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A study of bifurcation for non-autonomous ordinary differential equations.

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A STUDY OF BIFURCATION FOR NON-AUTONOMOUS ORDINARY DIFFERENTIAL EQUATIONS

by

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ABSTRACT OF DISSERTATION
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This dissertation is concerned with the study of the structure of the solution set for non-linear non-autonomous ordinary differential equations that involve a parameter. We study in particular the existence of bifurcation points and bifurcation continua. Many studies have been undertaken to investigate the existence of bifurcation for nonlinear autonomous differential equations; Krasnoselski's and Rabinowitz's theorems are some of these studies. They used the Leray-Schauder topological degree as a tool for their studies of nonlinear eigenvalue problems. In this work we use the Conley index theory, which has been used to study the global behavior of dynamical systems in the neighborhood of an isolated invariant set that can be applied to autonomous equations. The problem with non-autonomous equations is that the solutions do not form a useful dynamical system, but we can overcome this difficulty by defining skew-product flows and using the Conley index theory to study them. First, we consider non-autonomous ordinary differential equations involving a parameter that belongs to a finite interval of the real line and we prove the existence of bifurcation points and bifurcation continua for this kind of problems. Then we prove similar results for non-autonomous ordinary differential equations involving a parameter that belongs to the whole real line and we obtain some global results conserning parameter that belongs to the whole real line and we obtain some global results conserning the bifurcation continua. We also consider asymptotically autonomous ordinary differential equations that involve a parameter and we conclude that if bifurcation occurs for the autonomous limiting equation, then it also must occur for the original non-autonomous one. Finally, we prove a similar result on small perturbations. Throughout the dissertation, we examine a number of examples to see how we can apply our results.

Abstract Approved by: Committee Chairman James R. Ward

Program Director [illegible]

Date 4/17/96 Dean of Graduate School [illegible]

DEDICATION

I dedicate this dissertation to my parents, Salha Al-Ghalib and Abdulrahman Al-Sheikh. Words are not enough to express my gratitude to them for fostering within me the love of knowledge, for their continuous support and encouragement throughout my life, and for encouraging me to be the best that I can be. No matter what I say, I can never repay them.

I would also like to dedicate this dissertation to my husband, Khaled Al-Shaibi. He was there for me with his support during some difficult times. His support and encouragement helped me to finish my studies.

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CHAPTER 1

Introduction

Many non-linear differential equations in mathematics involve one or more parameters. A large number of studies have been undertaken to investigate the structure of the solution sets that exhibit branching phenomena. The study of these phenomena is known as bifurcation theory, a topic that has been of great interest to many researchers, [5, 7]. Bifurcation phenomena also occur in many parts of physics and have been intensively studied [8, 17]. Researchers have used many techniques and tools to approach the study of bifurcation. One of these techniques is the Leray-Schauder degree [11], which was used by Krasnoselski [10] and Rabinowitz [15, 16] to study bifurcation for a special class of nonlinear differential equations: Suppose $F : \mathbf{R} \times E \rightarrow E$, where E is a real Banach space, F is continuous, and F possesses a simple curve $\mathcal{C}$ of zeros: $\{(\lambda(\alpha), u(\alpha)) : \alpha \in I = (a, b) \subset \mathbf{R}\}$. If there exists $\overline{\alpha} \in I$, such that in every neighborhood of $(\lambda(\overline{\alpha}), u(\overline{\alpha}))$ F has a zero not in $\mathcal{C}$, then $(\lambda(\overline{\alpha}), u(\overline{\alpha}))$ is said to be a *bifurcation point* for F with respect to $\mathcal{C}$. Krasnoselski and Rabinowitz considered the special case where

$$F(\lambda, u) = u - (\lambda L u + H(\lambda, u)), \tag{1.1}$$

where L is a compact linear operator, and $H : \mathbf{R} \times E \rightarrow E$ is compact with $H = o(\|u\|)$ at $u = 0$ uniformly on bounded λ intervals. The zeros, $\mathcal{C} = \{(\lambda, 0) : \lambda \in \mathbf{R}\}$, of F are then called the line of trivial solutions of

$$u = \lambda Lu + H(\lambda, u). \tag{1.2}$$

In this context, Krasnoselski [10] has shown, that if μ is a real characteristic value of L of odd multiplicity, then $(\mu, 0)$ is a bifurcation point of F with respect to C. Rabinowitz [15] further refined Krasnoselski's result by showing that, in the above situation, bifurcation has global consequences. More precisely, if μ is of odd multiplicity, and S denotes the closure of the set of nontrivial solutions of (1.2), then S possesses a component that contains $(\mu, 0)$ and is either unbounded or meets $\left(\hat{\mu}, 0\right)$, where $\mu \neq \hat{\mu}$ is also a characteristic value of L. He also considered an application of these global results to bifurcation from infinity [16].

Another technique that has been used to approach the study of bifurcation theory is the Conley index [4, 8, 20]. The Conley index is a powerful topological tool that was developed by Charles Conley in the 1970s, and since then has been used by many researchers. Ward [25, 26] used the Conley index to study the existence and bifurcation of continua of non-stationary solutions in infinite dimensional semiflows.

The solutions to a non-autonomous equation, by themselves, do not naturally form a useful dynamical system. However, non-autonomous equations may be used to define skew-product flows. The theory of skew-product flows has been developed by Miller and Sell, see [13, 19]. The use of the Conley index theory to study skew-product flows associated with non-autonomous equations was introduced by Ward [12, 21-24]. In this dissertation, we focus on the study of bifurcation that occurs for non-autonomous ordinary differential equations. Suppose $f : \mathbf{R} \times W \times \mathbf{R} \to \mathbf{R}^n$, where W is an open and connected subset of $\mathbf{R}^n$, and f is continuous, we consider the following differential equation:

$$u'(t) = f(\mu, u(t), t). \tag{1.3}$$

Assume that equation (1.3) possesses a line of trivial solutions, that is, a line $\mathbf{R} \times \{q\}$, where $f(\mu, q, t) = 0 \ \forall \mu \in [a, b]$, and $t \in \mathbf{R}$. We consider this problem and look at the local and global behavior of the solution set, and we prove the existence of continua bifurcating from the line of trivial solutions at a special point (μ_o, q).

As was mentioned at the beginning of this introduction, bifurcation phenomena occur in many parts of physics, and here is an example (see Hale [6]):

Example 1.1 Consider a pendulum of mass m and length l, constrained to oscillate in a plane rotating with angular velocity ω about a vertical line. If u denotes the angular deviation of the pendulum from the vertical line, the moment of centrifugal force is $m\omega^2 l^2 \sin u \cos u$, the moment of the force due to gravity is $mgl \sin u$, and the moment of inertia is $I = ml^2$. The differential equation of the motion is

$$Iu'' - m\omega^2 l^2 \sin u \cos u + mgl \sin u = 0.$$

If $\mu = \frac{m\omega^2 l^2}{I}$ and $\lambda = \frac{g}{\omega^2 l}$, then this equation is equivalent to the system

$$\begin{aligned} u' &= v \\ v' &= \mu (\cos u - \lambda) \sin u. \end{aligned}$$

The equilibrium points of this system are the points $(n\pi, 0)$, where $n = 0, \mp 1, \mp 2, \ldots$ and $(\cos^{-1} \lambda, 0)$. So for any given λ, the equilibrium points are $(n\pi, 0)$ and $(\cos^{-1} \lambda, 0)$, the last one appearing only if $|\lambda| < 1$. A further examination of this system shows that the equilibrium point $(0,0)$ is unstable (a saddle point) when $\lambda < 1$, and stable (a center) when $\lambda > 1$. It also can be shown that, when $0 < \lambda < 1$, the stable equilibrium points are $(\cos^{-1}\lambda, 0)$. So the nature of the equilibrium point $(0,0)$ changes as we cross the value $\lambda = 1$, and there is a branch of non-trivial equilibrium points $\{(\cos^{-1}\lambda, 0, \lambda) : |\lambda| \leq 1\}$ in the u-v-λ space bifurcating from the point $(0,0,1)$. Therefore, the point $(0,0,1)$ is a bifurcation point.

