructed so that $\Lambda_{\eta,K} \subset \mathbf{C} - Q$; i.e., for each $z \in Q$,

(2.1) $$\Re[e^{i\eta}(z - K)] \geq 0.$$

Thus,

$$\bigcup_{(\eta,K)\in S} \Lambda_{\eta,K} \subset \mathbf{C} - Q.$$

That the reverse inclusion holds follows from geometric considerations. In brief, let $\lambda_0 \in \mathbf{C} - Q$ and let K denote the nearest point in Q to λ_0. Then the open disk about λ_0 with radius $|\lambda_0 - K|$ is contained in $\mathbf{C} - Q$. Let L_K denote the line through K perpendicular to the segment joining λ_0 and K. It follows that the open half-plane determined by L_K which contains λ_0 does not intersect Q (otherwise, the open disk about λ_0 with radius $|\lambda_0 - K|$ would intersect Q). Clearly, $K \in \partial Q$ and we have, from the above construction, a number η associated with L_K. This gives $(\eta, K) \in S$ and $\lambda_0 \in \Lambda_{\eta,K}$. Thus,

$$\mathbf{C} - Q = \bigcup_{(\eta,K)\in S} \Lambda_{\eta,K}.$$

Before presenting the nesting circle analysis, we introduce the following notation. For fixed $h_1, h_2, H_1, H_2 \in \mathbf{C}$ and for $\lambda \in \mathbf{C}$, let $\theta(\cdot,\lambda)$ and $\phi(\cdot,\lambda)$ be the unique solutions of

(2.2) $$L(y) = \lambda y$$

satisfying

$$\phi(0,\lambda) = h_1 \qquad \theta(0,\lambda) = h_2$$

$$\phi'(0,\lambda) = H_1 \qquad \theta'(0,\lambda) = H_2.$$

When confusion is unlikely, one or both arguments of ϕ and θ will be suppressed. For normalization purposes, we require that h_1, h_2, H_1, H_2 satisfy

$$h_1 H_2 - h_2 H_1 = 1$$

and

$$\overline{h_1} h_2 + \overline{H_1} H_2 = 0.$$

In particular, $\phi(\cdot, \lambda)$ and $\theta(\cdot, \lambda)$ are linearly independent. Furthermore, since $|h_1|^2 + |H_1|^2 > 0$, these two constraint equations set up a linear system for h_2, H_2 if h_1, H_1 are given.

Let H denote the quadruple (h_1, H_1, h_2, H_2) and define

$$S(H) = \{(\eta, K) \in S : \Re[e^{i\eta} \overline{h_1} H_1] \geq 0\}.$$

The condition

$$\Re[e^{i\eta} \overline{h_1} H_1] \geq 0 \tag{2.3}$$

appears naturally in the nesting circle analysis we present. Any pair $(\eta, K) \in S(H)$ as well as the corresponding half-plane $\Lambda_{\eta,K}$ is called *admissible*. Letting

$$Q(H) = \mathbf{C} - \bigcup_{(\eta,K) \in S(H)} \Lambda_{\eta,K},$$

we have $Q \subseteq Q(H)$. Furthermore, $Q(H)$ is closed and convex since it is the intersection of a family of closed half-planes. We shall assume henceforth that $Q(H) \neq \mathbf{C}$.

The purpose of this framework is that the nesting circle analysis is performed in $\mathbf{C} - Q(H)$. It is shown in [**3**] that $Q(H)$ contains the spectrum of several natural differential operators that are derived from the expression $L(y)$. The classical use of the nesting circle analysis is to determine, for a given $\lambda \in \mathbf{C}$, how many solutions of (2.2) lie in $L^2([0, b))$. This information is used, in turn, to define the Titchmarsh-Weyl m-function.

We shall make use below of the fact that, for $(\eta, K) \in S(H)$, a Möbius transformation

$$M(z) = \frac{Az + B}{Cz + D}$$

maps the line $\Re[ze^{i\eta}] = 0$ to a circle precisely when $\Re[\overline{C}De^{i\eta}] \neq 0$, and the radius of this circle is

$$r = \frac{|AD - BC|}{2|\Re[\overline{C}De^{i\eta}]|}.$$

Let $\lambda \in \mathbf{C} - Q(H)$. Then there is a pair $(\eta, K) \in S(H)$ such that $\lambda \in \Lambda_{\eta,K}$. Let $X \in (0, b)$. We now show that the Möbius transformation

$$M_X(z) = -\frac{\theta(X,\lambda)z + \theta'(X,\lambda)}{\phi(X,\lambda)z + \phi'(X,\lambda)}$$

maps the half-plane $\Re[ze^{i\eta}] \geq 0$ onto a closed disk $D_X(\lambda)$. We do this by showing first that the line $\Re[ze^{i\eta}] = 0$ is mapped by M_X to a circle and then that the singular point z_s of M_X has the property that $\Re[z_s e^{i\eta}] < 0$. Using the above fact about Möbius transformations, M_X maps the line $\Re[ze^{i\eta}] = 0$ to a circle precisely when $\Re[e^{i\eta}\phi'(X,\lambda)\overline{\phi(X,\lambda)}] \neq 0$. We show that, in fact, $\Re[e^{i\eta}\phi'(X,\lambda)\overline{\phi(X,\lambda)}] > 0$. Using integration by parts and the fact that $\phi(\cdot,\lambda)$ satisfies (2.2), we have

$$\int_0^X \lambda|\phi|^2 dx = -\phi'(X)\overline{\phi(X)} - \overline{h_1}H_1 + \int_0^X (|\phi'|^2 - q|\phi|^2)\,dx.$$

Taking real parts and rearranging gives

$$\Re[e^{i\eta}\phi'(X)\overline{\phi(X)}] = \Re[e^{i\eta}\overline{h_1}H_1] - \int_0^X \Re[e^{i\eta}((q-\lambda)|\phi|^2 + |\phi'|^2)]\,dx. \tag{2.4}$$

The first term on the right side of (2.4) is nonnegative since η satisfies (2.3). We rewrite the integrand in (2.4) as

$$\Re[e^{i\eta}((q - K - (\lambda - K))|\phi|^2 + |\phi'|^2)] = |\phi|^2\{-\Re[e^{i\eta}(\lambda - K)] + \Re[e^{i\eta}(q - K + \frac{|\phi'|^2}{|\phi|^2})]\}.$$

The first term in braces is positive since $\lambda \in \Lambda_{\eta,K}$. The second term in braces is nonnegative by (2.1). Thus, the integrand in (2.4) is nonnegative, and since ϕ is not identically zero on $[0, X]$, we see that the integral appearing in (2.4) is positive. Thus, (2.4) shows that

$$\Re[e^{i\eta}\phi'(X,\lambda)\overline{\phi(X,\lambda)}] > 0.$$

We have now shown that the line $\Re[ze^{i\eta}] = 0$ is mapped by M_X to a circle in $\mathbf{C}$. Furthermore, the singular point $z_s = -\phi'(X,\lambda)/\phi(X,\lambda)$ has the property that

$$\Re[e^{i\eta}z_s] = -\Re\left[e^{i\eta}\frac{\phi'(X,\lambda)\overline{\phi(X,\lambda)}}{|\phi(X,\lambda)|^2}\right] < 0$$

since $\Re[e^{i\eta}\phi'(X,\lambda)\overline{\phi(X,\lambda)}] > 0$. Therefore, the half-plane $\Re[ze^{i\eta}] \geq 0$ is mapped onto a closed disk $D_X(\lambda)$ in $\mathbf{C}$. Using again the above fact about Möbius transformations, the diameter $d_X(\lambda)$ of this disk is

$$d_X(\lambda) = \left|\frac{-\theta(X,\lambda)\phi'(X,\lambda) - \phi(X,\lambda)\theta'(X,\lambda)}{\Re[e^{i\eta}\phi'(X,\lambda)\overline{\phi(X,\lambda)}]}\right| = (\Re[e^{i\eta}\phi'(X,\lambda)\overline{\phi(X,\lambda)}])^{-1}.$$

Here, we have made use of the fact that $W(\phi,\theta)(X) = W(\phi,\theta)(0) = 1$.

We now establish that the disks $D_X(\lambda)$ are nested as $X \to b$. That is, if $0 < X < Y < b$, then $D_Y(\lambda) \subset D_X(\lambda)$. Given $w \in \mathbf{C}$, let $\psi_w(\cdot,\lambda) = \theta(\cdot,\lambda) + w\phi(\cdot,\lambda)$ and let $\psi_\infty(\cdot,\lambda) = \phi(\cdot,\lambda)$. From the previous paragraph, we have that $w \in D_X(\lambda)$ if and only if $\Re[e^{i\eta}M_X^{-1}(w)] \geq 0$. Since

$$M_X^{-1}(w) = -\frac{\psi_w'(X,\lambda)\overline{\psi_w(X,\lambda)}}{|\psi_w(X,\lambda)|^2},$$

$w \in D_X(\lambda)$ if and only if

$$\Re[e^{i\eta}\psi_w'(X,\lambda)\overline{\psi_w(X,\lambda)}] \leq 0.$$

An integration by parts shows that

$$\begin{aligned}\Re[e^{i\eta}\psi_w'(X,\lambda)\overline{\psi_w(X,\lambda)}] &= \Re[e^{i\eta}(H_2 + wH_1)\overline{(h_2 + wh_1)}] \\ &\quad - \int_0^X \Re[e^{i\eta}\{|\psi_w'|^2 + (q-\lambda)|\psi_w|^2\}]dx.\end{aligned}$$

By (2.1), the integrand appearing here is nonnegative. Now we have that $w \in D_X(\lambda)$ if and only if

$$(2.5)\qquad \int_0^X \Re[e^{i\eta}\{|v_w'|^2 + (q-\lambda)|v_w|^2\}]dx \le -\Re[e^{i\eta}(H_2 + wH_1)(\overline{h_2 + wh_1})].$$

It follows from this and the fact that the integrand appearing here is nonnegative that if $0 < X < Y < b$, then $D_Y(\lambda) \subset D_X(\lambda)$.

Therefore, $\bigcap_{0<X<b} D_X(\lambda)$ is either a single point or another disk. Note that the integral in (2.5) may be written as

$$(2.6)\qquad -\Re[e^{i\eta}(\lambda - K)]\int_0^X |v_w|^2 dx + \int_0^X \Re[e^{i\eta}(|v_w'|^2 + (q-K)|v_w|^2)]dx.$$

Let $m \in \bigcap_{0<X<b} D_X(\lambda)$. Substituting (2.6) into (2.5) and noting that, by (2.1), the integrand on the right in (2.6) is nonnegative, we have that for all $X \in [0,b)$,

$$(2.7)\qquad \int_0^X |v_m|^2 dx \le \frac{\Re[e^{i\eta}(H_2 + mH_1)(\overline{h_2 + mh_1})]}{\Re[e^{i\eta}(\lambda - K)]}.$$

Since the right side of (2.7) is independent of X, we have $v_m \in L^2([0,b))$. If $\bigcap_{0<X<b} D_X(\lambda)$ is a disk, then all solutions of (2.2) are in $L^2([0,b))$. This analysis shows that if $\lambda \in \mathbf{C} - Q(H)$, then there is an $L^2([0,b))$ solution of (2.2). We now make the following definition.

DEFINITION 2.1. *The expression L is in* Class I *at b if at most one solution of (2.2) (up to constant multiples) lies in $L^2([0,b))$. L is in* Class II *at b if all solutions of (2.2) lie in $L^2([0,b))$.*

The nesting circle analysis in [3] provides L with a three-fold classification which is a refinement of the classical, selfadjoint two-fold classification of L into *limit-point* or *limit-circle*. In [3], the limit-point case is that the intersection of disks is a point w, giving that only v_w and its multiples have the property that the integral in (2.5) has a finite limit as $X \to b$. This implies that $v_w \in L^2([0,b))$ by (2.6), but there may be another solution of (2.2), independent of v_w, lying in $L^2([0,b))$. This describes the limit-point subcases: either v_w is the only (up to multiples) solution of (2.2) lying in

$L^2([0,b))$, or all solutions of (2.2) lie in $L^2([0,b))$. The limit-circle case in [3] is that the intersection of disks is a disk, giving that all solutions of (2.2) have the property that the integral in (2.5) (with any solution replacing v_w) tends to a finite limit as $X \to b$. Again, this implies that all solutions of (2.2) are in $L^2([0,b))$.

The classification of L at b into Class I or Class II is independent of λ in the sense of the following theorem, which is a standard result. We include its proof for completeness.

THEOREM 2.2. *If $\lambda_0 \in \mathbf{C}$ has the property that all solutions of $L(y) = \lambda_0 y$ are in $L^2([0,b))$, then all $\lambda \in \mathbf{C}$ have this property.*

PROOF. Assume that u_1, u_2 are linearly independent, square-integrable solutions of $L(y) = \lambda_0 y$ such that $W(u_1, u_2) \equiv 1$.

Let $\lambda \neq \lambda_0$ and $M = \frac{1}{2}|\lambda - \lambda_0|^{-1/2}$. Then $x_0 \in (0,b)$ may be chosen such that $\int_{x_0}^b |u_k|^2 dx \leq M^2$ for $k = 1, 2$. Now introduce the notation

$$\|f\|_x = \left(\int_{x_0}^x |f|^2 dx\right)^{1/2}$$

for $x_0 \leq x < b$. Then for all such x we have $\|u_k\|_x \leq M$ for $k = 1, 2$.

Let y be any solution of $L(y) = \lambda y$. The rest of the proof consists in showing that y is square-integrable on $[x_0, b)$. By the variation of constants formula

$$(2.8) \qquad y(s) = c_1 u_1(s) + c_2 u_2(s) + (\lambda - \lambda_0) \int_{x_0}^s (u_1(s) u_2(t) - u_1(t) u_2(s)) y(t)\, dt$$

for all $s \in [0,b)$. The integral in (2.8) may now be estimated, using Schwarz's inequality, by

$$\left|\int_{x_0}^s (u_1(s) u_2(t) - u_1(t) u_2(s)) y(t)\, dt\right| \leq M(|u_1(s)| + |u_2(s)|)\|y\|_x$$

whenever $x_0 \leq s \leq x < b$. Applying the L^2-triangle inequality to (2.8) gives, for $x_0 \leq x < b$,

$$\|y\|_x \leq (|c_1| + |c_2|)M + 2|\lambda - \lambda_0| M^2 \|y\|_x = (|c_1| + |c_2|)M + \frac{1}{2}\|y\|_x.$$

This implies that $||y||_x$ is bounded by $2(|c_1|+|c_2|)M$, which is independent of x. Hence we have that y is square-integrable on $[x_0, b)$. Thus, $y \in L^2([0, b))$. □

CHAPTER 3

Maximal and Minimal Operators and Necessary Lemmas

In this chapter we define maximal and minimal operators and obtain a preliminary characterization of the domain of the closure of the minimal operator in terms of the maximal operator. We then use this to characterize the domain of the closure of the minimal operator in terms of a fundamental system of solutions of $-y'' + qy = \lambda y$. The purpose of this characterization is to provide necessary background in our analysis of the case that L is in Class II. The material in the first part of this chapter is standard, and a similar treatment can be found in [**6**].