In chapter 2, the notations that are used in this work, including some definitions of spaces and other concepts that the reader might not be familiar with, are given, as well as a statement of theorems without proof.

Chapter 3 is devoted to the explanation of the Conley index. due to its importance to this work.

In chapter 4 we first prove a general continuation theorem (Theorem 4.4). Then we consider equation (1.3), where $f : [a,b] \times W \times \mathbf{R} \to \mathbf{R}^n$, and we prove that if we have a special value $\mu_o \in [a,b]$, where the Conley index associated with the skew-product flow changes as we cross the value μ_o, then the point (μ_o, q) is a bifurcation point. The proof of the bifurcation theorem has some similarities to the proof of the global bifurcation theorem in [15] and other similarities to the main result in [25].

In chapter 5, we prove some global results, where we consider equation (1.3) with $f : \mathbf{R} \times W \times \mathbf{R} \to \mathbf{R}^n$. First, we prove a global continuation theorem similar to the result in [26]. Then we prove that, under some conditions on f and for a special value μ_o, there exists a global bifurcation continuum bifurcating from the point (μ_o, q).

In chapter 6, we examine a number of simple examples of our results in chapters 4 and 5.

In chapter 7, we establish a bifurcation theorem for asymptotically autonomous differential equations. We consider the following equation

$$u'(t) = f(\lambda, u(t)) + g(\lambda, u(t), t), \tag{1.4}$$

where $\lambda \in [a, b] \subset \mathbf{R}$, and $g_t \to 0$ as $|t| \to \infty$, where g_t is the translate of g, that is, $g_\tau(\lambda, x, t) = g(\lambda, x, t + \tau)$. We prove that, under some conditions on f and g, there is a bifurcation point for equation (1.4) whenever there is one for the limiting equation

$$u'(t) = f(\lambda, u(t)).$$

In chapter 8, we give a conclusion to this dissertation and discuss some possible future work that can be done.

CHAPTER 2

Preliminary Results

This chapter gives some definitions and states some theorems that are well known but presented here for the sake of completeness. See references [9, 14] for more details.

Definition. A *topology* on a set X is a collection $\Im$ of subsets of X having the following properties:

(1) ϕ and X are in $\Im$.

(2) The union of elements of any subcollection of $\Im$ is in $\Im$.

(3) The intersection of the elements of any finite subcollection of $\Im$ is in $\Im$.

A set X for which a topology $\Im$ has been specified is called a *topological space.*

Definition. If X is a set, a *basis* for a topology on X is a collection β of subsets of X (called the *basis elements*), such that

(1) For each $x \in X$, there is at least one basis element B containing x.

(2) If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x, such that $B_3 \subset B_1 \cap B_2$.

Definition. If β is a basis for a topology on X, the topology $\Im$ *generated by* β is described as follows: A subset U of X is said to be open in X if, for each $x \in U$, there is a basis element $B \in \beta$, such that $x \in B$ and $B \subset U$.

Definition. A *metric* on a set X is a function

$$d : X \times X \to \mathbf{R}$$

having the following properties:

(1) $d(x,y) \geq 0$ for all $x, y \in X$; equality holds if and only if $x = y$.

(2) $d(x,y) = d(y,x)$ for all $x, y \in X$.

(3) $d(x,y) + d(y,z) \geq d(x,z)$ for all $x, y, z \in X$.

Given a metric d on X, the number $d(x,y)$ is often called the *distance* between x and y in the metric d. Given $\varepsilon > 0$, consider the set

$$B_d(x,\varepsilon) = \{y : d(x,y) < \varepsilon\}$$

of all points y, whose distance from x is less than ε. It is called the ε-ball centered at x. Sometimes the metric d is omitted from the notation, and this ball is simply written as $B(x,\varepsilon)$ when no confusion will arise. The collection of all ε-balls $B_d(x,\varepsilon)$, for $x \in X$ and $\varepsilon > 0$, is a basis for a topology on X, called the *metric topology* induced by d.

Definition : A *norm* on a real vector space X is a function

$$\|\| : X \to \mathbf{R}$$

having the following properties:

(1) $\|x\| \geq 0$ for all $x \in X$; equality holds if and only if $x = 0$.

(2) $\|x\| + \|y\| \geq \|x + y\|$ for all $x, y \in X$.

(3) $\|\alpha x\| = |\alpha| \, \|x\|$ for all $\alpha \in \mathbf{R}$ and $x \in X$.

Next we define quotient spaces and homotopy, which are of great importance in the study of the Conley index, as is explained in the next chapter.

Definition : Let X and Y be topological spaces; let $p : X \to Y$ be a surjective map. The map p is said to be a *quotient map*, provided a subset U of Y is open in Y if and only if $p^{-1}(U)$ is open in X.

Definition : If X is a space and A is a set, and if $p : X \to A$ is a surjective map, then there exists exactly one topology $\Im$ on A, relative to which p is a quotient map; it is called the *quotient topology* induced by p.

Definition : Let X be a topological space, and let X^* be a partition of X into disjoint subsets whose union is X. Let $p : X \to X^*$ be a surjective map that carries each point of X to the element of X^* containing it. In the quotient topology induced by p, the space X^* is called a *quotient space* of X.

Definition : If f and f' are continuous maps of the space X into the space Y, we say that f is *homotopic* to f' if there is a continuous map $F : X \times [0, 1] \to Y$ such that

$$F(x, 0) = f(x) \text{ and } F(x, 1) = f'(x) \text{ for each } x \in X.$$

The map F is called a *homotopy* between f and f'.

Definition : A continuous map $f : X \to Y$ is called a *homotopy equivalence* if there is a continuous map $g : Y \to X$ such that $g \circ f$ is homotopic to the identity map i_X on X and $f \circ g$ is homotopic to the identity map i_Y on Y. The map g is said to be a *homotopy inverse* of the map f.

It can be shown that, given any collection C of topological spaces, the relation of homotopy equivalence is an equivalence relation on C. Two spaces that are homotopy equivalent are said to have the same homotopy type.

Definition : A space X is said to be *locally compact at* x if there is some compact set C of X that contains a neighborhood of x. If X is locally compact at each of its points, X is said to be locally compact.

Now we define a topology that is of special interest in this study.

Definition : Let (Y, d) be a metric space; let X be a topological space. Given a function $f : X \to Y$, a compact set C in X, and a number $\varepsilon > 0$, let $B_C(f, \varepsilon)$ denote the set of all those functions $g : X \to Y$ for which

$$lub\{d(f(x), g(x)) : x \in C\} < \varepsilon.$$

The sets $B_C(f, \varepsilon)$ form a basis for a topology on Y^X(the set of all functions $f : X \to Y$). It is called the *topology of uniform convergence on compact sets.*

Theorem 2.1 A sequence $f_n : X \to Y$ of functions converges to the function f in the topology of uniform convergence on compact sets if and only if, for each compact subset C of X, the sequence $f_n \mid C$ converges uniformly to $f \mid C$.

The following definitions are needed:

Definition : Let X be a topological space, and let $\mathbf{R}$ denote the real numbers. Let π be a continuous function from $X \times \mathbf{R}$ into X. This function is called a *flow* on X if the following conditions are satisfied for all $x \in X$ and $s, t \in \mathbf{R}$:

(1) $\pi(x,0) = x$.

(2) $\pi(x, s+t) = \pi(\pi(x,s), t)$.

Definition : Let X be a topological space, and let $\mathbf{R}$ denote the real numbers. Let $D \subset X \times \mathbf{R}$ be an open subset, and let $\pi : D \to X$ be a continuous function. Then π is called a *local flow* on X if the following conditions are satisfied for all $x \in X$ and $s, t \in \mathbf{R}$:

(1) $(x,0) \in D$ for all $x \in X$, and $\pi(x,0) = x$.

(2) For every $x \in X$, there exists an interval $I_x = (\alpha_x, \beta_x) \subset \mathbf{R}$ containing zero, such that $(x,t) \in D$ if and only if $t \in I_x$.

(3) If $(x,s) \in D$, and $(\pi(x,s), t) \in D$, then $(x, s+t) \in D$, and $\pi(x, s+t) = \pi(\pi(x,s), t)$.

Definition : A set $I \subset X$ is said to be *invariant* under the flow π if $\{\pi(x,t) : t \in I_x\} \subset I$ $\forall\, x \in I$. It is isolated if it is the maximal invariant set in some neighborhood of itself.

As an example of a local flow consider the differential equation

$$x' = f(x)$$

on the Euclidean space $\mathbf{R}^n$, where f is a C^1-function. Then, for every point $x \in \mathbf{R}^n$, there is one and only one solution $\phi(x,t)$ that satisfies the initial condition $\phi(x,0) = x$. We can define a local flow on $\mathbf{R}^n$ by setting $\pi(x,t) = \phi(x,t)$. One can check easily that the conditions in the definition above are satisfied.

Now let $v : W \times \mathbf{R} \to \mathbf{R}^n$ be a C^1 function, where W is an open set in $\mathbf{R}^n$, and consider the non-autonomous ordinary differential equation

$$x' = v(x,t).$$

Let $\phi(x_o, t_o, t)$ be the solution to this equation that satisfies $\phi(x_o, t_o, t_o) = x_o$, and let $I(x_o, t_o)$ be the interval of definition of the solution. Clearly if we set $\pi(x,t) = \phi(x, t_o, t)$, then this does not give us a local flow, since the third property in the definition above is not satisfied. Define

$$J(x_o, t_o) = \{t : t + t_o \in I(x_o, t_o)\}$$

and let $X = W \times \mathbf{R}$. Define π formally by

$$\pi(p,t) = (\phi(x_o, t_o, t_o + t), t_o + t),$$

where $p = (x_o, t_o) \in X$. The mapping π defines a local flow on X where the interval of definition of the motion $\pi(p,t)$ is $J(p)$. It should be noted that the flow π for this case has no rest points, no periodic points. We will overcome these deficiencies by giving a more appropriate definition of a flow for non-autonomous equations.

By $C_c = C_c(W \times \mathbf{R}, \mathbf{R}^n)$, we denote the linear space of all continuous functions mapping $W \times \mathbf{R}$ into $\mathbf{R}^n$ with the topology of uniform convergence on compact sets of $W \times \mathbf{R}$. For $f \in C_c(W \times \mathbf{R}, \mathbf{R}^n)$, consider the ordinary differential equation

$$x' = f(x,t). \tag{2.1}$$

By the *hull* of f, $H(f)$, we mean the closure in C_c of the set of all translates of f. That is, $H(f) = cl\{f_\tau : \tau \in \mathbf{R}\}$, where $f_\tau(x,t) = f(x, t+\tau)$ for all $(x,t) \in W \times \mathbf{R}$.

Assume that for each $g \in H(f)$ there is a unique solution ϕ of

$$x' = g(x,t)$$

that satisfies $\phi(0) = x$. Let $\phi(x,g,t)$ denote that solution, that is, $\phi(x,g,0) = x$, and let $I(x,g)$ denote the maximal interval of definition for this solution. If we let

$$D = \{(x,g,t) \in W \times H(f) \times \mathbf{R} : t \in I(x,g)\},$$

then we shall show that a flow $\pi : D \to W \times H(f)$ is given by

$$\pi(x,g,t) = (\phi(x,g,t), g_t). \tag{2.2}$$

Now we are going to prove that (2.2) actually defines a local flow on $W \times H(f)$, (see reference [19]).

Theorem 2.2 The mapping π given by (2.2) defines a local flow on $W \times H(f)$ called the *skew-product flow*.

Proof : We only need to check the third property. Assume that $\tau \in I(x,g)$ for some $g \in H(f)$ and $\sigma \in I(\pi(x,g,\tau))$ and define

$$\phi(t) = \phi(x,g,t),\ \psi(t) = \phi(\phi(\tau), g_\tau, t),\ \zeta(t) = \phi(\tau + t).$$

Then ϕ is the solution of $x' = g(x,t)$ that satisfies $\phi(0) = x$, and ψ is the solution of

$$x' = g_\tau(x,t) = g(x, t+\tau) \tag{2.3}$$

that satisfies $\psi(0) = \phi(\tau) = \phi(x,g,\tau)$. However, ζ is also a solution of (2.3) and $\zeta(0) = \phi(\tau)$. Hence, by the uniqueness of solutions we have

$$\zeta(t) = \phi(t+\tau) = \psi(t) \text{ for all } t \in I(\pi(x,g,\tau)).$$

Consequently,

$$\begin{aligned}\pi\left(\pi\left(x,g,\tau\right),\sigma\right) &= \pi\left(\left(\phi\left(\tau\right),g_{\tau}\right),\sigma\right) = \left(\psi\left(\sigma\right),g_{\tau+\sigma}\right) \\ &= \left(\phi\left(\tau+\sigma\right),g_{\tau+\sigma}\right) = \pi\left(x,g,\tau+\sigma\right).\end{aligned}$$

The continuity of π follows from the following lemma of Kamke.

Lemma 2.3 (Kamke): Let A denotes the collection of all functions $f \in C\left(W \times \mathbf{R}, \mathbf{R}^n\right)$ that admit a unique solution of equation (2.1). Then the solution function $\phi\left(x,f,t\right)$ is continuous on the subset of $W \times A \times \mathbf{R}$ for which it is defined.

The proof of this lemma can be found in Appendix A in [19].

CHAPTER 3

The Conley Index

In this chapter we describe a powerful topological tool, the Conley index, which is a generalization of the Morse index. Most of the material in this chapter can be found in more details in references [4, 18, 20].