Let L be the differential expression defined by

$$L(y) = -y'' + qy,$$

where $q \in L^1_{loc}([0,b))$ and $0 < b \leq \infty$. L is *regular* at b if $b < \infty$ and $q \in L^1([0,b])$. Otherwise, L is called *singular* at b. If $g \in L^1_{loc}([0,b))$ and $z \in \mathbf{C}$, then all solutions of $L(y) = zy + g$ are given by

$$c_1u_1(x) + c_2u_2(x) + \int_c^x \frac{u_1(x)u_2(t) - u_2(x)u_1(t)}{W(u_1,u_2)(t)} g(t)\,dt,$$

where u_1 and u_2 are a fundamental system of solutions of $Ly = zy$, c_1 and c_2 are arbitrary complex constants, and c is some point in $[0,b)$.

The formal adjoint of L is the differential expression given by

$$L^+(y) = -y'' + \bar{q}y.$$

For brevity, define

$$[f,g](x) = W(f,g)(x) = f(x)g'(x) - f'(x)g(x).$$

An integration by parts shows that if f, g, f', g' are absolutely continuous on a compact interval $[\alpha, \beta] \subset [0, b)$, then

$$\int_\alpha^\beta L(f) \cdot g dx - \int_\alpha^\beta f \cdot L(g) dx = [f, g](\beta) - [f, g](\alpha). \tag{3.1}$$

We now wish to define the maximal and minimal operators associated with L.

Let T be the operator whose domain is

$$D(T) = \{f \in L^2([0, b)) : f, f' \in AC_{loc}([0, b)) \text{ and } L(f) \in L^2([0, b))\},$$

and define $T(f) = L(f)$ for all $f \in D(T)$. Then T is called the maximal operator associated with L. Let T_0 be the operator whose domain is

$$D(T_0) = \{f \in D(T) : f(0) = f'(0) = 0 \text{ and } \operatorname{supp}(f) \text{ is compact in } [0, b)\},$$

and define $T_0(f) = L(f)$ for all $f \in D(T_0)$. Then T_0 is called the minimal operator associated with L. From Theorem 10.7 of [**6**], we have that T_0 (hence, T) is densely defined (that is, $D(T_0)$ is a dense subspace of $L^2([0, b))$). The operators T^+ and T_0^+, the maximal and minimal operators associated with L^+, are defined similarly. Note that if $f \in D(T)$, then $\bar{f} \in D(T^+)$, and if $g \in D(T^+)$, then $\bar{g} \in D(T)$.

LEMMA 3.1. *If $f, g \in D(T)$ then $[f, g](b) = \lim_{x \to b}[f, g](x)$ exists and*

$$\int_0^b T(f) g dx - \int_0^b f T(g) dx = [f, g](b) - [f, g](0). \tag{3.2}$$

PROOF. Let $x \in (0, b)$. Since f, f', g, g' are absolutely continuous on $[0, x]$, we have, by (3.1),

$$\int_0^x (T(f)g - fT(g))dt = \int_0^x L(f) g dt - \int_0^x f L(g) dt = [f, g](x) - [f, g](0).$$

Since $T(f)g - fT(g)$ is in $L^1([0, b))$,

$$\lim_{x \to b} \int_0^x (T(f)g - fT(g))dt$$

exists. Together with the previous equation, this implies that $\lim_{x \to b}[f, g](x)$ exists and $\int_0^b (T(f)g - fT(g))dx = [f, g](b) - [f, g](0)$. $\square$

We remark here that (3.1) and (3.2) will be used frequently in what follows. We now show that $T_0^* = T^+$. The proof is carried out in several steps.

LEMMA 3.2. *$T_0^* = T^+$ and $(T_0^+)^* = T$.*

PROOF. 1. Let u_1, u_2 be a fundamental system of solutions of $L(y) = 0$. For $j = 1, 2$ define the following linear functionals on $D(T_0)$:

$$F_j(f) = \int_0^b f u_j dx.$$

Then

$$\ker(F_1) \cap \ker(F_2) = \{f \in D(T_0) : \int_0^b f u dx = 0 \text{ for all } u \text{ satisfying } L(u) = 0\}$$

since u_1, u_2 are a fundamental system of solutions of $L(y) = 0$.

2. $\operatorname{Im}(T_0) = \ker(F_1) \cap \ker(F_2)$.

Let $f \in D(T_0)$ so that $T_0(f) \in \operatorname{Im}(T_0)$. Let $\operatorname{supp}(f) \subset [0, \beta] \subset [0, b)$, and let u be a solution of $L(u) = 0$. Then

$$\int_0^\beta T_0(f) u dx = \int_0^\beta T_0(f) u dx - \int_0^\beta f L(u) dx = [f, u](\beta) - [f, u](0) = 0.$$

and since $f(0) = f'(0) = f(\beta) = f'(\beta) = 0$,

$$\int_0^\beta T_0(f) u dx = \int_0^b T_0(f) u dx = 0.$$

Since f has compact support, $T_0(f)$ does, so $T_0(f) \in D(T_0)$. Thus, $T_0(f) \in \ker(F_1) \cap \ker(F_2)$.

Let $f \in \ker(F_1) \cap \ker(F_2)$. Let $\operatorname{supp}(f) \subset [0, \beta] \subset [0, b)$. Define h to be the unique solution of $L(h) = f$ for which $h(0) = h'(0) = 0$. Then h, h' are locally absolutely continuous. Let $x \in [\beta, b)$. Now, let u be the solution of $L(y) = 0$ with $u(x) = 0$ and $u'(x) = 1$. Then

$$\int_0^x f u dt = \int_0^x L(h) u dt - \int_0^x h L(u) dt = [h, u](x) - [h, u](0) = h(x).$$

Since $\int_0^x f u dt = \int_0^b f u dt = 0$, we have $h(x) = 0$. Thus, $h = 0$ on $[\beta, b)$, so we have $h \in D(T_0)$. Thus, $f = L(h) = T_0(h) \in \operatorname{Im}(T_0)$.

3. $T_0^* \subset T^+$.

Let $f \in D(T_0^*)$. Note that $T_0^*(f) \in L^2([0,b))$, so $T_0^*(f) \in L^1_{loc}([0,b))$. Let h be a solution of $L(h) = \overline{T_0^*(f)}$. Then h, h' are locally absolutely continuous. Let $k \in D(T_0)$ and let $\operatorname{supp}(k) \subset [0,\beta] \subset [0,b)$. Using that $k(\beta) = k'(\beta) = 0$, we calculate

$$\begin{aligned}
\int_0^b T_0(k)(\bar{f} - h)dx &= (T_0(k), f) - \int_0^\beta L(k)h dx \\
&= (k, T_0^*(f)) - \int_0^\beta k L(h) dx \\
&= (k, T_0^*(f)) - (k, T_0^*(f)) \\
&= 0.
\end{aligned}$$

Let $F : D(T_0) \to \mathbf{C}$ be the linear functional defined by

$$F(g) = \int_0^b g(\bar{f} - h)dx.$$

Then we have just shown that $\operatorname{Im}(T_0) \subset \ker(F)$. Using Theorem 4.1 from [14], there are complex constants c_1 and c_2 such that $F = c_1 F_1 + c_2 F_2$.

Let $[0,\beta] \subset [0,b)$ be a compact interval, and let $x \in [0,\beta]$. Then $\chi_{[0,x]}$, the characteristic function of $[0,x]$, is in $L_0^2([0,b))$, so

$$\begin{aligned}
\int_0^x (\bar{f} - h)dt &= \int_0^b \chi_{[0,x]}(\bar{f} - h)dt \\
&= F(\chi_{[0,x]}) \\
&= c_1 \int_0^b \chi_{[0,x]} u_1 dt + c_2 \int_0^b \chi_{[0,x]} u_2 dt \\
&= c_1 \int_0^x u_1 dt + c_2 \int_0^x u_2 dt.
\end{aligned}$$

Thus,

$$\int_0^x (\bar{f} - h - c_1 u_1 - c_2 u_2)dt = 0.$$

so the mapping

$$x \mapsto \int_0^x (\bar{f} - h - c_1 u_1 - c_2 u_2) dt$$

is identically zero on $[0, \beta]$, and its derivative equals $\bar{f} - h - c_1 u_1 - c_2 u_2$ almost everywhere. Thus, $\bar{f} = h - c_1 u_1 - c_2 u_2$ almost everywhere on $[0, \beta]$. Since β was arbitrary, we have $\bar{f} = h - c_1 u_1 - c_2 u_2$ almost everywhere on $[0, b)$. Note that $\bar{f}, \bar{f}' \in AC_{loc}([0, b))$ since $h, u_1, u_2 \in AC_{loc}([0, b))$. Since $f \in L^2([0, b))$ and $L(\bar{f}) = \overline{T_0^*(f)} \in L^2([0, b))$, we have $\bar{f} \in D(T)$, so $f \in D(T^-)$. Finally, note that $T_0^*(f) = \overline{L(\bar{f})} = L^-(f) = T^-(f)$.

4. $T^- \subset T_0^*$.

Let $f \in D(T^-)$ and $g \in D(T_0)$. Since $\bar{f} \in D(T)$, we have

$$\begin{aligned} (T_0(g), f) - (g, T^-(f)) &= \int_0^b T_0(g) \bar{f} dx - \int_0^b g \overline{T^-(f)} dx \\ &= \int_0^b T(g) \bar{f} dx - \int_0^b g T(\bar{f}) dx \\ &= [g, \bar{f}](b) - [g, \bar{f}](0) \\ &= 0 \end{aligned}$$

using (3.2) and the fact that $g \in D(T_0)$. Thus, $f \in D(T_0^*)$ and $T^-(f) = T_0^*(f)$.

By replacing g with $\bar{g}$, our argument just given shows that $(T_0^-)^* = T$. □

Thus, T and T^- are closed operators. Also, T_0 is closable since T_0^* is densely defined in $L^2([0, b))$. Note that $(T^-)^* = \overline{T_0} \subset T$.

We now state a few lemmas and a theorem, the last two of which will be of principal use in defining the Titchmarsh-Weyl m-function for Class II. Lemma 3.3 and its proof come from [**6**]. We include it for completeness.

LEMMA 3.3. *If L is regular on a compact interval $[A, B]$ and $a_1, a_2, b_1, b_2 \in \mathbb{C}$, then there is a $\phi \in D(T)$ such that*

$$\phi(A) = a_1, \phi'(A) = a_2, \phi(B) = b_1, \phi'(B) = b_2.$$

PROOF. Let v_1, v_2 be the fundamental system of solutions of $L^+(y) = 0$ with

$$v_1(B) = 1, v_1'(B) = 0, v_2(B) = 0, v_2'(B) = 1.$$

We claim that the determinant

$$\begin{vmatrix} (v_1, v_1) & (v_1, v_2) \\ (v_2, v_1) & (v_2, v_2) \end{vmatrix} \tag{3.3}$$

is nonzero. Suppose that this is not true. Then there is a pair $(c_1, c_2) \in \mathbf{C}^2$ such that $(c_1, c_2) \neq (0, 0)$ and

$$\begin{aligned} c_1(v_1, v_1) + c_2(v_1, v_2) &= 0 \\ c_1(v_2, v_1) + c_2(v_2, v_2) &= 0. \end{aligned}$$

Thus, $\overline{c_1}v_1 + \overline{c_2}v_2$ is orthogonal to both v_1 and v_2, which implies that the solution space of $L^+(y) = 0$ is at least three-dimensional (a contradiction).

Thus, there is a pair $(\mu_1, \mu_2) \in \mathbf{C}^2$ such that

$$\begin{aligned} \mu_1(v_1, v_1) + \mu_2(v_1, v_2) &= -\overline{b_2} \\ \mu_1(v_2, v_1) + \mu_2(v_2, v_2) &= \overline{b_1}. \end{aligned}$$

Letting $f = \overline{\mu_1}v_1 + \overline{\mu_2}v_2$, we have $(f, v_1) = -b_2$ and $(f, v_2) = b_1$. Let u be the solution of $L(u) = f$ with $u(A) = u'(A) = 0$. Then $u \in D(T)$ and, using (3.1), we calculate for $j = 1, 2$

$$\begin{aligned} (f, v_j) &= \int_A^B f\overline{v_j}dx \\ &= \int_A^B L(u)\overline{v_j}dx \\ &= \int_A^B uL(\overline{v_j})dx - [u, \overline{v_j}](B) - [u, \overline{v_j}](A) \\ &= [u, \overline{v_j}](B). \end{aligned}$$

Thus, $(f, v_1) = [u, \overline{v_1}](B) = -u'(B)$ and $(f, v_2) = [u, \overline{v_2}](B) = u(B)$, so $u'(B) = b_2$ and $u(B) = b_1$. We apply a similar procedure (starting with the choice of (μ_1, μ_2)) to obtain a $v \in D(T)$ such that

$$v(A) = a_1, v'(A) = a_2, v(B) = v'(B) = 0.$$

The function $\phi = u - v$ satisfies the conclusion of the lemma. $\square$

LEMMA 3.4. $D(\overline{T_0}) = \{f \in D(T) : \forall g \in D(T) : f(0) = f'(0) = [f, g](b) = 0\}$.

PROOF. Let $f \in D(T)$ with $f(0) = f'(0) = [f, h](b) = 0$ for all $h \in D(T)$. Let $g \in D(T^+)$. Then $\bar{g} \in D(T)$ and

$$\begin{aligned}
(f, T^+(g)) - (T(f), g) &= \int_0^b \overline{T^+(g)} f dx - \int_0^b \bar{g} T(f) dx \\
&= \int_0^b T(\bar{g}) f dx - \int_0^b \bar{g} T(f) dx \\
&= [\bar{g}, f](b) - [\bar{g}, f](0) \\
&= -[f, \bar{g}](b) \\
&= 0.
\end{aligned}$$

Thus, $(T^+(g), f) = (g, T(f))$ for all $g \in D(T^+)$, so $f \in D((T^+)^*) = D(\overline{T_0})$ and $T(f) = (T^+)^*(f) = \overline{T_0}(f)$.

Let $f \in D(\overline{T_0}) = D((T^+)^*)$. Since $(T^+)^* \subset T$, we have $f \in D(T)$ and $T(f) = (T^+)^*(f)$. Let $g \in D(T)$. We now show that $[f, g](b) = 0$. Note that $\bar{g} \in D(T^+)$. Let $X \in (0, b)$ and T_X^+ denote the maximal operator for $-y'' + \bar{q}y$ over $[0, X]$. Applying Lemma 3.3, there is a $g_1 \in D(T_X^+)$ such that

$$g_1(0) = -\overline{g(0)}, g_1'(0) = -\overline{g'(0)}, g_1(X) = g_1'(X) = 0.$$

Let $\tilde{g_1} = g_1$ on $[0, X]$ and $\tilde{g_1} = 0$ on (X, b). Then $\tilde{g_1} \in D(T^*)$. Finally, let $\bar{g} = \tilde{g_1} + \tilde{g}$. Then $\bar{g} \in D(T^*)$ and

$$\begin{aligned}
[f, g](b) &= [f, \bar{\bar{g}}](b) - [f, \bar{\bar{g}}](0) \\
&= \int_0^b T(f)\bar{\bar{g}}dx - \int_0^b fT(\bar{\bar{g}})dx \\
&= \int_0^b T(f)\bar{\bar{g}}dx - \int_0^b f\overline{T^*(\bar{g})}dx \\
&= (T(f), \bar{g}) - (f, T^*(\bar{g})) \\
&= ((T^*)^*(f), \bar{g}) - (f, T^*(\bar{g})) \\
&= 0.
\end{aligned}$$

In the next paragraph, we show that $[f, g](0) = 0$.