1 Isolated Invariant Sets and Continuation

The basic objects of study are the isolated invariant sets of a differential equation.

Definition : A set is called *invariant* if it is a union of solution curves. It is *isolated* if it is the maximal invariant set in some compact neighborhood of itself. Such a neighborhood is called an *isolating neighborhood.*

The interest in isolated invariant sets comes from the fact that they can be continued to nearby equations in a natural way; in this sense they are stable objects. The continuation is defined in terms of isolating neighborhoods as follows. A compact set N is an isolating neighborhood of the maximal invariant set S contained in N if and only if $S \subset N^o$ (the interior of N). Equivalently, N is an isolating neighborhood if and only if no point on ∂N is on a solution curve that is contained completely in N. This last condition is obviously stable under small changes in the equation. Therefore, if N is an isolating neighborhood of some equation, then N will be an isolating neighborhood for all equations near the given one. The isolated invariant sets thus determined by N are the continuations. Specifically, if N is an isolating neighborhood for a connected set of equations, then the corresponding isolated invariant sets are said to be related by continuation. We can extend this relation to non-nearby equations by making the relation transitive.

We illustrate these ideas by means of a simple example.

Example 3.1 Consider the scalar equation on $\mathbf{R}$ given by

$$\frac{dx}{dt} = x\left(1 - x^2\right) - \lambda = f(x, \lambda),$$

where λ is to be thought of as a parameter. In figure 1, we have sketched the curve $f(x,\lambda) = 0$. Observe that this curve meets each horizontal line in the set of critical points of the equation with the corresponding value of λ; for each fixed $\lambda = \lambda_o$, the horizontal line $\lambda = \lambda_o$ is the phase space of the equation

$$\frac{dx}{dt} = f(x, \lambda_o).$$

At each of the three λ-levels, the marked off intervals are isolating neighborhoods. The rest points are all examples of isolated invariant sets. But more generally, any interval, each of whose endpoints is a rest point, is also an isolated invariant set: a slightly larger interval will serve as an isolating neighborhood.

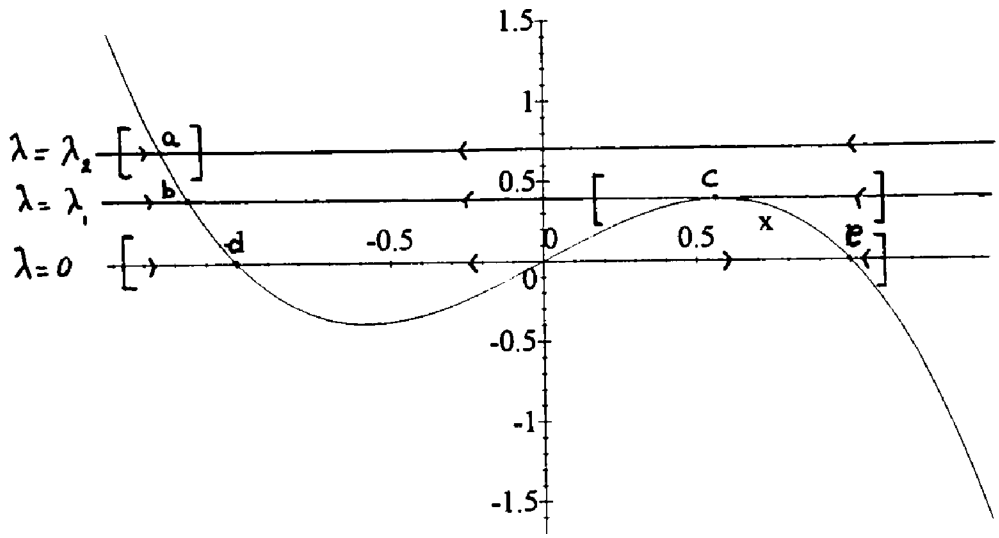

Figure 1. Graph of $f(x, \lambda) = 0$ for example 3.1.

For example, the closed interval $[d, 0]$ is an isolated invariant set for the $\lambda = 0$ equation. Since disjoint union of isolated invariant sets are isolated sets, we can easily find all of them. Thus, for the $\lambda = \lambda_2$ equation, there is only one (non-void) isolated invariant set: namely, the point a. Similarly, for the $\lambda = \lambda_1$ equation, there are four isolated invariant sets: $b, c, \{b, c\}$, and $[b, c]$. Finally, the $\lambda = 0$ equation has 12 isolated invariant sets. If we choose N to be the interval marked off on $\lambda = \lambda_2$ level, we see that the left-hand rest points in phase portrait are related by continuation. If we choose N to be the interval depicted on $\lambda = 0$ level, we see that a is related by continuation to the full set of bounded orbits in the other two phase portraits. It follows that d is related by continuation to the set of bounded orbits for the $\lambda = 0$ equation. The same statement is true for the rest point e, but not all isolated invariant sets are in this class; for example, 0 is not, since no choice of N will continue 0 to a or e.

The Conley index takes the form of the homotopy type of a pointed topological space. For our purposes, it suffices to think of a pointed space simply as a pair (X, x), where X is a topological space, and x is a point that belongs to X that is often called the *distinguished point*. The pointed spaces (X, x) and (Y, y) are said to be *homotopically equivalent*, written $(X, x) \sim (Y, y)$ if there is a homotopy from X into Y which takes x into y. The Conley index is computed from special isolating neighborhoods called *isolating blocks*, having the property that the solution through each boundary point of a block goes immediately out of the block in one or the other time directions. If B is an isolating block, the subset of the boundary of B, ∂B, consisting of points that leave B in positive time (the exit set) is denoted by B^-. Denoting by S the maximal invariant set in B, the Conley index of S, $h(S)$, is defined to be the homotopy equivalence class of the (pointed) quotient space B/B^-, that is,

$$h(S) = \left(\left[B/B^-\right], \left[B^-\right]\right).$$

Of course, this may be viewed as the pointed space obtained from B on collapsing B^- to a point. It is also important to note that isolating blocks always exist in locally compact metric spaces, and that the index is independent of the choice of the isolating block (see references [4, 20]).

The Conley index is invariant under continuation in the sense that, if S_1 and S_2 are isolated invariant sets that are related by continuation, then $h(S_1) = h(S_2)$.

Let us return to the example and compute the indices of some of the isolated invariant sets. Consider first the rest point 0, taking for B any proper subinterval of the interval $[d, e]$, we find that B^- is the complete boundary of this subinterval. If we identify these two endpoints, we see that B/B^- has the homotopy type of (pointed) circle, which we denote as $\sum^1$; that is, $h(0) = \sum^1$. Now consider the rest point a, taking for B the interval indicated in figure 1, we see that B^- is void. Now when the empty set of a space is collapsed to a point, the resulting space is homeomorphic to the disjoint union of the space and a point. We may deform the interval to a point without changing the homotopy type; this gives a pointed two-point space. Therefore, $h(a) = \sum^0$, the pointed zero-sphere, which is sometimes denoted by $\overline{1}$. Since the zero sphere is not homotopically equivalent to the one-sphere, we see that a cannot be continued to 0. Finally, consider the rest point c, taking for B the interval indicated in figure 1, we see that B^- is the left end point. Thus B/B^- has the homotopy type of a pointed interval. Since a simpler representative can be obtained by collapsing the interval to a point, we see that $h(c) = \overline{0}$, a (pointed) one-point space. Now consider the interval about c, if we raise the interval (i.e., increase λ), we see that c continues to the empty set. The empty set is always an invariant set: in fact, it is an isolating neighborhood and an isolating block of itself. The exit set is thus empty and the space obtained by collapsing the exit set to a point is therefore the one-point (pointed) space, $\overline{0}$. This agrees with our above calculations since we have continued the rest point c to the empty set. Now suppose that we start with λ slightly larger than λ_1, and decrease λ to λ_1. Here we see that the empty set continues to the isolated invariant set c: that is, we may alternately say that c *bifurcates* out of the empty set. For λ slightly smaller than λ_1, we see that c, itself, continues to an interval. It follows that $h([0,e]) = \bar{0}$.

2 Sums and Products of Indices

Since the intersection and disjoint union of two isolated invariant sets are isolated, a relationship between the various indices might be expected.

Consider first the case of the disjoint union of two isolated invariant sets, S_1 and S_2. We may assume that each S_i is contained in an isolating block B_i, with $B_1 \cap B_2 = \phi$. Then $B_1 \cup B_2$ will be a block for $S_1 \cup S_2$. To calculate $h(S_1 \cup S_2)$, we collapse the exit set of each block individually, and then collapse the resulting two distinguished points to a single point. The first collapse gives the disjoint union of B_1/B_1^- with B_2/B_2^-. The second, upon identifying B_1^- with B_2^-, gives a space, which we denote by $B_1/B_1^- \vee B_2/B_2^-$. This is called the *wedge* or *sum* of the two pointed spaces, and is the pointed space that results from gluing the two pointed spaces together at their distinguished points. Thus

$$h(S_1 \cup S_2) = h(S_1) \vee h(S_2).$$

This operation is always well defined on pointed spaces. Note that the zero element is just $\bar{0}$, that is, $(X, x) \vee \bar{0} = (X, x)$. However there are no inverses under the wedge operation. In fact the index is non-negative in the sense that, if the sum of two indices is zero, then they are both zero.

Indices can also be multiplied; the *smash product* $(X,x) \wedge (Y,y)$ of two pointed spaces is defined by

$$(X,x) \wedge (Y,y) = X \times Y / (X \times y) \cup (x \times Y),$$

that is, the space obtained from the topological product upon collapsing to a point the set of pairs, either of whose entries is a distinguished point. The pointed zero-sphere $\sum^0$ (or $\bar{1}$) is the multiplicative identity.

Lemma 3.2 Let X and Y be two connected topological spaces, and consider the pointed spaces (X,x) and (Y,y), where $x \in X$ and $y \notin Y$. Then $(X,x) \vee (Y,y)$ is disconnected and $(X,x) \wedge (Y,y)$ is connected.

Proof : First it should be noted that we think of the space (Y,y) as the pointed space $(Y \dot{\cup} y, y)$, where $Y \dot{\cup} y$ is the disjoint union of the space Y and the point y. From the definition of the sum of two pointed spaces above, the space $(X,x) \vee (Y,y)$ is the pointed space that results from gluing the pointed spaces (X,x) and (Y,y) at their distinguished points x and y, or in other words, we identify the point x with the point y, so the distinguished point for the new pointed space $(X,x) \vee (Y,y)$ still does not belong to the space Y, therefore $(X,x) \vee (Y,y)$ is disconnected.

Now $(X,x) \wedge (Y,y)$ is the space obtained by collapsing to a point the set of pairs, either of whose entries is a distinguished point, that is, the set $(X \times y) \cup (x \times Y)$. But $x \times Y \subset X \times Y$, so the distinguished point of the new space $(X,x) \wedge (Y,y)$ belongs to the connected space $X \times Y$. Therefore $(X,x) \wedge (Y,y)$ is connected.

3 More About the Conley Index

Theorem 3.3 Let $B \subset \mathbf{R}^n$ and f be a function that maps B into $\mathbf{R}^n$, consider the ordinary differential equation

$$\frac{dx}{dt} = f(x), \tag{3.1}$$

which admits x_o as a rest point. Suppose that the linearized equations about x_o are given by

$$\frac{d\xi}{dt} = A\xi, \tag{3.2}$$

where A is a matrix, none of whose eigenvalues has zero real parts; that is, x_o is a hyperbolic rest point. Then x_o is an isolated invariant set of equation (3.1), and $h(x_o) = \sum^k$, the pointed k-sphere, where k is the number of eigenvalues of A with positive real parts.

The proof of this theorem can be found in [20] and is omitted here.

Example 3.4 This is an example of bifurcation. Suppose we are given a one-parameter family of differential equations in $\mathbf{R}^n$

$$\frac{dx}{dt} = f(x,\lambda), \qquad |\lambda| \leq 1, \tag{3.3}$$

and that the origin is a rest point for all λ, that is, $f(0,\lambda) = 0$, for $|\lambda| \leq 1$. Suppose also that this rest point is an attractor if $\lambda < 0$, and a repellor if $\lambda > 0$. Thus if $\lambda \neq 0$, the origin is an isolated invariant set, and we have

$$h(0) = \begin{cases} \sum^n & if \quad \lambda > 0 \\ \sum^0 & if \quad \lambda < 0 \end{cases}.$$

This change of index reflects a change in structure of the solution set as λ crosses 0. In fact if N is any compact neighborhood of the origin in $\mathbf{R}^n$, which is an isolating neighborhood for the origin when $\lambda = \pm 1$, then for some $\lambda \in (-1, 1)$, there is a solution of the equation that lies in N for all time and that passes through a boundary point of N. In particular, this solution is not the rest point at the origin, otherwise N would define a continuation of the origin to itself, for $\lambda = \mp 1$, and this would force the indices to be the same.

Proposition 3.5 Let X be a locally compact metric space, and let $I \subset X$ be a compact isolated invariant set for the local flow π, with compact isolating neighborhood N. Then
(1) The Conley index $h(\pi, I)$ is defined, and $h(\pi, I) = ([B/B^-], [B^-])$, where $I \subset B \subset N$, and B is an isolating block.
(2) If $h(\pi, I) \neq \overline{0}$, then $I \neq \phi$.

Remark 3.6 We put the compact-open (c-o) topology on the set of functions from $X \times \mathbf{R} \to \mathbf{X}$, that is, $f_n \to f$ if and only if f_n converges to f uniformly on compact subsets of $X \times \mathbf{R}$.

Theorem 3.7 Let X be a locally compact metric space and $a, b \in \mathbf{R}$, and let π_μ be a family of local flows on X, where $\mu \in [a, b]$. Suppose that
(1) The map $\mu \to \pi_\mu$ is continuous in the sense that, if $\{\mu_n\} \subset [a, b]$, $\{x_n\} \subset X$, and $\{t_n\} \subset \mathbf{R}$ are sequences with $\mu_n \to \mu$, $x_n \to x$, $t_n \to t$, and $\pi_\mu(x, t)$ is defined, then

$\pi_{\mu_n}(x_n, t_n)$ is defined for large n, and

$$\pi_{\mu_n}(x_n, t_n) \to \pi_\mu(x, t) \qquad \text{as } n \to \infty$$

(2) There is a compact set N in X such that for each $\mu \in [a, b]$, N is an isolating neighborhood for the flow π_μ.