By Lemma 3.3, there is a $g_1 \in D(T_X^*)$ such that

$$g_1(0) = \overline{g(0)}, g_1'(0) = \overline{g'(0)}, g_1(X) = g_1'(X) = 0.$$

Let $\bar{g} = g_1$ on $[0, X]$ and $\bar{g} = 0$ on (X, b). Then $\bar{g} \in D(T^*)$ and

$$\begin{aligned}
-[f, g](0) &= [f, \bar{\bar{g}}](b) - [f, \bar{\bar{g}}](0) \\
&= \int_0^b T(f)\bar{\bar{g}}dx - \int_0^b fT(\bar{\bar{g}})dx \\
&= (T(f), \bar{g}) - (f, T^*(\bar{g})) \\
&= ((T^*)^*(f), \bar{g}) - (f, T^*(\bar{g})) \\
&= 0.
\end{aligned}$$

Thus, we have that $[f, g](0) = 0$ for all $g \in D(T)$. Using Lemma 3.3 in the same way as in the previous paragraph, there is an $h \in D(T)$ such that $h(0) = 0$ and $h'(0) = 1$. Thus, $0 = [f, h](0) = f(0)$. Similarly, there is a $k \in D(T)$ such that $k(0) = 1$ and $k'(0) = 0$. Then $0 = [f, k](0) = -f'(0)$. Thus, $f(0) = f'(0) = 0$. We now conclude that f is in the set defined by the right-hand side of the statement of the lemma. □

For the rest of this chapter, we assume that $\phi(\cdot, \lambda)$ and $\theta(\cdot, \lambda)$ are a fundamental system of solutions of $L(y) = \lambda y$ with $[\phi, \theta](0) = 1$. We further assume that L is in Class II at b (i.e., all solutions of $L(y) = \lambda y$ are in $L^2([0, b))$ for all $\lambda \in \mathbf{C}$). For the purpose of the lemmas, λ does not need to be specified, so we shall abbreviate ϕ for $\phi(\cdot, \lambda)$ and θ for $\theta(\cdot, \lambda)$. Note that $\phi, \theta \in D(T)$. Lemmas 3.5 through 3.8 below are based on a series of lemmas given in [**16**].

LEMMA 3.5. *If $f, g \in D(T)$ then $[f, g](x) = [f, \phi](x)[g, \theta](x) - [f, \theta](x)[g, \phi](x)$ for all $x \in [0, b]$.*

PROOF. For $x < b$, this is a mere calculation. Now, Lemma 3.1 and the fact that the equation

$$[f, g](x) = [f, \phi](x)[g, \theta](x) - [f, \theta](x)[g, \phi](x)$$

holds for $x < b$ imply that $[f, g](b) = [f, \phi](b)[g, \theta](b) - [f, \theta](b)[g, \phi](b)$. $\square$

LEMMA 3.6. *If $g \in D(T)$, then there are numbers $\alpha, \beta \in \mathbf{C}$ such that $[f, g](b) = [f, \alpha\phi - \beta\theta](b)$ for all $f \in D(T)$.*

PROOF. Fix $g \in D(T)$. Let $\alpha = [g, \theta](b)$ and $\beta = -[g, \phi](b)$. If $f \in D(T)$ we have

$$\begin{aligned} [f, g](b) &= [f, \phi](b)[g, \theta](b) - [f, \theta](b)[g, \phi](b) \\ &= [f, \alpha\phi](b) - [f, \beta\theta](b) \\ &= [f, \alpha\phi - \beta\theta](b). \end{aligned}$$

$\square$

LEMMA 3.7. $D(\overline{T_0}) = \{f \in D(T) : f(0) = f'(0) = [f, \phi](b) = [f, \theta](b) = 0\}$.

PROOF. $D(\overline{T_0})$ is a subset of the right-hand set by taking $g = \theta, \phi$. Now, let f lie in the right-hand set. Let $g \in D(T)$. By Lemma 3.6, there are $\alpha, \beta \in \mathbf{C}$ such that $[f, g](b) = [f, \alpha\phi - \beta\theta](b) = \alpha[f, \phi](b) - \beta[f, \theta](b) = 0$. Thus, $[f, g](b) = 0$, so $f \in D(\overline{T_0})$. $\square$

LEMMA 3.8. *Given $a_1, a_2, b_1, b_2 \in \mathbf{C}$, there is an element $f \in D(T)$ such that*

$$f(0) = a_1 \qquad [f,\phi](b) = b_1$$

$$f'(0) = a_2 \qquad [f,\theta](b) = b_2.$$

PROOF. Fix $a_1, a_2, b_1, b_2 \in \mathbf{C}$. Let $f_1 = b_2\phi - b_1\theta$. Then $f_1 \in D(T)$ and $[f_1,\theta](b) = b_2$ and $[f_1,\phi](b) = b_1$. Let $X \in (0,b)$. Let T_X denote the maximal operator for q on $[0,X]$. By Lemma 3.3, there is an element $f_2 \in D(T_X)$ such that

$$f_2(0) = a_1 \qquad f_2(X) = b_2\phi(X) - b_1\theta(X)$$

$$f_2'(0) = a_2 \qquad f_2'(X) = b_2\phi'(X) - b_1\theta'(X).$$

Let $f = f_2$ on $[0,X]$ and $f = f_1$ on (X,b). Then $f \in D(T)$, $f(0) = f_2(0) = a_1$, $f'(0) = f_2'(0) = a_2$, $[f,\theta](b) = [f_1,\theta](b) = b_2$, and $[f,\phi](b) = [f_1,\phi](b) = b_1$. Thus, f is a function satisfying the conditions of the lemma. $\square$

Finally, we have the following.

THEOREM 3.9. *Let $u \in D(T)$. If for some $\lambda_0 \in \mathbf{C}$ it is true that $[\theta(\cdot,\lambda_0),u](b) = [\phi(\cdot,\lambda_0),u](b) = 0$, then for all $\lambda \in \mathbf{C}$ it is true that $[\theta(\cdot,\lambda),u](b) = [\phi(\cdot,\lambda),u](b) = 0$.*

PROOF. Fix $\lambda \in \mathbf{C}$. Then by Lemma 3.5 we have

$$[\theta(\cdot,\lambda),u](b) = [\theta(\cdot,\lambda),\phi(\cdot,\lambda_0)](b)[u,\theta(\cdot,\lambda_0)](b) - [\theta(\cdot,\lambda),\theta(\cdot,\lambda_0)](b)[u,\phi(\cdot,\lambda_0)](b) = 0$$

using the hypothesis. Similarly, we get $[\phi(\cdot,\lambda),u](b) = 0$. $\square$

CHAPTER 4

The Titchmarsh-Weyl m-Function

The classical (q real-valued) use of the nesting circle analysis is to demonstrate that the differential expression $L(y) = -y'' - qy$ is in the limit-point (at most one linearly independent $L^2([0,b))$ solution of (2.2) for all λ) or limit-circle (all solutions of (2.2) are in $L^2([0,b))$ for all λ) case at the endpoint b. Recall from Chapter 2 that L is in Class I if there is at most one linearly independent $L^2([0,b))$ solution of (2.2) for all λ and that L is in Class II if all solutions of (2.2) are in $L^2([0,b))$ for all λ. In Chapter 2, it is shown that if $Q(H)$ is a proper subset of **C**, then for any $\lambda \in \mathbf{C} - Q(H)$ we have at least one solution of (2.2) lying in $L^2([0,b))$. For both Class I and Class II, we may define an *m-function*.

To define the m-function for Class I and Class II, we will make use of the following notation, which was introduced in Chapter 2. For fixed $h_1, h_2, H_1, H_2 \in \mathbf{C}$ and for $\lambda \in \mathbf{C}$, let $\phi(\cdot,\lambda)$ and $\theta(\cdot,\lambda)$ be the solutions of

$$-y'' - qy = \lambda y \tag{4.1}$$

satisfying

$$\phi(0,\lambda) = h_1 \qquad \theta(0,\lambda) = h_2$$

$$\phi'(0,\lambda) = H_1 \qquad \theta'(0,\lambda) = H_2.$$

We also require that

$$h_1H_2 - h_2H_1 = 1 \text{ and } \overline{h_1}h_2 - \overline{H_1}H_2 = 0.$$

When confusion is unlikely, one or both arguments of ϕ and θ will be suppressed.

We are now ready to define the m-function for Class I. From the nesting circle analysis, we associate to each $\lambda \in \mathbf{C} - Q(H)$ the intersection $\bigcap_{0<X<b} D_X(\lambda)$ of nesting

disks. Since we are in Class I, $\bigcap_{0<X<b} D_X(\lambda)$ is a single point for all $\lambda \in \mathbf{C} - Q(H)$ (otherwise, all solutions are in $L^2([0,b))$ for all $\lambda \in \mathbf{C}$). Fix $\lambda_0 \in \mathbf{C} - Q(H)$. Since there is only one solution of $L(y) = \lambda_0 y$ in $L^2([0,b))$, there is exactly one number $l \in \mathbf{C}_\infty$ such that $v_l = v_l(\cdot, \lambda_0)$ is in $L^2([0,b))$. From the nesting circle analysis, (2.7), the number $\tilde{l} \in \bigcap_{0<X<b} D_X(\lambda_0)$ has the property that $v_{\tilde{l}}$ is in $L^2([0,b))$. Thus, $l = \tilde{l}$.

DEFINITION 4.1. *Fix a quadruple h_1, H_1, h_2, H_2 as above. The Class I m-function is the function*

$$m : \mathbf{C} - Q(H) \longrightarrow \mathbf{C}$$

whose value at λ_0 is the number $m(\lambda_0) \in \mathbf{C}$ with the property that $v_{m,\lambda_0} \in L^2([0,b))$.

Note that the domain of the Class I m-function, denoted by $\mathrm{dom}(m)$, is $\mathbf{C} - Q(H)$. By the remarks preceding Definition 4.1, we have that for $\lambda_0 \in \mathbf{C} - Q(H)$,

$$m(\lambda_0) \in \bigcap_{0<X<b} D_X(\lambda_0).$$

We now define the Class II m-function. Note that for Class II, $\theta(\cdot, \lambda)$ and $\phi(\cdot, \lambda)$ are in $L^2([0,b))$ (hence, in $D(T)$) for all $\lambda \in \mathbf{C}$. We claim that there is an element $u \in D(T)$ such that if $\lambda \in \mathbf{C}$ then $[\theta(\cdot, \lambda), u](b) \neq 0$ or $[\phi(\cdot, \lambda), u](b) \neq 0$. To see this, let $\lambda_0 \in \mathbf{C}$. By Lemma 3.8, there is an element $u \in D(T)$ such that

$$[\theta(\cdot, \lambda_0), u](b) \neq 0 \text{ or } [\phi(\cdot, \lambda_0), u](b) \neq 0.$$

The contrapositive of Theorem 3.9 implies now that for any $\lambda \in \mathbf{C}$ we have

$$[\theta(\cdot, \lambda), u](b) \neq 0 \text{ or } [\phi(\cdot, \lambda), u](b) \neq 0.$$

We call such an element u a boundary condition function for b. Recall that $v_\infty(\cdot, \lambda) = \phi(\cdot, \lambda)$.

THEOREM 4.2. *Let u be a boundary condition function for b. For each $\lambda \in \mathbf{C}$, there is exactly one number $l(\lambda) \in \mathbf{C}_\infty$ such that $v_{l(\lambda)}(\cdot, \lambda) = \theta(\cdot, \lambda) - l(\lambda)\phi(\cdot, \lambda)$*

satisfies the boundary condition $[v_{l(\lambda)}, u](b) = 0$. *Furthermore,*

$$l(\lambda) = -\frac{[\theta(\cdot,\lambda), u](b)}{[\phi(\cdot,\lambda), u](b)}$$

for all λ.

PROOF. Note that if $\lambda \in \mathbf{C}$, then, by the remark in the previous paragraph, the number

$$z = -\frac{[\theta(\cdot,\lambda), u](b)}{[\phi(\cdot,\lambda), u](b)}$$

is an element of $\mathbf{C}_\infty$. Using Lemma 3.1, this equation gives that $[v_z(\cdot,\lambda), u](b) = 0$. This establishes existence. To establish uniqueness, assume that there are numbers $l_1, l_2 \in \mathbf{C}_\infty$ such that $l_1 \neq l_2$ and

$$[v_{l_1}, u](b) = [v_{l_2}, u](b) = 0.$$

Then v_{l_1} and v_{l_2} are linearly independent solutions of (4.1). Thus, $[\theta(\cdot,\lambda), u](b) = [\phi(\cdot,\lambda), u](b) = 0$, which gives a contradiction. $\square$

Theorem 4.2 states that for each $\lambda \in \mathbf{C}$, there is only one solution, up to constant multiples, of (4.1) which satisfies the boundary condition $[f, u](b) = 0$ (for a given u).

DEFINITION 4.3. *Fix a quadruple* h_1, H_1, h_2, H_2 *and a boundary condition function* u. *The Class II m-function is the function*

$$m : \mathbf{C} \longrightarrow \mathbf{C}_\infty$$

whose value at λ *is the number, say* $m(\lambda)$, *with the property that* $v_{m(\lambda)}(\cdot,\lambda)$ *satisfies the boundary condition* $[v_{m(\lambda)}, u](b) = 0$.

We remark here that, by Theorem 4.2, the Class II m-function has the value

$$m(\lambda) = -\frac{[\theta(\cdot,\lambda), u](b)}{[\phi(\cdot,\lambda), u](b)}$$

at each number λ. Also, note that the domain of the Class II m-function, denoted by dom(m), is $\mathbf{C}$.