Let $I_\mu = \{x \in N : \pi_\mu(x, t) \in N \ \forall t \in \mathbf{R}\}$. Then the Conley index $h(\pi_\mu, I_\mu)$ is independent of $\mu \in [a, b]$.

See [21] for details.

The preceding is an overview of the main features of the Conley index, together with an indication of how it can be applied. As previously noted, the material in this chapter can be found in Charles Conley's book [4] or in [20], and the reader who wants to know more about the Conley index is referred to these two references.

CHAPTER 4

The Existence of Bifurcation Continua

In this chapter, we prove the existence of continua bifurcating from a line of trivial solutions for non-autonomous ordinary differential equations. We use the Conley index to prove our results.

Let $\mathbf{R}$ denote the real numbers, and $W \subset \mathbf{R}^n$ $(n \geq 1)$ be an open and connected subset. Recall that by $C_c = C_c(W \times \mathbf{R}, \mathbf{R}^n)$, we denote the linear space of all continuous functions mapping $W \times \mathbf{R}$ into $\mathbf{R}^n$ with the topology of uniform convergence on compact subsets of $W \times \mathbf{R}$. This space is metrizable, and we will let $d = d(.,.)$ denote a fixed metric on C_c.

For $f \in C_c(W \times \mathbf{R}, \mathbf{R}^n)$, consider the ordinary differential equation

$$\frac{dx}{dt} = f(x,t). \tag{4.1}$$

Definition : By the *hull* of f, $H(f)$, we mean the closure in C_c of the set of all translates of f, that is, $H(f) = cl\{f_\tau : \tau \in \mathbf{R}\}$, where

$$f_\tau(x,t) = f(x,t+\tau) \qquad \forall (x,t) \in W \times \mathbf{R}.$$

For $g \in H(f)$ and $x \in W$, consider the initial value problem

$$u'(t) = g(u(t), t) \tag{4.2}$$

$$u(0) = x \tag{4.3}$$

and assume:

(A1) For each $g \in H(f)$ and $x \in W$, (4.2) and (4.3) has a unique solution.

Let $I(x, g)$ be the maximal interval of existence of this solution, and let

$$D = \{(x, g, t) \in W \times H(f) \times \mathbf{R} : t \in I(x, g)\}.$$

Define $\pi : D \to W \times H(f)$ by

$$\pi(x, g, t) = (u(t; x, g), g_t), \tag{4.4}$$

where $u(t; x, g)$ is the solution to (4.2) and (4.3), and $g_t(y, s) = g(y, s + t)$ for $(y, s) \in W \times \mathbf{R}$. It was explained in chapter 2 that (4.4) defines a local flow on $W \times H(f)$ called a skew-product flow.

It follows from Ascoli's theorem and is shown in [19] that for $f \in C_c$, the hull $H(f)$ is compact if and only if f is bounded and uniformly continuous on every subset of the form $K \times \mathbf{R}$, where K is a compact subset in W. Thus, we have the following result which enables us to use the Conley index.

Theorem 4.1 Let $f \in C_c$. If f is bounded and uniformly continuous on sets of the form $K \times \mathbf{R}$, where K is a compact set in W, then $W \times H(f)$ is a locally compact metric space.

Now we look at parameter dependent equations. For each $\mu \in [a,b]$, where $[a,b] \subset \mathbf{R}$, let $f(\mu,.,.) \in C_c(W \times \mathbf{R}, \mathbf{R}^n)$, and consider the ordinary differential equation

$$\frac{dx}{dt} = f(\mu, x, t). \tag{4.5}$$

We assume

(A2) Let $f : [a,b] \times W \times \mathbf{R} \to \mathbf{R}^n$ be bounded and uniformly continuous on sets of the form $[a,b] \times K \times \mathbf{R}$, where K is compact in W, and assume that for each $\mu \in [a,b]$, $x \in W$ and $g \in H(f(\mu,.,.))$, the initial value problem

$$\begin{aligned} u'(t) &= g(\mu, u(t), t) \\ u(0) &= x \end{aligned} \tag{4.6}$$

has a unique solution.

For each $\mu \in [a,b]$, we may define a skew-product flow on $W \times H(f(\mu,.,.))$ as above; the latter space depends upon μ. In order that our flows should all be in the same space we make the following assumption

(A3) There is a function

$$\varphi : [a,b] \times H(f(b,.,.)) \to \bigcup_{a \le \mu \le b} H(f(\mu,.,.))$$

continuous in the product topology, such that for all $\tau \in \mathbf{R}$, and $\mu \in [a,b]$, we have

$$\varphi(\mu, f_\tau(b,.,.)) = f_\tau(\mu,.,.).$$

It follows from (A3) that $\varphi(\mu, g_\tau) = (\varphi(\mu, g))_\tau$ for all $\mu \in [a,b]$, $\tau \in \mathbf{R}$, and $g \in H(f(b,.,.))$. As an example, if $f(\mu, x, t) = \mu A(x,t) + (1-\mu) B(x)$ for some functions

A and B, then we let

$$\varphi(\mu, f_\tau(1,.,.)) = \varphi(\mu, A_\tau) = \mu A_\tau + (1-\mu) B.$$

Now let $F = f(b,.,.)$. For each $\mu \in [a,b]$, let

$$D(\mu) = \{(x,g,t) \in W \times H(F) \times \mathbf{R} : t \in I(x, \varphi(\mu, g))\},$$

and define $\pi_\mu : D(\mu) \to W \times H(F)$ by

$$\pi_\mu(x,g,t) = (u(t;x,\varphi(\mu,g)), g_t), \tag{4.7}$$

where $u(t;x,\varphi(\mu,g))$ is the solution to the initial value problem

$$\begin{aligned} u'(t) &= \varphi(\mu,g)(u(t),t) \qquad (4.8) \\ u(0) &= x. \end{aligned}$$

Notice that the flow π_μ does depend upon μ but they all lie in the same space $W \times H(F)$.

Theorem 4.2 If f satisfies (A2) and (A3), then for each $\mu \in [a,b]$ the mapping π_μ given by (4.7) defines a continuous flow on the locally compact metric space $W \times H(F)$.

Theorem 4.3 Let f satisfy (A2) and (A3), and let $\pi_\mu, \mu \in [a,b]$, be the flow defined by (4.7). Suppose there is a non-empty compact set $N \subset W \times H(F)$, which is an isolating neighborhood for π_μ for each $\mu \in [a,b]$. Let $I_\mu = \{(x,g) \in N : \pi_\mu(x,g,t) \in N, \forall t \in \mathbf{R}\}$, that is, I_μ is the maximal invariant set in N for the flow π_μ. Then the Conley index $h(\pi_\mu, I_\mu)$ is independent of $\mu \in [a,b]$.

The proof of theorem (4.3) is omitted here; it can be found in [21].

Definition : Let $U \subset [a,b] \times \mathbf{R}^n \times H(F)$, by U_μ we mean the section

$$U_\mu = \{(x,g) \in \mathbf{R}^n \times H(F) : (\mu, x, g) \in U\}.$$

Now we prove a continuation theorem analogous to the one in [25].

Theorem 4.4 Let $A \subset [a,b] \times W$ be an open and bounded set, and let $O = A \times H(F)$. Let $I_\mu = \left\{(x,g) \in \overline{O}_\mu : \pi_\mu(x,g,t) \in \overline{O}_\mu \forall t \in \mathbf{R}\right\}$. Suppose that, if $(x,g) \in I_\mu$ for some $\mu \in [a,b]$, then $(\mu, x, g) \notin \partial O$. Then the Conley index $h(\pi_\mu, I_\mu)$ is independent of $\mu \in [a,b]$.

Proof : If $(x,g) \in I_\mu$ for some $\mu \in [a,b]$, then from our assumption we have $(\mu, x, g) \notin \partial O$, which implies that $(x,g) \notin (\partial O)_\mu$. So $\overline{O}_\mu$ is an isolating neighborhood of I_μ. Also we know from elementary ordinary differential equations (ODE) (see [3] or [6]) that I_μ is closed, so I_μ is a closed subset of the compact set $\overline{A}_\mu \times H(F)$. Hence I_μ, itself, is compact, and so is $\{\mu\} \times I_\mu$ $(\subset O)$. Let $(\mu, x, g) \in \{\mu\} \times I_\mu \subset O = A \times H(F)$, then $(\mu, x) \in A$, which is open. So there exists a number $\delta_x > 0$, and an open set $V_x \subset \mathbf{R}^n$, such that

$$(\mu, x) \in [\mu - \delta_x, \mu + \delta_x] \times V_x \subset A.$$

Let $U_x = [\mu - \delta_x, \mu + \delta_x] \times V_x$, then we have $(\mu, x, g) \in U_x \times H(F)$. Now $\{U_x \times H(F) : (x,g) \in I_\mu \text{ for some g} \in H(F)\}$ is a cover of $\{\mu\} \times I_\mu$, and since it is compact, there exists a finite subcover$\{U_j \times H(F) : 1 \leq j \leq q\}$, where $U_j = [\mu - \delta_j, \mu + \delta_j] \times V_j$. Hence,

$$\{\mu\} \times I_\mu \subset \bigcup_{j=1}^{q} (U_j \times H(F)) = \bigcup_{j=1}^{q} U_j \times H(F).$$

Let $V_\mu = \bigcup_{j=1}^{q} V_j$, and $\delta_\mu = \min\{\delta_j : 1 \le j \le q\}$. Then we have

$$\{\mu\} \times I_\mu \subset [\mu - \delta_\mu, \mu + \delta_\mu] \times V_\mu \times H(F) \subset A \times H(F) = O.$$

Now $\overline{V}_\mu \times H(F)$ is an isolating neighborhood for the flow π_μ. We claim that it is an isolating neighborhood for π_η for all η sufficiently close to μ, as we show. Suppose by way of contradiction that there exists a sequence $\{\mu_m\} \subset [a, b]$, such that $\mu_m \to \mu$ as $m \to \infty$, and $(x_m, g_m) \in I_m = I_{\mu_m}$ with $(x_m, g_m) \notin V_\mu \times H(F)$ (i.e. $x_m \notin V_\mu$). Let $A_m = A_{\mu_m}$, $\pi_m = \pi_{\mu_m}$, and $O_m = O_{\mu_m}$. Now we have

$$\pi_m(x_m, g_m, t) \in I_m \subset A_m \times H(F) \qquad\qquad \forall t \in \mathbf{R}.$$

Recall that $\pi_m(x_m, g_m, t) = (u(t; x_m, \varphi(t_m, g_m)), (g_m)_t)$. Let $P : R^n \times H(F) \to \mathbf{R}^n$ be the projection mapping, and let $B = \bigcup_{\mu \in [a,b]} PI_\mu \subset R^n$, then B is bounded since $[a, b] \times B \subset A$, and A is bounded by assumption, also we have $u(t; x_m, \varphi(\mu_m, g_m)) \in B$, $\forall m \in \mathbf{N}$, and $t \in \mathbf{R}$, so in particular $u(0; x_m, \varphi(0, g_m)) \in B$, $\forall m \in \mathbf{N}$. But $u(0, x_m, \varphi(0, g_m)) = x_m$, which means that the sequence $\{x_m\}$ is bounded (since $\{x_m\} \subset B$) and, therefore, it has a convergent subsequence, which we relabel as $\{x_m\}$.

Let $\overline{x} = \lim_{m \to \infty} x_m$. We claim that $\overline{x} \in V_\mu$. To prove this claim, we will prove that there exists a function $\overline{g} \in H(F)$ such that $\pi_\mu(\overline{x}, \overline{g}, t) \in \overline{O}_\mu$ $\forall t \in \mathbf{R}$, which implies that $(\overline{x}, \overline{g}) \in I_\mu \subset V_\mu \times H(F)$, and so $\overline{x} \in V_\mu$. Now the sequence $\{g_m\}$ lies in $H(F)$, which is compact; therefore, it has a subsequence, which we relabel as $\{g_m\}$, that converges to $\overline{g} \in H(F)$, so $\varphi(\mu, \overline{g})$ is defined and, hence, by (A2) the initial value problem (4.8) (4.3) has a unique solution. So $\pi_\mu(\overline{x}, \overline{g}, t)$ is defined for every $t \in I(\overline{x}, \varphi(\overline{\mu}, g))$. Now

$(x_m, g_m) \in I_m$. This implies that $\pi_m(x_m, g_m, t) \in \overline{O}_m$ $\forall t \in \mathbf{R}$, also, since $x_m \to \overline{x}$ and $g_m \to \overline{g}$, we know from elementary ODE, that $\pi_m(x_m, g_m, t) \to \pi_\mu(\overline{x}, \overline{g}, t)$ $\forall t \in \mathbf{R}$. Therefore, we have

$$(\mu_m, \pi_m(x_m, g_m, t)) \to (\mu, \pi_\mu(\overline{x}, \overline{g}, t)) \qquad \forall t \in \mathbf{R}.$$

But $(\mu_m, \pi_m(x_m, g_m, t)) \in \overline{O}$ $\forall t \in \mathbf{R}$. Therefore $(\mu, \pi_\mu(\overline{x}, \overline{g}, t)) \in \overline{O}$ $\forall t \in \mathbf{R}$, which implies that $\pi_\mu(\overline{x}, \overline{g}, t) \in \overline{O}_\mu$ $\forall t \in \mathbf{R}$. This proves our second claim, that is, $\overline{x} \in V_\mu$, which contradicts our previous assumption that the sequence $\{x_m\}$ does not lie in V_μ. Hence, $\overline{V}_\mu \times H(F)$ is an isolating neighborhood for π_η for all η sufficiently close to μ. Therefore, by theorem (4.3), the Conley index $h(\pi_\eta, I_\eta)$ is independent of $\eta \in (\mu - \eta_o, \mu + \eta_o)$, where $\eta_0 > 0$ is small enough. But since $[a, b]$ is connected and compact, and μ was arbitrary, we can find a finite cover $\{(\mu_i - \eta_i, \mu_i + \eta_i) : 1 \le i \le m\}$ of the interval $[a, b]$ where on each of these intervals $I_i = (\mu_i - \eta_i, \mu_i + \eta_i)$ the Conley index is independent of $\mu \in I_i$. But the intervals I_i are overlapping; therefore, we conclude that

$$h(\pi_\mu, I_\mu) \equiv \text{constant for } \mu \in [a, b].$$

This proves our theorem.