For a regular problem on $[0, b]$ (which is necessarily a Class II problem), specifying a boundary condition function u at b is equivalent to specifying two complex numbers α and β, not both of which are zero. Specifically, suppose that u is a boundary condition function for b. Then $\alpha = -u(b)$ and $\beta = u'(b)$ are two complex numbers, not both of which are zero, with the property that for all $\lambda \in \mathbf{C}$

$$\beta\theta(b,\lambda) + \alpha\theta'(b,\lambda) \neq 0 \text{ or } \beta\phi(b,\lambda) + \alpha\phi'(b,\lambda) \neq 0.$$

Conversely, suppose that α and β are two complex numbers, not both of which are zero. Assume by way of contradiction that there is a $\lambda_0 \in \mathbf{C}$ such that $\beta\theta(b,\lambda_0) + \alpha\theta'(b,\lambda_0) = 0$ and $\beta\phi(b,\lambda_0) + \alpha\phi'(b,\lambda_0) = 0$. Then, if $\beta \neq 0$, $1 = \phi(b,\lambda_0)\theta'(b,\lambda_0) - \phi'(b,\lambda_0)\theta(b,\lambda_0) = -(\alpha/\beta)\phi'(b,\lambda_0)\theta'(b,\lambda_0) - (\alpha/\beta)\phi'(b,\lambda_0)\theta'(b,\lambda_0) = 0$, which gives a contradiction. Clearly, if $\alpha \neq 0$ then we reach a similar contradiction. Thus, it is the case that for all $\lambda \in \mathbf{C}$

$$\beta\theta(b,\lambda) + \alpha\theta'(b,\lambda) \neq 0 \text{ or } \beta\phi(b,\lambda) + \alpha\phi'(b,\lambda) \neq 0.$$

By Lemma 3.3 there is an element $u \in D(T)$ such that $u(b) = -\alpha$ and $u'(b) = \beta$. Thus, for all $\lambda \in \mathbf{C}$,

$$[\theta(\cdot,\lambda), u](b) \neq 0 \text{ or } [\phi(\cdot,\lambda), u](b) \neq 0.$$

The previous paragraph shows that for a regular problem on $[0, b]$, we may choose to define the Class II m-function in terms of a pair α, β of complex numbers, not both of which are zero:

$$m(\lambda) = -\frac{\beta\theta(b,\lambda) + \alpha\theta'(b,\lambda)}{\beta\phi(b,\lambda) + \alpha\phi'(b,\lambda)}.$$

We now discuss the dependence of the Class I and Class II m-functions on the quadruple h_1, h_2, H_1, H_2. Consider two quadruples h_1, h_2, H_1, H_2 and $\tilde{h}_1, \tilde{h}_2, \tilde{H}_1, \tilde{H}_2$. Let θ and ϕ be the solutions of (4.1) satisfying the initial conditions as given with (4.1), and let $\tilde{\theta}$ and $\tilde{\phi}$ be the solutions of (4.1) satisfying the initial conditions with $\tilde{h}_1$ replacing h_1, etc. Finally, let m denote the m-function for the quadruple h_1, etc., and let $\tilde{m}$ denote the m-function for the quadruple $\tilde{h}_1$, etc. In the present analysis, we assume only that

$$h_1H_2 - h_2H_1 \neq 0 \text{ and } \tilde{h}_1\tilde{H}_2 - \tilde{h}_2\tilde{H}_1 \neq 0,$$

which merely is a statement that θ, ϕ and $\tilde{\theta}, \tilde{\phi}$ are two fundamental systems of solutions of (4.1). Thus, we have that there are constants c_1, c_2, d_1, d_2 such that

$$\tilde{\theta} = c_1\theta + c_2\phi \text{ and } \tilde{\phi} = d_1\theta + d_2\phi.$$

From these equations it follows that

$$\begin{aligned}
\tilde{h}_2 &= c_1h_2 - c_2h_1\\
\tilde{H}_2 &= c_1H_2 + c_2H_1\\
\tilde{h}_1 &= d_1h_2 - d_2h_1\\
\tilde{H}_1 &= d_1H_2 - d_2H_1.
\end{aligned}$$

Solving the above system for c_1, c_2, d_1, d_2 gives

$$\begin{aligned}
c_1 &= (h_1\tilde{H}_2 - \tilde{h}_2H_1)/(h_1H_2 - h_2H_1)\\
c_2 &= (\tilde{h}_2H_2 - h_2\tilde{H}_2)/(h_1H_2 - h_2H_1)\\
d_1 &= (h_1\tilde{H}_1 - \tilde{h}_1H_1)/(h_1H_2 - h_2H_1)\\
d_2 &= (\tilde{h}_1H_2 - h_2\tilde{H}_1)/(h_1H_2 - h_2H_1).
\end{aligned}$$

Note that

$$\tilde{\theta} - \tilde{m}\tilde{\phi} = (c_1 - \tilde{m}d_1)\theta + (c_2 - \tilde{m}d_2)\phi = (c_1 - \tilde{m}d_1)\left[\theta + \frac{c_2 - \tilde{m}d_2}{c_1 - \tilde{m}d_1}\phi\right].$$

so by uniqueness of m we have

$$m = \frac{c_2 - \tilde{m}d_2}{c_1 - \tilde{m}d_1} = \frac{\tilde{h}_2H_2 - h_2\tilde{H}_2 + (\tilde{h}_1H_2 - h_2\tilde{H}_1)\tilde{m}}{h_1\tilde{H}_2 - \tilde{h}_2H_1 + (h_1\tilde{H}_1 - \tilde{h}_1H_1)\tilde{m}}.$$

Thus, m and $\tilde{m}$ are related by a Möbius transformation since its "determinant" is

$$(h_1H_2 - h_2H_1)(\tilde{h}_1\tilde{H}_2 - \tilde{h}_2\tilde{H}_1) \neq 0.$$

We now use the result of the previous paragraph to justify the two conditions

$$h_1H_2 - h_2H_1 = 1 \text{ and } \overline{h_1}h_2 + \overline{H_1}H_2 = 0$$

that we wish to place on the quadruple h_1, h_2, H_1, H_2. Fix $h_1, h_2, H_1, H_2 \in \mathbf{C}$ such that $h_1H_2 - h_2H_1 \neq 0$. Since $|h_1|^2 + |H_1|^2 > 0$, there is a pair $\tilde{h}_2, \tilde{H}_2 \in \mathbf{C}$ such that

$$\begin{aligned} h_1\tilde{H}_2 - \tilde{h}_2H_1 &= 1 \\ \overline{h_1}\tilde{h}_2 - \overline{H_1}\tilde{H}_2 &= 0. \end{aligned}$$

Letting $\tilde{m}$ denote the m-function for the quadruple $h_1, H_1, \tilde{h}_2, \tilde{H}_2$ and m denote the m-function for the quadruple h_1, H_1, h_2, H_2, we have from the previous paragraph that $\tilde{m}$ and m are related by a Möbius transformation. That is, there is a Möbius transformation T such that

$$m = T \circ \tilde{m}.$$

Thus, no generality is lost by placing the conditions

$$h_1H_2 - h_2H_1 = 1 \text{ and } h_1\overline{h_2} - H_1\overline{H_2} = 0$$

on the quadruple h_1, h_2, H_1, H_2 chosen as the initial values for θ and ϕ.

Define the Class I and Class II Dirichlet m-function, denoted by m_D, to be the m-function in the case that $h_1 = 0$. We shall prove that m_D has a certain asymptotic behavior. Our proof requires two lemmas. The first allows asymptotic estimates of the solutions ϕ and θ, and the second gives an asymptotic estimate of $d_a(\lambda)$, the diameter of the Weyl disks, when $a \in (0, b)$ is fixed. The square roots that appear are from the principal branch: that is, $\lambda \in \mathbf{C} \Rightarrow \Re\sqrt{\lambda} \geq 0$. The proof of Lemma 4.4 is based on that given in [**1**].

LEMMA 4.4. *Suppose that $u(\cdot, \lambda)$ solves (4.1) and that $u(0, \lambda), u'(0, \lambda)$ are independent of λ. Denote $k = \sqrt{-\lambda}$ ($\Re(k) \geq 0$). Let*

$$u_0(x, \lambda) = u(0)\cosh(kx) + u'(0)\frac{\sinh(kx)}{k}$$

and $c(\lambda) = |u(0)| + |u'(0)|/|k|$. *Then*

$$(4.2) \qquad |u(x) - u_0(x)| \leq c(\lambda) \cdot |e^{kx}| \cdot (e^{\int_0^x \frac{|q|}{|k|} dt} - 1)$$

and

$$(4.3) \qquad |u'(x) - u_0'(x)| \leq c(\lambda) \cdot |k| \cdot |e^{kx}| \cdot (e^{\int_0^x \frac{|q|}{|k|} dt} - 1).$$

These estimates hold for all $x \in [0, b)$ *and all* $\lambda \in \mathbf{C}$.

PROOF. The variation of constants formula gives

$$u(x) = u_0(x) + \int_0^x \frac{\sinh[k(x-t)]}{k} q(t)u(t)dt.$$

Let

$$g(x) = |e^{-kx}| \cdot \left| \int_0^x \frac{\sinh[k(x-t)]}{k} q(t)u(t)dt \right| = |e^{-kx}| \cdot |u(x) - u_0(x)|.$$

Estimating g yields

$$\begin{aligned} g(x) \leq \; & |e^{-kx}| \cdot \int_0^x \frac{|\sinh[k(x-t)]|}{|k|} |q(t)| g(t) |e^{kt}| dt + \\ & |e^{-kx}| \cdot \int_0^x \frac{|\sinh[k(x-t)]|}{|k|} |q(t)| \left| u(0) \cosh(kt) + \frac{u'(0)}{k} \sinh(kt) \right| dt. \end{aligned}$$

Noting that all three inequalities

$$\begin{aligned} |e^{-k(x-t)} \sinh[k(x-t)]| &\leq 1 \\ |e^{-kx} \sinh[k(x-t)] \cosh(kt)| &\leq 1 \\ |e^{-kx} \sinh[k(x-t)] \sinh(kt)| &\leq 1 \end{aligned}$$

hold for $t \leq x$ and using that

$$|\cosh(kt)| \leq |\sinh(kt)| \text{ or } |\sinh(kt)| \leq |\cosh(kt)|.$$

we may write

$$g(x) \leq \frac{1}{|k|} \int_0^x |q| g dt + \frac{c(\lambda)}{|k|} \int_0^x |q| dt.$$

We now multiply both sides of this inequality by $|q(x)|e^{-\int_0^x |\frac{q}{k}|dt}$ and bring the second term of the right-hand side to the left-hand side. The left-hand side then becomes

$$\left[e^{-\int_0^x |\frac{q}{k}| dt} \cdot \int_0^x |q| g dt\right]'$$

and the right-hand side is

$$c(\lambda)|k| \left[e^{-\int_0^x |\frac{q}{k}| dt} \left(-\frac{1}{|k|} \int_0^x |q| dt - 1\right)\right]'.$$

Integrating both sides from 0 to x gives

$$e^{-\int_0^x |\frac{q}{k}| dt} \cdot \int_0^x |q| g dt \leq c(\lambda)|k| e^{-\int_0^x |\frac{q}{k}| dt} \cdot \left(-\frac{1}{|k|} \int_0^x |q| dt - 1\right) + c(\lambda)|k| .$$

Multiplying both sides by $(1/|k|)e^{\int_0^x |\frac{q}{k}| dt}$ and rearranging terms gives

$$\frac{1}{|k|} \int_0^x |q| g dt + \frac{c(\lambda)}{|k|} \int_0^x |q| dt \leq c(\lambda) \cdot \left(e^{\int_0^x |\frac{q}{k}| dt} - 1\right).$$

Since the left-hand side of this inequality dominates $g(x)$, we obtain

$$g(x) \leq c(\lambda) \cdot \left(e^{\int_0^x |\frac{q}{k}| dt} - 1\right).$$

Using the definition of g, we now have (4.2).

Differentiating the formula for u we get

$$u'(x) = u_0'(x) + \int_0^x \cosh[k(x-t)] q(t) u(t) dt.$$

We now estimate

$$\begin{aligned} |u'(x) - u_0'(x)| &\leq \int_0^x |\cosh[k(x-t)]| |q(t)| |u(t)| dt \\ &\leq e^{|kx|} \int_0^x |\cosh[k(x-t)]| |q(t)| c(\lambda) (e^{\int_0^t |\frac{q}{k}| ds} - 1) e^{kt} |e^{-kx}| dt \\ &\quad + |e^{kx}| \int_0^x |\cosh[k(x-t)]| |q(t)| |e^{-kx}| |u(0)| \cosh(kt) dt \\ &\quad + e^{kx}| \int_0^x |\cosh[k(x-t)]| |q(t)| |e^{-kx} \frac{|u'(0)|}{|k|} |\sinh(kt)| dt. \end{aligned}$$

Noting that all three inequalities

$$
\begin{aligned}
|\cosh[k(x-t)]e^{-k(x-t)}| &\leq 1 \\
|\cosh[k(x-t)]\cosh(kt)e^{-kx}| &\leq 1 \\
|\cosh[k(x-t)]\sinh(kt)e^{-kx}| &\leq 1
\end{aligned}
$$

hold for $t \leq x$ and using that

$$
|\cosh(kt)| \leq |\sinh(kt)| \text{ or } |\sinh(kt)| \leq |\cosh(kt)|,
$$

we may write

$$
\begin{aligned}
|u'(x) - u_0'(x)| &\leq |e^{kx}| \int_0^x |c(\lambda)||q(t)||e^{\int_0^t \frac{q}{k}\,ds} - 1|dt + |e^{kx}| \int_0^x |c(\lambda)||q(t)|dt \\
&= |e^{kx}| \int_0^x |c(\lambda)||q(t)| e^{\int_0^t |\frac{q}{k}|\,ds} dt \\
&= |c(\lambda)| \cdot |k| \cdot |e^{kx}| \cdot \left(e^{\int_0^x |\frac{q}{k}|\,dt} - 1 \right).
\end{aligned}
$$

so we now have (4.3). ∎

In the following, we continue to use the notation $k = \sqrt{-\lambda}$, where $\Re(k) \geq 0$, as well as the notation in Chapter 2. Henceforth, unless otherwise indicated, all rays referred to emanate from the origin in $\mathbf{C}$. The next lemma gives an asymptotic estimate of the diameters of the Weyl disks $D_X(\lambda)$ on certain rays in $\mathbf{C}$.

LEMMA 4.5. *Let $a \in (0, b)$ be fixed. If $\mathcal{R}$ is a non-real ray which eventually stays in some admissible $\Lambda_{\eta,K}$ plane, then*

$$
d_a(\lambda) = \frac{1}{|\Im(k)|\,|\varphi(0) + \frac{\varphi'(0)}{k}|^2} O(e^{-2a\Re(k)})
$$

as $\lambda \to \infty$ on $\mathcal{R}$. It follows from this that for any $\epsilon > 0$ and for any non-real ray $\mathcal{R}$ which eventually stays in some admissible $\Lambda_{\eta,K}$ plane we have

$$
d_a(\lambda) = O(e^{-2(a-\epsilon)\Re(k)})
$$

as $\lambda \to \infty$ along $\mathcal{R}$.

PROOF. Fix a non-real ray $\mathcal{R}$ which eventually stays in some admissible $\Lambda_{\eta,K}$ plane. Then the set $\mathcal{S} = \{k = \sqrt{-\lambda} : \lambda \in \mathcal{R}\}$ is a ray which lies in the right half-plane and which coincides with neither the positive real axis nor the positive imaginary axis; thus, $\Re(k)/\Im(k)$, where $k \in \mathcal{S}$, is a nonzero real constant, say r. We now demonstrate that $\tan(\eta) \neq r$. Represent $\mathcal{R}$ as the set of numbers $e^{i\tau}t$, where τ is fixed in $(-\pi,\pi) - \{0\}$ and t varies in $[0,\infty)$. Recall that $\eta \in [-\pi/2, \pi/2]$. By hypothesis, if t is large enough, then $e^{i\tau}t \in \Lambda_{\eta,K}$. That is, if t is large enough then

$$\Re[(e^{i\tau}t - K)e^{i\eta}] < 0,$$

so

$$t\cos(\tau - \eta) - \Re[Ke^{i\eta}] < 0.$$

Thus, $\cos(\tau + \eta) \leq 0$. Note that $\tau - \eta \in (-3\pi/2, 3\pi/2)$. The condition $\cos(\tau + \eta) \leq 0$ implies that

$$\tau + \eta \in (-3\pi/2, -\pi/2] \cup [\pi/2, 3\pi/2),$$

so $\pi/2 \leq |\tau + \eta| < 3\pi/2$. We claim that $\tau \neq -2\eta$. Suppose that this is not true. Then substituting $\tau = -2\eta$ into the inequality

$$\pi/2 \leq |\tau + \eta|$$

and using that $|\eta| \leq \pi/2$ gives $|\eta| = \pi/2$, so $|\tau| = \pi$, which gives a contradiction.

Note that if $k \in \mathcal{S}$ then

$$\frac{\Re(k)}{\Im(k)} = \frac{\cos[(\tau \pm \pi)/2]}{\sin[(\tau \pm \pi)/2]} = -\tan(\tau/2).$$

Since $-\tau/2 \neq \eta$ and $-\tau/2, \eta \in [-\pi/2, \pi/2]$, we have

$$r = \tan(-\tau/2) \neq \tan(\eta).$$

From Chapter 2, we have $d_a(\lambda) = (\Re[e^{i\eta}\phi'(a,\lambda)\overline{\phi(a,\lambda)}])^{-1}$. We now calculate $d_a(\lambda)$. First, we abbreviate $\phi = \phi(a,\lambda)$ and $\phi' = \phi'(a,\lambda)$. Here and until the conclusion of our proof of the lemma, $\lambda \in \mathcal{R} \cap \Lambda_{\eta,K}$.

Note that $\Re[e^{i\eta}\phi'\bar{\phi}] = \cos(\eta)\cdot\Re[\phi'\bar{\phi}] - \sin(\eta)\cdot\Im[\phi'\bar{\phi}]$. Letting ϕ_0 be defined analogously to u_0 in Lemma 4.4, we write $\bar{\phi} = \bar{\phi}_0 + (\overline{\phi - \phi_0})$ and $\phi' = \phi_0' + (\phi' - \phi_0')$. Thus, $\Im[\phi'\bar{\phi}] = \Im[\phi_0'\bar{\phi}_0] + \Im[\phi_0'(\overline{\phi - \phi_0})] + \Im[\bar{\phi}_0(\phi' - \phi_0')] + \Im[(\phi' - \phi_0')(\overline{\phi - \phi_0})]$. Now we wish to show that

$$(4.4) \qquad \Im[\phi_0'(\overline{\phi - \phi_0})] = o\left(\Im(k)|e^{ka}|^2|\phi(0) + \frac{\phi'(0)}{k}|^2\right).$$

Using Lemma 4.4 and the fact that $|\phi_0'| \le c(\lambda)|k||e^{ka}|$, we have

$$\begin{aligned}
\frac{|\Im[\phi_0'(\overline{\phi - \phi_0})]|}{|\Im(k)||e^{ka}|^2|\phi(0) + \frac{\phi'(0)}{k}|^2} &\le \frac{|\phi_0'||\phi - \phi_0|}{|\Im(k)||e^{ka}|^2|\phi(0) + \frac{\phi'(0)}{k}|^2} \\
&\le \frac{|k|}{|\Im(k)|}\frac{[c(\lambda)]^2}{|\phi(0) + \frac{\phi'(0)}{k}|^2}\left(e^{\int_0^a |\frac{q}{k}|\,dt} - 1\right).
\end{aligned}$$

Since the first two factors in the last expression are bounded as $\lambda \to \infty$ and the last expression goes to zero as $\lambda \to \infty$, we have shown (4.4). Using Lemma 4.4 and the fact that $|\phi_0| \le c(\lambda)|e^{ka}|$, we show similarly that

$$(4.5) \qquad \Im[\bar{\phi}_0(\phi' - \phi_0')] = o\left(\Im(k)|e^{ka}|^2|\phi(0) + \frac{\phi'(0)}{k}|^2\right)$$

and

$$(4.6) \qquad \Im[(\phi' - \phi_0')(\overline{\phi - \phi_0})] = o\left(\Im(k)|e^{ka}|^2|\phi(0) + \frac{\phi'(0)}{k}|^2\right).$$

Clearly, (4.4) - (4.6) hold with $\Im$ on the left-hand side replaced by $\Re$. For convenience in the next calculation, let

$$c = \frac{\phi(0) - \frac{\phi'(0)}{k}}{\phi(0) + \frac{\phi'(0)}{k}}.$$

By writing ϕ_0 in terms of exponentials and using the fact that $\Re k > 0$ and $\Im k \ne 0$, we have

$$\begin{aligned}
\Im[\phi_0'\bar{\phi}_0] &= \Im\left[\frac{k}{4}|e^{ka}|^2|\phi(0) + \frac{\phi'(0)}{k}|^2(1 - ce^{-2ka})(1 - \bar{c}e^{-\overline{2ka}})\right] \\
&= \frac{|e^{ka}|^2}{4}|\phi(0) + \frac{\phi'(0)}{k}|^2\Im[k + o(1)] \\
&= \frac{|e^{ka}|^2}{4}|\phi(0) + \frac{\phi'(0)}{k}|^2\Im(k)(1 + o(1)).
\end{aligned}$$

Arguing in the same way, we show that

$$\Re[\phi_0'\bar{\phi}_0] = \frac{|e^{ka}|^2}{4}|\phi(0) - \frac{\phi'(0)}{k}|^2\Im(k)\left[\frac{\Re(k)}{\Im(k)} + o(1)\right].$$

Thus,

$$\Im[\phi'\bar{\phi}] = \Im(k)|e^{ka}|^2|\phi(0) - \frac{\phi'(0)}{k}|^2\left(\frac{1}{4} - o(1)\right)$$

and

$$\Re[\phi'\bar{\phi}] = \Im(k)|e^{ka}|^2|\phi(0) - \frac{\phi'(0)}{k}|^2\left(\frac{r}{4} - o(1)\right).$$

Thus,

$$\begin{aligned} d_a(\lambda) &= (\Re[e^{i\eta}\phi'\bar{\phi}])^{-1} \\ &= \frac{1}{\Im(k)e^{2a\Re(k)}|\phi(0) - \frac{\phi'(0)}{k}|^2\left[\frac{r}{4}\cos(\eta) - \frac{1}{4}\sin(\eta) - o(1)\right]} \\ &= \frac{e^{-2a\Re(k)}}{\Im(k)||\phi(0) - \frac{\phi'(0)}{k}|^2}\cdot\frac{4}{\cos(\eta)\cdot r - \sin(\eta) - o(1)}. \end{aligned}$$

Since $\tan(\eta) = r$, the last equality shows that

$$d_a(\lambda) = \frac{1}{|\Im(k)||\phi(0) - \frac{\phi'(0)}{k}|^2}O(e^{-2a\Re(k)}).$$

Now, if $\epsilon > 0$ and $\mathcal{R}$ is a non-real ray which eventually stays in some admissible $\Lambda_{\eta,K}$ plane, then for $\lambda \in \mathcal{R}$ large enough, we have

$$d_a(\lambda) \leq \frac{C}{e^{2\epsilon\Re(k)}|\Im(k)||\phi(0) - \frac{\phi'(0)}{k}|^2}\cdot e^{-2(a-\epsilon)\Re(k)},$$

where C is a constant. If $\lambda \in \mathcal{R}$ is large enough, then

$$e^{2\epsilon\Re(k)}|\Im(k)||\phi(0) - \frac{\phi'(0)}{k}|^2 \geq 1$$

(to get this inequality, we use the fact that $\Re(k)/\Im(k)$ is constant on the ray $\mathcal{R}$).

Thus, we have that if $\lambda \in \mathcal{R}$ is large enough, then

$$d_a(\lambda) \leq C\cdot e^{-2(a-\epsilon)\Re(k)},$$

which shows that $d_a(\lambda) = O(e^{-2(a-\epsilon)\Re(k)})$. ∎

Henceforth, we will consider m-functions only for those problems which fall into one of the three following types:

Type 1: Class I;

Type 2: Class II, b not regular, the boundary condition function u satisfying the property that $-u'/u$ lies in some half-plane $\Re[e^{i\eta}z] \geq 0$;

Type 3: Class II, b regular.

Thus, we treat all problems of Class I but only two types of problems of Class II. We emphasize that we have defined Type 2 and Type 3 so that they are distinct categories. We shall remark about Type 2 here. Suppose that (η, K) is admissible, $\lambda \in \Lambda_{\eta,K}$, and $-u'/u$ lies in the half-plane $\Re[ze^{i\eta}] \geq 0$. In this case, the m-function m is given by

$$\begin{aligned} m(\lambda) &= -\frac{[\theta(\cdot,\lambda),u](b)}{[\phi(\cdot,\lambda),u](b)} \\ &= \lim_{X\to b} -\frac{(-u'(X)/u(X))\theta(X,\lambda)-\theta'(X,\lambda)}{(-u'(X)/u(X))\phi(X,\lambda)-\phi'(X,\lambda)} \\ &= \lim_{X\to b} M_X(-u'(X)/u(X)). \end{aligned}$$

We have from Chapter 2 that for each $X \in (0,b)$,

$$M_X(-u(X)/u(X)) \in D_X(\lambda),$$

so $m(\lambda) \in \bigcap_{0<X<b} D_X(\lambda)$. Thus, the hypotheses of Type 2 are set so that m lies in the intersection of the Weyl disks.

The next two lemmas give an asymptotic behavior of m_D along certain rays in $\mathbf{C}$.

LEMMA 4.6. *Let m_D denote the Dirichlet m-function for a Type 1 or Type 2 problem. If $\mathcal{R}$ is a non-real ray which eventually stays in some admissible $\Lambda_{\eta,K}$ plane and if, in Type 2, we have that $-u'/u$ lies in the half-plane $\Re[ze^{i\eta}] \geq 0$, then $m_D(\lambda) = -(h_2/H_1)\sqrt{-\lambda} - o(\sqrt{-\lambda})$ as $\lambda \to \infty$ along $\mathcal{R}$.*

PROOF. Denote $k = \sqrt{-\lambda}$. We are considering the Dirichlet case, so $h_1 = H_2 = 0$. Fix $a \in (0,b)$. For any $\lambda \in \mathcal{R} \cap \Lambda_{\eta,K}$ we have $m_D(\lambda) \in D_a(\lambda)$ since $m_D(\lambda) \in$

$\bigcap_{0<\lambda'<b} D_{\lambda'}(\lambda)$. Recall that $D_a(\lambda)$ is the image of the half-plane $\Re[ze^{i\gamma}] \geq 0$ under the Möbius transformation

$$M_a(z) = -\frac{\theta(a,\lambda)z + \theta'(a,\lambda)}{\phi(a,\lambda)z + \phi'(a,\lambda)}.$$

Elementary properties of Möbius transformations show that ∞ is mapped by M_a to a point on $\partial D_a(\lambda)$; i.e., $-\theta(a,\lambda)/\phi(a,\lambda)$ is a point on the circle bounding $D_a(\lambda)$. Thus,

$$\left| m_D(\lambda) - \frac{\theta(a,\lambda)}{\phi(a,\lambda)} \right| \leq d_a(\lambda).$$

Letting ϕ_0 and θ_0 be defined analogously to u_0 in Lemma 4.4, we use Lemma 4.4 to calculate

$$\begin{aligned}
-\frac{\theta(a,\lambda)}{\phi(a,\lambda)} &= -\frac{\theta_0(a,\lambda) + [\theta(a,\lambda) - \theta_0(a,\lambda)]}{\phi_0(a,\lambda) + [\phi(a,\lambda) - \phi_0(a,\lambda)]} \\
&= -k \cdot \frac{h_2 \cosh(ka) + o(e^{ka})}{H_1 \sinh(ka) + o(e^{ka})} \\
&= -\frac{h_2}{H_1} k \cdot \frac{1 + e^{-2ka} + o(1)}{1 - e^{-2ka} + o(1)}
\end{aligned}$$

as $\lambda \to \infty$ along $\mathcal{R}$. Since

$$\lim_{\mathcal{R} \ni \lambda \to \infty} \frac{1 + e^{-2ka} + o(1)}{1 - e^{-2ka} + o(1)} = 1,$$

we have

$$-\frac{\theta(a,\lambda)}{\phi(a,\lambda)} = -\frac{h_2}{H_1} k + o(k)$$

as $\lambda \to \infty$ along $\mathcal{R}$. Now, note that

$$\frac{m_D(\lambda)}{-(h_2/H_1)k} = \frac{m_D(\lambda) - \frac{\theta(a,\lambda)}{\phi(a,\lambda)}}{(-h_2/H_1)k} + \frac{-\frac{\theta(a,\lambda)}{\phi(a,\lambda)}}{-(h_2/H_1)k}.$$

We have just shown that the second term tends to 1 as $\lambda \to \infty$ along $\mathcal{R}$. The first term tends to 0, using $\phi(0) = 0$, $\phi'(0) = H_1$, and Lemma 4.5, since

$$\left| \frac{m_D(\lambda) - \frac{\theta(a,\lambda)}{\phi(a,\lambda)}}{-(h_2/H_1)k} \right| \leq \frac{d_a(\lambda)}{|(h_2/H_1)k|} \leq C \cdot \frac{e^{-2a\Re(k)}}{\Im(k)|k|\left|\phi(0) - \frac{\phi'(0)}{k}\right|^2} = C \cdot \frac{|k|}{\Im(k)} \cdot e^{-2a\Re(k)}$$

and the last expression tends to 0 as $\lambda \to \infty$ along $\mathcal{R}$. Thus,

$$\frac{m_D(\lambda)}{-(h_2/H_1)k} \to 1$$

as $\lambda \to \infty$ along $\mathcal{R}$. $\square$

We now state a result which gives identical asymptotic behavior of the Dirichlet m-function for Type 3 problems.

LEMMA 4.7. *Let m_D denote the Dirichlet m-function for a problem of Type 3. If $\mathcal{R}$ is a non-real ray which eventually stays in some admissible $\Lambda_{a,K}$ plane, then $m_D(\lambda) = -(h_2/H_1)\sqrt{-\lambda} + o(\sqrt{-\lambda})$ as $\lambda \to \infty$ along $\mathcal{R}$.*

PROOF. Again, since we are considering the Dirichlet case, we have $h_1 = H_2 = 0$. Denote $k = \sqrt{-\lambda}$. Using Lemma 4.4 then gives, for any $x \in [0, b]$ and for any $\lambda \in \mathbf{C}$,

$$\begin{aligned}
\phi(x,\lambda) &= H_1 \frac{\sinh(kx)}{k} + O\left(\frac{H_1}{k} e^{kx}(e^{\int_0^x \frac{|q|}{|k|}\,dt} - 1)\right) \\
\phi'(x,\lambda) &= H_1 \cosh(kx) + O\left(H_1 e^{kx}(e^{\int_0^x \frac{|q|}{|k|}\,dt} - 1)\right) \\
\theta(x,\lambda) &= h_2 \cosh(kx) + O\left(h_2 e^{kx}(e^{\int_0^x \frac{|q|}{|k|}\,dt} - 1)\right) \\
\theta'(x,\lambda) &= k h_2 \sinh(kx) + O\left(h_2 k e^{kx}(e^{\int_0^x \frac{|q|}{|k|}\,dt} - 1)\right).
\end{aligned}$$

Since we are considering the regular case, we have

$$m_D(\lambda) = -\frac{\frac{u'(b)}{u(b)}\theta(b,\lambda) - \theta'(b,\lambda)}{\frac{u'(b)}{u(b)}\phi(b,\lambda) - \phi'(b,\lambda)}.$$

Substituting the above expressions for $\phi(b,\lambda), \phi'(b,\lambda), \theta(b,\lambda), \theta'(b,\lambda)$ and factoring $h_2 e^{kb} k$ from the numerator and $e^{kb} H_1$ from the denominator, we get

$$m_D(\lambda) = -\frac{h_2 e^{kb} k}{e^{kb} H_1} \cdot \frac{\frac{u'(b)}{u(b)}\left[\frac{1-e^{-2kb}}{2k} + o(1)\right] - \frac{1-e^{-2kb}}{2} + o(1)}{\frac{u'(b)}{u(b)}\left[\frac{1-e^{-2kb}}{2k} + o(1)\right] - \frac{1-e^{-2kb}}{2} + o(1)}.$$

Since the right-hand fraction here converges to 1 as $\lambda \to \infty$, this equation shows that $m_D(\lambda) = -(h_2/H_1)k + o(k)$ as $\lambda \to \infty$ along $\mathcal{R}$. Note that if $u(b) = 0$ then

$$m_D(\lambda) = -\frac{\theta(b,\lambda)}{\phi(b,\lambda)},$$

and, as in the proof of Lemma 4.6, we may conclude that

$$m_D(\lambda) = -\frac{h_2}{H_1} k + o(k)$$

as $\lambda \to \infty$ along $\mathcal{R}$.

The following results will be used in the next chapter.

THEOREM 4.8. *Let m denote the m-function for a problem of Type 1, Type 2, or Type 3 generated by (4.1). Let $a \in (0, b)$, and let m_a denote the m-function for a regular problem on $[0, a]$ generated by (4.1). Then for any $\epsilon > 0$ and for any non-real ray $\mathcal{R}$ which eventually stays in some admissible $\Lambda_{\eta,K}$ plane,*

$$m(\lambda) - m_a(\lambda) = O(e^{-2(a-\epsilon)\Re(k)})$$

as $\lambda \to \infty$ along $\mathcal{R}$.

PROOF. For a Type 2 or Type 3 problem, let u denote the boundary condition function for m. We prove the theorem in several steps. Fix a non-real ray $\mathcal{R}$ which eventually stays in some admissible $\Lambda_{\eta,K}$ plane.

For the first step, let $\tilde{m}$ denote the m-function for a regular problem on $[0, a]$ generated by (4.1) and with boundary condition function $\tilde{u}$ satisfying $\tilde{u}(a) = 0$. Then

$$\tilde{m}(\lambda) = -\frac{\theta(a, \lambda)}{\phi(a, \lambda)}.$$

We now prove that $m(\lambda) - \tilde{m}(\lambda) = O(e^{-2(a-\epsilon)\Re(k)})$.

If the problem for m is Type 1, then from the nesting circle analysis, we have $|m(\lambda) - \tilde{m}(\lambda)| \leq d_a(\lambda)$ for $\lambda \in \mathcal{R}$ large enough. If the problem for m is Type 2 (that is, $-u'/u$ lies in the half-plane $\Re[ze^{i\eta}] \geq 0$), then for λ large enough, we have

$$m(\lambda) \in \bigcap_{0<X<b} D_X(\lambda).$$

Thus,

$$|m(\lambda) - \tilde{m}(\lambda)| = \left|m(\lambda) - \frac{\theta(a, \lambda)}{\phi(a, \lambda)}\right| \leq d_a(\lambda).$$

We now have that if the problem for m is Type 1 or Type 2, then

$$m(\lambda) - \tilde{m}(\lambda) = O(e^{-2(a-\epsilon)\Re(k)})$$

by Lemma 4.5.

Assume now that the problem for m is Type 3. Only to handle this case, we introduce a regular problem on $[0, b]$ generated by (4.1) and with boundary condition function u_0 satisfying $u_0(b) = 0$. Let m_0 denote the m-function for this problem. Then $m_0(\lambda) = -\theta(b, \lambda)/\phi(b, \lambda)$ and

$$\begin{aligned} |m(\lambda) - m_0(\lambda)| &= \left| -\frac{(-u'(b)/u(b))\theta(b,\lambda) - \theta'(b,\lambda)}{(-u'(b)/u(b))\phi(b,\lambda) - \phi'(b,\lambda)} - \frac{\theta(b,\lambda)}{\phi(b,\lambda)} \right| \\ &= \frac{1}{|\phi(b,\lambda)[\phi'(b,\lambda) - (u'(b)/u(b))\phi(b,\lambda)]|}. \end{aligned}$$

Using Lemma 4.4, it follows that this last fraction is less than or equal to $Ce^{-2(a-\epsilon)\Re(k)}$ for λ large enough on $\mathcal{R}$. Since $a < b$, this gives that

$$|m_0(\lambda) - m(\lambda)| \leq Ce^{-2(a-\epsilon)\Re(k)}.$$

From the nesting circle analysis, we have $m_0(\lambda) \in \partial D_b(\lambda)$ and $\tilde{m}(\lambda) \in \partial D_a(\lambda)$. Since $D_b(\lambda) \subset D_a(\lambda)$, we have $|m_0(\lambda) - \tilde{m}(\lambda)| \leq d_a(\lambda) \leq C_1e^{-2(a-\epsilon)\Re(k)}$ using also Lemma 4.5. The triangle inequality now gives

$$|m(\lambda) - \tilde{m}(\lambda)| \leq C_2e^{-2(a-\epsilon)\Re(k)},$$

so $m(\lambda) - \tilde{m}(\lambda) = O(e^{-2(a-\epsilon)\Re(k)})$.

Now, for the second step, let $\hat{m}$ be the m-function for a regular problem on $[0, a]$ with q and with boundary condition function $\hat{u}$ satisfying $\hat{u}(a) = 0$. Then

$$\begin{aligned} |\hat{m}(\lambda) - \tilde{m}(\lambda)| &= \left| -\frac{(\hat{u}'(a)/\hat{u}(a))\theta(a,\lambda) - \theta'(a,\lambda)}{(\hat{u}'(a)/\hat{u}(a))\phi(a,\lambda) - \phi'(a,\lambda)} - \frac{\theta(a,\lambda)}{\phi(a,\lambda)} \right| \\ &= \frac{1}{|\phi(a,\lambda)[\phi'(a,\lambda) - (\hat{u}'(a)/\hat{u}(a))\phi(a,\lambda)]|}. \end{aligned}$$

Using Lemma 4.4, it follows that this last fraction is bounded by $Ce^{-2(a-\epsilon)\Re(k)}$ for λ large enough on $\mathcal{R}$, so

$$\hat{m}(\lambda) - \tilde{m}(\lambda) = O(e^{-2(a-\epsilon)\Re(k)}).$$

Let m_a be as in the hypothesis of the theorem, and let u_a be its boundary condition function. If $u_a(a) = 0$, then the first step above shows that

$$m(\lambda) - m_a(\lambda) = O(e^{-2(a-\epsilon)\Re(k)}).$$

If, on the other hand, $u_a(a) \neq 0$, then introduce a regular problem on $[0,a]$ with boundary condition function $\tilde{u}$ satisfying $\tilde{u}(a) = 0$. Let $\tilde{m}$ denote the m-function for this introduced problem. Then the previous three paragraphs give that

$$|m(\lambda) - m_a(\lambda)| \leq |m(\lambda) - \tilde{m}(\lambda)| + |\tilde{m}(\lambda) - m_a(\lambda)| \leq Ce^{-2(a-\epsilon)\Re(k)},$$

so $m(\lambda) - m_a(\lambda) = O(e^{-2(a-\epsilon)\Re(k)})$. □

We finish this chapter with a result that states a manner in which the Green's function for a given problem converges to zero.

LEMMA 4.9. *Let m denote the m-function for a problem of Type 1, Type 2, or Type 3 generated by (4.1). Let $\psi_{m(\lambda)}(x,\lambda) = \theta(x,\lambda) + m(\lambda)\phi(x,\lambda)$ and $a \in (0,b)$ be fixed. If $\mathcal{R}$ is a non-real ray which eventually stays in some admissible $\Lambda_{\eta,K}$ plane, then $\phi(a,\lambda)\psi_{m(\lambda)}(a,\lambda) \to 0$ as $\lambda \to \infty$ along $\mathcal{R}$.*

PROOF. Consider a Dirichlet problem on $[a,b)$:

$$\phi_a(a,\lambda) = 0 \qquad \theta_a(a,\lambda) = 1$$

$$\phi_a'(a,\lambda) = -1 \qquad \theta_a'(a,\lambda) = 0,$$

where θ_a, ϕ_a solve (4.1) on $[a,b)$. Note that the two problems (the one involving m and the Dirichlet problem) are both Class I or both Class II. Since for any λ each of θ and ϕ may be written as a linear combination of θ_a and ϕ_a on $[a,b)$, we may use the boundary condition function associated with m to define an m-function m_a for the Dirichlet problem if they are both Class II. Let

$$\psi_{a,m_a(\lambda)}(\cdot,\lambda) = \theta_a(\cdot,\lambda) + m_a(\lambda)\phi_a(\cdot,\lambda).$$

Note that $v_{a,m_a(\lambda)}(a,\lambda) = 1$ and $v'_{a,m_a(\lambda)}(a,\lambda) = -m_a(\lambda)$. Thus,

$$m_a(\lambda) = -\frac{v'_{a,m_a(\lambda)}(a,\lambda)}{v_{a,m_a(\lambda)}(a,\lambda)}.$$

We now show via two cases that for $\lambda \in \mathcal{R}$ large enough,

$$m_a(\lambda) = -\frac{v'_{m(\lambda)}(a,\lambda)}{v_{m(\lambda)}(a,\lambda)}.$$

Case 1: Both problems are Class I. Then from the nesting circle analysis we have $\mathrm{dom}(m) \subset \mathrm{dom}(m_a)$. If $\lambda \in \mathcal{R}$ is large enough, then $\lambda \in \mathrm{dom}(m) \cap \mathrm{dom}(m_a)$. Since both $v_{m(\lambda)}|[a,b)$ and $v_{a,m_a(\lambda)}$ are square-integrable on $[a,b)$, they are multiples of each other on $[a,b)$. Thus,

$$m_a(\lambda) = -\frac{v'_{a,m_a(\lambda)}(a,\lambda)}{v_{a,m_a(\lambda)}(a,\lambda)} = -\frac{v'_{m(\lambda)}(a,\lambda)}{v_{m(\lambda)}(a,\lambda)}.$$

Case 2: Both problems are Class II. Then $\mathrm{dom}(m_a) = \mathrm{dom}(m) = \mathbf{C}$. Since both $v_{m(\lambda)}|[a,b)$ and $v_{a,m_a(\lambda)}$ satisfy the boundary condition at b, they are multiples of one another. Thus, as in Case 1, $m_a(\lambda) = -v'_{m(\lambda)}(a,\lambda)/v_{m(\lambda)}(a,\lambda)$.

Thus, for $\lambda \in \mathcal{R}$ large enough, we have that

$$-\frac{v'_{m(\lambda)}(a,\lambda)}{v_{m(\lambda)}(a,\lambda)}$$

is a Dirichlet m-function (it coincides with m_a for $\lambda \in \mathcal{R}$ large enough). By Lemmas 4.6 and 4.7,

$$\frac{v'_{m(\lambda)}(a,\lambda)}{v_{m(\lambda)}(a,\lambda)} = -\sqrt{-\lambda} + o(\sqrt{-\lambda})$$

as $\lambda \to \infty$ along $\mathcal{R}$.

By Lemma 4.4,

$$-\frac{\phi'(a,\lambda)}{\phi(a,\lambda)} = -\sqrt{-\lambda} + o(\sqrt{-\lambda})$$

as $\lambda \to \infty$ along $\mathcal{R}$. To complete our proof, note that

$$\frac{1}{\phi(a,\lambda)v_{m(\lambda)}(a,\lambda)} = \frac{\phi(a,\lambda)v'_{m(\lambda)}(a,\lambda) - \phi'(a,\lambda)v_{m(\lambda)}(a,\lambda)}{\phi(a,\lambda)v_{m(\lambda)}(a,\lambda)} = \frac{v'_{m(\lambda)}(a,\lambda)}{v_{m(\lambda)}(a,\lambda)} - \frac{\phi'(a,\lambda)}{\phi(a,\lambda)}.$$

Since both terms in the last expression are asymptotic to $-\sqrt{-\lambda}$ as $\lambda \to \infty$ along $\mathcal{R}$, we have that the reciprocal of the Green's function tends to infinity like $-2\sqrt{-\lambda}$, so the Green's function $\phi(a, \lambda)\psi_{m(\lambda)}(a, \lambda)$ tends to 0 as $\lambda \to \infty$ along $\mathcal{R}$. $\square$

CHAPTER 5

A Local Borg-Marchenko Theorem for Complex Potentials

This chapter considers the Sturm-Liouville spectral problem defined by the equation

$$Ly = -y'' + qy = \lambda y \tag{5.1}$$

over the interval $[0, b)$, where $0 < b \leq \infty$ and q is complex-valued and locally integrable. Recall the functions $\theta(\cdot, \lambda)$ and $\phi(\cdot, \lambda)$ defined in Chapters 2 and 4. The endpoint 0 is assumed regular, and at b the equation may be either regular or singular. A boundary condition $H_1 y(0) - h_1 y'(0) = 0$ is imposed and, in the case of a Class II problem, a boundary condition $[y, u](b) = 0$ in terms of a boundary condition function $u \in D(T)$ is imposed at b. Here, h_1 and H_1 are as in the previous chapters.

The purpose of this chapter is to prove a generalization of the theorem in [**1**]. The generalization consists in allowing the potential q and the boundary parameters h_1, H_1, u to be **C**-valued, thereby making (5.1) non-selfadjoint. Our proof follows in the spirit of Bennewitz and incorporates the language of the Sturm-Liouville spectral problem found in [**3**] (in particular, the nesting circle analysis). The results we present in this chapter involve directly the Titchmarsh-Weyl m-function, which has come to play an important role in the spectral analysis of Sturm-Liouville problems in the selfadjoint case. They state, roughly, that two potentials coincide on a compact interval if and only if the corresponding m-functions are exponentially close on certain rays in **C**.

To state our first theorem, we introduce another similar Sturm-Liouville problem defined by a potential $\tilde{q}$ over the interval $[0, \tilde{b})$. We introduce the quantities $\tilde{Q}(H_1)$, $\tilde{m}$, and, if the problem is Class II, a boundary condition $[y, \tilde{u}](\tilde{b}) = 0$ where $\tilde{u}$ is a boundary condition function in the domain of the maximal operator for $\tilde{q}$. We denote by $\tilde{\theta}$ and $\tilde{\phi}$ solutions of (5.1) with $\tilde{q}$ replacing q and the same initial conditions as for θ, ϕ.

THEOREM 5.1. *Suppose that the problems associated with q and $\tilde{q}$ are Type 1, Type 2, or Type 3 (the two problems need not be of the same type). Let $a \in \mathbf{R}$ with $a \in (0, \min(b, \tilde{b}))$. If $q = \tilde{q}$ on $[0, a]$, then for any $\epsilon > 0$ and for any non-real ray $\mathcal{R}$ which eventually stays in the intersection $\Lambda_{\eta,K} \cap \Lambda_{\tilde{\eta},\tilde{K}}$ of admissible half-planes,*

$$m(\lambda) - \tilde{m}(\lambda) = O(e^{-2(a-\epsilon)\Re k})$$

as $\lambda \to \infty$ on $\mathcal{R}$.

PROOF. Assume that $q = \tilde{q}$ on $[0, a]$. Let $\epsilon > 0$ and $\mathcal{R}$ be a ray as stated in the hypotheses of the theorem. Let $\hat{m}$ be the m-function for a regular problem on $[0, a]$ generated by (5.1) and boundary condition function $\hat{u}$. By Theorem 4.8, we have

$$m(\lambda) - \hat{m}(\lambda) = O(e^{-2(a-\epsilon)\Re k})$$

as $\lambda \to \infty$ along $\mathcal{R}$. Since $q = \tilde{q}$ on $[0, a]$, $\hat{m}$ is the m-function for the same regular problem on $[0, a]$ generated by (5.1), with $\tilde{q}$ replacing q (same function $\hat{u}$). By Theorem 4.8 again, we have

$$\tilde{m}(\lambda) - \hat{m}(\lambda) = O(e^{-2(a-\epsilon)\Re k})$$

as $\lambda \to \infty$ along $\mathcal{R}$. The triangle inequality now gives that

$$m(\lambda) - \tilde{m}(\lambda) = O(e^{-2(a-\epsilon)\Re(k)})$$

as $\lambda \to \infty$ along $\mathcal{R}$. $\square$

We now present a sort of converse of Theorem 5.1. Recall the notation $\psi_{m(\lambda)}(\cdot, \lambda) = \theta(\cdot, \lambda) + m(\lambda)\phi(\cdot, \lambda)$, where m is the m-function for the problem generated by (5.1). Taking into account the comments immediately preceding Theorem 5.1, we write

$\tilde{v}_{\tilde{m}} = \tilde{\theta} + \tilde{m}\tilde{\phi}$ and we use the analogue of Lemma 4.9 with $\phi, v, Q(H)$ replaced by $\tilde{\phi}, \tilde{v}, \tilde{Q}(H)$, respectively.

THEOREM 5.2. *Suppose that the problems associated with q and $\tilde{q}$ are Type 1, Type 2, or Type 3 (the two problems need not be of the same type). If there are two distinct non-real rays, say $\mathcal{R}_1$ and $\mathcal{R}_2$, which eventually stay in the intersections $\Lambda_{\eta_1,K_1} \cap \Lambda_{\tilde{\eta}_1,\tilde{K}_1}$ and $\Lambda_{\eta_2,K_2} \cap \Lambda_{\tilde{\eta}_2,\tilde{K}_2}$, respectively, of admissible half-planes, and for which it is true that given any $\epsilon > 0$, $m(\lambda) - \tilde{m}(\lambda) = O(e^{-2(a-\epsilon)\Re k})$ as $\lambda \to \infty$ along each ray, then $q = \tilde{q}$ on $[0, a]$.*

PROOF. Fix $x \in (0, a)$. For the next calculation, we abbreviate $\phi = \phi(x, \lambda)$ and similarly for $\tilde{\phi}$. Using Lemma 4.4 and the fact that $\Re k > 0$, we have

$$\begin{aligned}
\frac{\phi}{\tilde{\phi}} &= \frac{\phi_0 - (\phi - \phi_0)}{\tilde{\phi}_0 - (\tilde{\phi} - \tilde{\phi}_0)} \\
&= \frac{\phi_0 - O(c(\lambda)e^{kx}(e^{\int_0^x \frac{|q|}{|k|}dt} - 1))}{\tilde{\phi}_0 - O(\tilde{c}(\lambda)e^{kx}(e^{\int_0^x \frac{|\tilde{q}|}{|k|}dt} - 1))} \cdot \frac{e^{-kx}}{e^{-kx}} \\
&= \frac{\frac{1}{2}\left[\phi(0) - \frac{\phi'(0)}{k} - \left(\phi(0) - \frac{\phi'(0)}{k}\right)e^{-2kx}\right] - e^{-kx}O(c(\lambda)e^{kx}(e^{\int_0^x \frac{|q|}{|k|}dt} - 1))}{\frac{1}{2}\left[\tilde{\phi}(0) - \frac{\tilde{\phi}'(0)}{k} - \left(\tilde{\phi}(0) - \frac{\tilde{\phi}'(0)}{k}\right)e^{-2kx}\right] - e^{-kx}O(\tilde{c}(\lambda)e^{kx}(e^{\int_0^x \frac{|\tilde{q}|}{|k|}dt} - 1))} \cdot \frac{c(\lambda)}{\tilde{c}(\lambda)} \\
&= \frac{\frac{\phi(0)-\phi'(0)/k}{c(\lambda)} + o(1)}{\frac{\tilde{\phi}(0)-\tilde{\phi}'(0)/k}{\tilde{c}(\lambda)} + o(1)},
\end{aligned}$$

where $c(\lambda) = |\phi(0)| + |\phi'(0)/k|$. The last expression converges to 1 as $\lambda \to \infty$ on $\mathcal{R}_1$ or $\mathcal{R}_2$, so $\phi(x, \lambda)/\tilde{\phi}(x, \lambda) \to 1$ as $\lambda \to \infty$ on $\mathcal{R}_1$ or $\mathcal{R}_2$. Using this fact and Lemma 4.9 yields

$$\tilde{\phi}(x, \lambda)v_{m(\lambda)}(x, \lambda) = \frac{\tilde{\phi}(x, \lambda)}{\phi(x, \lambda)}\left(\phi(x, \lambda)v_{m(\lambda)}(x, \lambda)\right) \to 0$$

and

$$\phi(x, \lambda)\tilde{v}_{\tilde{m}(\lambda)}(x, \lambda) = \frac{\phi(x, \lambda)}{\tilde{\phi}(x, \lambda)}\left(\tilde{\phi}(x, \lambda)\tilde{v}_{\tilde{m}(\lambda)}(x, \lambda)\right) \to 0$$

as $\lambda \to \infty$ on $\mathcal{R}_1$ or $\mathcal{R}_2$. Thus, their difference

$$\tilde{\phi}(x, \lambda)\theta(x, \lambda) - \phi(x, \lambda)\tilde{\theta}(x, \lambda) + (m(\lambda) - \tilde{m}(\lambda))\phi(x, \lambda)\tilde{\phi}(x, \lambda)$$

converges to 0 along $\mathcal{R}_1$ and $\mathcal{R}_2$. Using Lemma 4.4 and the fact that $\Re k > 0$ gives that $\varphi(x,\lambda)\tilde{\varphi}(x,\lambda) = \mathcal{O}(1)e^{2kx}$ as $\lambda \to \infty$ on $\mathcal{R}_1$ or $\mathcal{R}_2$. Let $0 < \epsilon < a - x$. Now we make use of the hypothesis that $m(\lambda) - \tilde{m}(\lambda) = O(e^{-2(a-\epsilon)\Re k})$ to see that

$$|(m(\lambda) - \tilde{m}(\lambda))\varphi(x,\lambda)\tilde{\varphi}(x,\lambda)| \leq Ce^{-2(a-\epsilon-x)\Re k} \to 0$$

as $\lambda \to \infty$ on $\mathcal{R}_1$ or $\mathcal{R}_2$ since $\Re k > 0$ and $a - \epsilon - x > 0$. Thus, the entire function

$$g(\lambda) = \tilde{\varphi}(x,\lambda)\theta(x,\lambda) - \varphi(x,\lambda)\tilde{\theta}(x,\lambda)$$

converges to 0 as $\lambda \to \infty$ on $\mathcal{R}_1$ or $\mathcal{R}_2$, so it is bounded on these two rays. By a corollary of the Phragmén-Lindelöf theorem as stated in [5], we may conclude that g is bounded in $\mathbf{C}$, so it is constant by Liouville's theorem. Since g converges to 0 along $\mathcal{R}_1$, it is identically 0. That is,

$$\tilde{\varphi}(x,\lambda)\theta(x,\lambda) = \varphi(x,\lambda)\tilde{\theta}(x,\lambda).$$

Since $x \in (0,a)$ was arbitrary, this equation holds for all $x \in (0,a)$ and for all $\lambda \in \mathbf{C}$. By continuity in x, the equation holds for all $x \in [0,a]$ and for all $\lambda \in \mathbf{C}$. Let $\lambda \in \mathbf{C}$ be fixed for the rest of this proof. We now suppress both arguments and write the previous equation as

$$\tilde{\varphi}\theta = \varphi\tilde{\theta},$$

where this equation holds for all $x \in [0,a]$. The fact that $q = \tilde{q}$ on $[0,a]$ now follows from a short sequence of calculations. We rewrite the previous equation as $\theta/\varphi = \tilde{\theta}/\tilde{\varphi}$ and differentiate both sides (relative to x) to get

$$\frac{\varphi\theta' - \varphi'\theta}{\varphi^2} = \frac{\tilde{\varphi}\tilde{\theta}' - \tilde{\varphi}'\tilde{\theta}}{\tilde{\varphi}^2}.$$

Since the numerators on both sides are 1 (that is, the Wronskians are 1), we have $\varphi^2 = \tilde{\varphi}^2$. Differentiating both sides of this equation gives $2\varphi\varphi' = 2\tilde{\varphi}\tilde{\varphi}'$. Dividing this equation by the equation $\varphi^2 = \tilde{\varphi}^2$ gives

$$\frac{\varphi'}{\varphi} = \frac{\tilde{\varphi}'}{\tilde{\varphi}}.$$

Differentiating again finally gives

$$\frac{\phi''}{\phi} = \frac{\tilde{\phi}''}{\tilde{\phi}}.$$

Since ϕ and $\tilde{\phi}$ solve (5.1) with q and $\tilde{q}$, respectively, the previous equation implies that $q - \lambda = \tilde{q} - \lambda$, so $q = \tilde{q}$ on $[0, a]$. $\square$

CHAPTER 6

An Application

In this chapter we will use Theorem 5.2 to provide a proof of the Borg-Levinson theorem for complex potentials. The strategy is to characterize the m-function (i.e., demonstrate what information comprises the m-function) using ideas from [15] and then to call upon Theorem 5.2. In this chapter we consider only a regular problem on $[0, b]$ with the following Neumann conditions at 0:

$$h_1 = 1, H_1 = 0, h_2 = 0, H_2 = 1.$$

We continue to use the notation $k = \sqrt{-\lambda}$, where $\Re(k) \geq 0$. Let $u_1 \in \mathbf{C}_\infty$. The associated m-function is then

$$m(\lambda) = -\frac{\theta(b, \lambda)u_1 + \theta'(b, \lambda)}{\phi(b, \lambda)u_1 + \phi'(b, \lambda)}.$$

(If $u_1 = \infty$ then $m(\lambda) = -\theta(b, \lambda)/\phi(b, \lambda)$.)

LEMMA 6.1. *When $u_1 \neq \infty$ we have*

$$|m(\lambda)| \leq \frac{C}{|k|}$$

for k on the semicircles $|k| = (n - 1/2)\pi/b$ for all positive integers n large enough, and when $u_1 = \infty$ we have the same inequality for m for k on the semicircles $|k| = n\pi/b$ for all n large enough.

PROOF. By Lemma 4.4 we may express m as

$$m(\lambda) = -\frac{1}{k} \cdot \frac{1 - e^{-2kb} + o(1)}{1 + e^{-2kb} + o(1)} \tag{6.1}$$

when $u_1 \neq \infty$ and as

$$m(\lambda) = -\frac{1}{k} \cdot \frac{1 + e^{-2kb} + o(1)}{1 - e^{-2kb} + o(1)} \tag{6.2}$$

when $u_1 = \infty$, as $\lambda \to \infty$. In the case $u_1 \neq \infty$, a calculation shows that $|1 - e^{-2kb}| \geq 1/2$ for all k on the semicircles $|k| = (n + 1/2)\pi/b$. Thus, if n is large enough (so that λ is large enough), the denominator in (6.1) is bounded away from zero on the semicircles $|k| = (n + 1/2)\pi/b$. Clearly, the numerator in (6.1) is bounded on these semicircles for large enough n. Thus, for n large enough, we have $k \cdot m(\lambda)$ bounded on the semicircles $|k| = (n + 1/2)\pi/b$. In the case that $u_1 = \infty$, a similar argument shows that $k \cdot m(\lambda)$ is bounded on the semicircles $|k| = n\pi/b$ for sufficiently large n. $\Box$

Let m be an m-function for the type of problem described before Lemma 6.1. Let $\{\lambda_n : n \in \mathbb{N}\}$ denote the set of distinct poles of m (that is, the zeroes of

$$g(\lambda) = \phi(b, \lambda)u_1 - \phi'(b, \lambda),$$

the denominator of m) and let j_n be the multiplicity of λ_n as a zero of g (if $u_1 = \infty$ then $g(\lambda) = \phi(b, \lambda)$). A standard result is that g is an entire function whose growth order is $1/2$. It follows immediately that g has infinitely many zeroes. Furthermore, Hadamard's Factorization Theorem as stated in [5] gives that the λ_n and the j_n determine g up to a constant factor. This factor, however, is determined by the known asymptotic behavior of g (via Lemma 4.4). Since g is not identically zero, the λ_n's do not have a finite cluster point. We assume that the λ_n's are enumerated so that for each n, $|\lambda_{n+1}| \geq |\lambda_n|$. For each n let

$$m(\lambda) = \frac{m_n(\lambda)}{(\lambda - \lambda_n)^{j_n}},$$

where m_n is analytic at λ_n and $m_n(\lambda_n) \neq 0$. From Lemma 6.1, we may choose a sequence of (counterclockwise-oriented) circles Γ_n, with radius r_n and center 0, in the λ-plane so that $\lim_{n\to\infty} r_n = \infty$ and the estimate

$$|m(\lambda)| \leq \frac{C}{|k|}$$

holds for λ on any Γ_n. For each n, let N_n denote the number of λ_p's interior to Γ_n.

Theorem 6.2. *If $\lambda \notin \{\lambda_n : n \in \mathbf{N}\}$, then*

$$m(\lambda) = -\lim_{n\to\infty} \sum_{p=1}^{N_n} \left(\frac{1}{(j_p-1)!} \left(\frac{m_p(\mu)}{\mu-\lambda}\right)^{(j_p-1)} (\lambda_p) \right).$$

Proof. Fix $\lambda \notin \{\lambda_n : n \in \mathbf{N}\}$. Let N be chosen so that if $n \geq N$ then $r_n > |\lambda|$. If $n \geq N$ then by the residue theorem

$$\text{(6.3)} \qquad \frac{1}{2\pi i}\int_{\Gamma_n} \frac{m(\mu)}{\mu-\lambda} d\mu = \sum_{p=1}^{N_n} \operatorname{Res}\left(\frac{m(\mu)}{\mu-\lambda}, \mu=\lambda_p\right) + \operatorname{Res}\left(\frac{m(\mu)}{\mu-\lambda}, \mu=\lambda\right).$$

The second term easily evaluates to $m(\lambda)$. Note that for each p,

$$\operatorname{Res}\left(\frac{m(\mu)}{\mu-\lambda}, \mu=\lambda_p\right) = \frac{1}{(j_p-1)!}\left(\frac{m_p(\mu)}{\mu-\lambda}\right)^{(j_p-1)} (\lambda_p).$$

We remark here that since

$$\left(\frac{m_p(\mu)}{\mu-\lambda}\right)^{(j_p-1)} (\lambda_p) = \sum_{r=0}^{j_p-1} \left[\binom{j_p-1}{r} m_p^{(r)}(\lambda_p) \left(\frac{1}{\mu-\lambda}\right)^{(j_p-1-r)} (\lambda_p) \right],$$

each residue depends principally on the numbers $m_p^{(r)}(\lambda_p)$, for $r = 0, \cdots, j_p - 1$. We now show that

$$\lim_{n\to\infty} \int_{\Gamma_n} \frac{m(\mu)}{\mu-\lambda} d\mu = 0.$$

To do this, we make use of the estimate $|m(\mu)| \leq C/\sqrt{|-\mu|}$ for $\mu \in \Gamma_n$. We have

$$\left|\frac{m(\mu)}{\mu-\lambda}\right| \leq \frac{C/\sqrt{|-\mu|}}{|\mu| - |\lambda|} = \frac{C/\sqrt{r_n}}{r_n - |\lambda|},$$

so

$$\left|\int_{\Gamma_n} \frac{m(\mu)}{\mu-\lambda} d\mu\right| \leq \frac{C}{\sqrt{r_n}} \cdot \frac{1}{r_n - |\lambda|} \cdot 2\pi r_n \to 0$$

as $n \to \infty$. By taking limits on both sides of (6.3) we obtain our result. □

Note that Theorem 6.2 characterizes m in the sense that if the λ_n and the numbers $m_n^{(p)}(\lambda_n)$ $(p = 0, \cdots, j_n - 1)$ are specified, then m is uniquely determined.

Let $u_1, u_2 \in \mathbf{C}_\infty$. For $j = 1, 2$ we define the operator T_j whose domain is

$$D(T_j) = \{y \in D(T) : y'(0) = 0 \text{ and } y(b)u_j + y'(b) = 0\}$$

if $u_j \neq \infty$, or

$$D(T_j) = \{y \in D(T) : y'(0) = 0 \text{ and } y(b) = 0\}$$

if $u_j = \infty$ and which is defined by $T_j(y) = L(y)$ for $y \in D(T_j)$. For the problems considered here, the spectrum of T_j consists only of eigenvalues, which are precisely the poles of m. Furthermore, the functions $\phi(\cdot, \lambda_n)$ are eigenfunctions of T_j (i.e., $\phi(\cdot, \lambda_n) \in D(T_j)$ and $T_j(\phi(\cdot, \lambda_n)) = \lambda_n \phi(\cdot, \lambda_n)$). For the function ϕ, a prime denotes differentiation relative to x and superscripts denote differentiation relative to λ. An induction shows that

$$(T_j - \lambda_n)(\phi^{(p)}(\cdot, \lambda_n)) = p\phi^{(p-1)}(\cdot, \lambda_n)$$

for each positive integer p. It follows that the functions $\phi^{(p)}(\cdot, \lambda_n)$, $p = 0, \cdots, j_n - 1$, are generalized eigenfunctions of T_j. In particular, we have that for each $p = 0, \cdots, j_n - 1$,

$$(T_j - \lambda_n)^{p+1}(\phi^{(p)}(\cdot, \lambda_n)) = 0.$$

From Appendix IV of [8], we have that j_n is the dimension of the generalized eigenspace (algebraic eigenspace) corresponding to λ_n. Define

$$S_j = \{(\lambda_n, j_n) : \lambda_n \text{ is a zero of } \phi(b, \lambda)u_j + \phi'(b, \lambda) \text{ of multiplicity } j_n\}$$

if $u_j \neq \infty$ and

$$S_j = \{(\lambda_n, j_n) : \lambda_n \text{ is a zero of } \phi(b, \lambda) \text{ of multiplicity } j_n\}$$

if $u_j = \infty$.

We now come to our main theorems in this chapter; they state equivalences of certain pieces of spectral information. For their purpose we make a few notes. We define "generalized norming constants" to be the numbers

$$X_{n,p} = \int_0^b (p\phi^{(j_n)}(x, \lambda_n)\phi^{(p-1)}(x, \lambda_n) - j_n\phi^{(p)}(x, \lambda_n)\phi^{(j_n-1)}(x, \lambda_n))dx,$$

where $p = 0, \cdots, j_n - 1$. A hypothesis of the next theorem is the restriction $u_1 \neq \infty$, but Theorem 6.4 handles the case $u_1 = \infty$. Finally, we point out that the potential

q in the next theorem is restricted to a smaller class than $L^1_{loc}([0,b])$ so that we may make a straightforward use of Theorem 5.2.

THEOREM 6.3. *Let $u_1, u_2 \in \mathbf{C}_\infty$ with $u_1 \neq u_2$ and $u_1 \neq \infty$. Let S denote a sector in $\mathbf{C}$ and for $q \in L^1([0,b])$, let Q_q denote the closed convex hull for q as defined at the beginning of Chapter 2. Then the following pieces of information are equivalent:*

1. *The sets S_1, S_2;*
2. *The sets S_1 and $\{\phi^{(p)}(b,\lambda_n) : n \in \mathbf{N}, p = 0,\cdots,j_n - 1\}$;*
3. *The sets S_1 and $\{m_n^{(p)}(\lambda_n) : n \in \mathbf{N}, p = 0,\cdots,j_n - 1\}$;*
4. *The sets S_1 and $\{X_{n,p} : n \in \mathbf{N}, p = 0,\cdots,j_n - 1\}$;*
5. *The potential $q \in \{q \in L^1([0,b]) : S \subset \mathbf{C} - Q_q\}$.*

PROOF. We remark here that the purpose of the sector S is to ensure the existence of two distinct rays as in the hypotheses of Theorem 5.2. We shall assume for this proof that $u_2 \neq \infty$ (the case that $u_2 = \infty$ is treated almost identically).

We first show that (2) and (3) are equivalent. Let $f(\lambda) = \theta(b,\lambda)u_1 - \theta'(b,\lambda)$ and $g(\lambda) = \phi(b,\lambda)u_1 - \phi'(b,\lambda)$. Then $m(\lambda) = -f(\lambda)/g(\lambda)$. Let $\{\lambda_n : n \in \mathbf{N}\}$ denote the poles of m, which are the eigenvalues of T_1. As explained before Theorem 6.2, q is determined uniquely from S_1, and for each n and each $p = 0,\cdots,j_n - 1$ we have $g^{(p)}(\lambda_n) = 0$. Fix n. Let $g(\lambda) = g_n(\lambda)(\lambda - \lambda_n)^{j_n}$, where $g_n(\lambda_n) \neq 0$. Then $-f = g_n m_n$. We now show that knowing the set of numbers $\{m_n^{(p)}(\lambda_n) : p = 0,\cdots,j_n - 1\}$ is knowing the set $\{\phi^{(p)}(b,\lambda_n) : p = 0,\cdots,j_n - 1\}$. Note that

$$\phi(b,\lambda)\theta'(b,\lambda) - \phi'(b,\lambda)\theta(b,\lambda) = 1 \tag{6.4}$$

for all $\lambda \in \mathbf{C}$. Evaluating at λ_n and using that $g(\lambda_n) = 0$, we get

$$\phi(b,\lambda_n)f(\lambda_n) = 1. \tag{6.5}$$

Let $p = 1, \cdots, j_n - 1$. Differentiating both sides of (6.4) p times relative to λ, evaluating at λ_n, and using that $g^{(p)}(\lambda_n) = 0$, we get

$$\sum_{r=0}^{p} \left[\binom{p}{r} \phi^{(r)}(b, \lambda_n) f^{(p-r)}(\lambda_n) \right] = 0. \tag{6.6}$$

Since $f(\lambda_n) \neq 0$ and $\phi(b, \lambda_n) \neq 0$, the system of j_n equations defined by (6.5) and (6.6) is nonsingular. In fact, on inspection we see that for each $p = 0, \cdots, j_n - 1$, $f^{(p)}(\lambda_n)$ is expressed in terms of $\phi(b, \lambda_n), \cdots, \phi^{(p)}(b, \lambda_n)$ and $\phi^{(p)}(b, \lambda_n)$ is expressed in terms of $f(\lambda_n), \cdots, f^{(p)}(\lambda_n)$.

Using that $g_n(\lambda_n) \neq 0$, we have that the numbers $f^{(p)}(\lambda_n)$ are related to the numbers $m_n^{(p)}(\lambda_n)$ by a nonsingular system (the system is generated by differentiating p times relative to λ both sides of $-f = g_n m_n$, $p = 0, \cdots, j_n - 1$). Thus, the set $\{m_n^{(p)}(\lambda_n) : p = 0, \cdots, j_n - 1\}$ is related by a nonsingular system to the set $\{\phi^{(p)}(b, \lambda_n) : p = 0, \cdots, j_n - 1\}$. This system depends only on g, so knowing one is knowing the other.

We now show that (2) and (4) are equivalent. Let λ_n denote the eigenvalues of T_1. Again, S_1 determines g (as defined two paragraphs above) and, hence, the numbers $g^{(j_n)}(\lambda_n)$ uniquely. Differentiating p times relative to λ both sides of $-\phi'' - q\phi = \lambda\phi$ gives

$$-(\phi'')^{(p)} - q\phi^{(p)} = \lambda\phi^{(p)} + p\phi^{(p-1)}. \tag{6.7}$$

Fix n, set $\lambda = \lambda_n$ and let $p = 0, \cdots, j_n - 1$. With the aid of (6.7), a direct calculation gives that

$$(\phi^{(p)}(x, \lambda_n)(\phi')^{(j_n)}(x, \lambda_n) - \phi^{(j_n)}(x, \lambda_n)(\phi')^{(p)}(x, \lambda_n))'$$

is equal to

$$p\phi^{(j_n)}(x, \lambda_n)\phi^{(p-1)}(x, \lambda_n) - j_n\phi^{(p)}(x, \lambda_n)\phi^{(j_n-1)}(x, \lambda_n).$$

Using the fundamental theorem of calculus and the fact that

$$g^{(p)}(\lambda_n) = (\phi')^{(j_n)}(0, \lambda_n) = \phi^{(j_n)}(0, \lambda_n) = 0.$$

we get

$$\begin{aligned} X_{n,p} &= \phi^{(p)}(b,\lambda_n)(\phi')^{(j_n)}(b,\lambda_n) - \phi^{(j_n)}(b,\lambda_n)(\phi')^{(p)}(b,\lambda_n) \\ &= \phi^{(p)}(b,\lambda_n)g^{(j_n)}(\lambda_n). \end{aligned}$$

Now, the equation

$$X_{n,p} = \phi^{(p)}(b,\lambda_n)g^{(j_n)}(\lambda_n)$$

gives that knowing the set

$$\{X_{n,p} : n \in \mathbf{N}, p = 0, \cdots, j_n - 1\}$$

is equivalent to knowing the set $\{\phi^{(p)}(b,\lambda_n) : n \in \mathbf{N}, p = 0, \cdots, j_n - 1\}$.

We now show that the information in (1) gives the information in (2). Let $\tilde{g}_r(\lambda) = \phi'(b,\lambda) - u_r\phi(b,\lambda)$ (if $u_2 = \infty$, then $\tilde{g}_2(\lambda) = \phi(b,\lambda)$). Again, the set S_r uniquely determines $\tilde{g}_r$, whose zeroes give the eigenvalues of T_r. Let λ_n denote the distinct zeroes of g_1, with j_n the multiplicity of λ_n (so that (λ_n, j_n) are the elements of S_1). Note that $\tilde{g}_1^{(p)}(\lambda_n) = (\phi')^{(p)}(b,\lambda_n) - u_1\phi^{(p)}(b,\lambda_n) = 0$ for $p = 0, \cdots, j_n - 1$. Let $p = 0, \cdots, j_n - 1$. Since $(\phi')^{(p)}(b,\lambda_n) = -u_1\phi^{(p)}(b,\lambda_n)$, we have

$$\tilde{g}_2^{(p)}(\lambda_n) = (\phi')^{(p)}(b,\lambda_n) + u_2\phi^{(p)}(b,\lambda_n) = (u_2 - u_1)\phi^{(p)}(b,\lambda_n).$$

Since $\tilde{g}_2$ and hence the numbers $\tilde{g}_2^{(p)}(\lambda_n)$ are determined uniquely from S_2, the equation

$$\tilde{g}_2^{(p)}(\lambda_n) = (u_2 - u_1)\phi^{(p)}(b,\lambda_n)$$

then determines the numbers $\phi^{(p)}(b,\lambda_n)$, $p = 0, \cdots, j_n - 1$.

We now show that the information in (3) gives the information in (5). By Theorem 6.2, the information in (3) determines uniquely the m-function for the problem associated with u_1. By Theorem 5.2, the potential q is determined uniquely.

Clearly, the information in (5) gives any of the other pieces of information (in particular, (5) gives the information in (1)). Thus, we have that (1) gives (2), (2) gives (3). (3) gives (4). (4) gives (5), and (5) gives (1). The Borg-Levinson Theorem for complex potentials is that (1) gives (5). □

We conclude with a theorem analogous to Theorem 6.3. Our proof is nearly identical to the one for Theorem 6.3, so we omit it. We use the same notation as in the statement of Theorem 6.3.

THEOREM 6.4. *Suppose that $u_1 = \infty$ and $u_2 \in \mathbf{C}$. Let S denote a sector in $\mathbf{C}$ and let Q_q be defined exactly as in the statement of Theorem 6.3. Then the following pieces of information are equivalent:*

1. *The sets S_1, S_2;*
2. *The sets S_1 and $\{(\phi')^{(p)}(b,\lambda_n) : n \in \mathbf{N}, p = 0, \cdots, j_n - 1\}$;*
3. *The sets S_1 and $\{m_n^{(p)}(\lambda_n) : n \in \mathbf{N}, p = 0, \cdots, j_n - 1\}$;*
4. *The sets S_1 and $\{X_{n,p} : n \in \mathbf{N}, p = 0, \cdots, j_n - 1\}$;*
5. *The potential $q \in \{q \in L^1([0,b]) : S \subset \mathbf{C} - Q_q\}$.*

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