For our next result we need to make the following definitions.

Definitions : Let $q \in W$, and $\mu_o \in [a, b]$,

(1) We say that $[a, b] \times \{q\}$ is the *line of trivial solutions* if

$$f(\mu, q, t) = 0 \qquad \forall \mu \in [a, b], \text{ and } t \in \mathbf{R},$$

which implies that

$$\pi_\mu (q, g, t) = (q, g_t) \qquad \forall t \in \mathbf{R},\ \mu \in [a, b],\ \text{and } g \in H(F),$$

that is,

$$u(t; q, \varphi(\mu, g)) = q \qquad \forall t \in \mathbf{R},\ \mu \in [a, b],\ \text{and } g \in H(F).$$

(2) The pair (μ_o, q) is said to be a *bifurcation point* if for every $\varepsilon > 0$, there exists a pair $(\mu, x) \in [a, b] \times W$, with $x \neq q$, and $g \in H(F)$, such that

$$|\mu - \mu_o| + \|\pi_\mu (x, g, t) - (q, g_t)\|_{\mathbf{R}^n \times H(F)} \leq \varepsilon \qquad \forall t \in \mathbf{R},$$

that is,

$$|\mu - \mu_o| + \|u(t; x, \varphi(\mu, g)) - q\|_{\mathbf{R}^n} \leq \varepsilon, \qquad \forall t \in \mathbf{R}$$

where the norm on $\mathbf{R}^n \times H(F)$ is the sum of the norms for $\mathbf{R}^n$ and $H(F)$.

Before proving our next results we need the following lemma:

Lemma 4.5 Assume that there is a line of trivial solutions, $[a, b] \times \{q\}$, and that $\{q\} \times H(F)$ is an isolated invariant set for each flow π_μ, where $\mu \in [\alpha, \beta] \subset [a, b]$. Then there exists a number $r > 0$ such that $\overline{B}(q, r) \times H(F)$ is an isolating neighborhood of $\{q\} \times H(F)$ for each flow π_μ, where $\mu \in [\alpha, \beta]$.

Proof : Let $\overline{\mu} \in [\alpha, \beta]$, then there exists a number $r(\overline{\mu}) > 0$ such that $\overline{B}(q, r(\overline{\mu})) \times H(F)$ is an isolating neighborhood of $\{q\} \times H(F)$ for the flow $\pi_{\overline{\mu}}$. It is also an isolating neighborhood for each flow π_μ for μ in a small neighborhood around $\overline{\mu}$, otherwise we would have $\{\mu_n\} \subset [\alpha,\beta]$ converging to $\overline{\mu}$, $\{g_n\} \subset H(F)$, and a sequence $\{y_n(t)\} \subset W$ of solutions to the problem

$$\begin{aligned} u'(t) &= \varphi(\mu_n, g_n)(u(t), t) \\ u(0) &= x, \end{aligned} \tag{4.9}$$

such that $y_n(t) \subset \overline{B}(q, r(\overline{\mu}))$ $\forall t \in \mathbf{R}$ and $y_n\left(\hat{t}_n\right) \in \partial B(q, r(\overline{\mu}))$ for some $\hat{t}_n > 0$. Now $H(F)$ is compact, therefore the sequence $\{g_n\}$ has a convergent subsequence that we relabel as $\{g_n\}$, which converges to $\overline{g} \in H(F)$, so if the sequence $\{\hat{t}_n\}$ converges to a number $\hat{t} \in \mathbf{R}$, then from elementary ODE we know that $\{y_n(t)\}$ will converge to a solution $\overline{y}(t)$ to the problem

$$\begin{aligned} u'(t) &= \varphi(\overline{\mu}, \overline{g})(u(t), t) \\ u(0) &= x, \end{aligned} \tag{4.10}$$

such that $\overline{y}\left(\hat{t}\right) \in \partial B(q, r(\overline{\mu}))$, which contradicts the fact that $\overline{B}(q, r(\overline{\mu})) \times H(F)$ is an isolating neighborhood for the flow $\pi_{\overline{\mu}}$. But if $\hat{t}_n \rightarrow \infty$ then we let $\hat{y}_n(t) = y_n\left(\hat{t}_n + t\right)$, and we have another sequence $\{\hat{y}_n(t)\} \subset W$ of solutions to

$$u'(t) = \varphi\left(\mu_n, (g_n)_{\hat{t}_n}\right)(u(t), t),$$

such that $\hat{y}_n(t) \in \overline{B}(q, r(\overline{\mu}))$ $\forall t \in \mathbf{R}$ and $\hat{y}_n(0) \in \partial B(q, r(\overline{\mu}))$ $\forall n \in \mathbf{N}$, and again the sequence $\{(g_n)_{\hat{t}_n}\}$ has a convergent subsequence that converges to $\widetilde{g} \in H(F)$. Thus from elementary ODE $\{\hat{y}_n(t)\}$ will converge to a solution $\widetilde{y}(t)$ to

$$u'(t) = \varphi(\overline{\mu}, \widetilde{g})(u(t), t),$$

such that $\widetilde{y}(0) \in \partial B(q, r(\overline{\mu}))$. Therefore $\overline{B}(q, r(\overline{\mu})) \times H(F)$ is an isolating neighborhood for each flow π_μ for μ in a small neighborhood around $\overline{\mu}$. Now for each $\mu \in [\alpha, \beta]$ there is a neighborhood W_μ of μ such that for each flow π_μ, $\mu \in W_\mu$, we have the same isolating neighborhood of $\{q\} \times H(F)$. Consider the set $\{W_\mu : \mu \in [\alpha, \beta]\}$. It forms an open cover of the interval $[\alpha, \beta]$, which is compact, so it has a finite subcover $\{W_{\mu_j} : 1 \leq j \leq m\}$. Note that W_{μ_j} are overlapping. Therefore for each j we have an isolating neighborhood $\overline{B}(q, r(\mu_j)) \times H(F)$ for π_μ, where $\mu \in W_{\mu_j}$. Let $r_1 = \min\{r(\mu_j) : 1 \leq j \leq m\}$, then $\overline{B}(q, r_1) \times H(F) \subset \overline{B}(q, r(\mu_j)) \times H(F)$ for every j. Therefore $\overline{B}(q, r_1) \times H(F)$ is an isolating neighborhood for each flow π_μ, where $\mu \in [\alpha, \beta]$. This proves the lemma.

Now we prove the main results of this chapter. As was mentioned in the introduction, the proofs have some similarities to those in [25].

Theorem 4.6 Assume that there is a line of trivial solutions, $[a, b] \times \{q\}$, and that $\{q\} \times H(F)$ is an isolated invariant set for each flow π_μ, where $\mu \in [a, b] \backslash \{\mu_o\}$, and that $h(\pi_a, \{q\} \times H(F)) \neq h(\pi_b, \{q\} \times H(F))$. Then for every $\varepsilon > 0$, there exists $(\mu, x, g) \in [a, b] \times W \times H(F)$, $x \neq q$, such that

$$|\mu - \mu_o| + \|\pi_\mu(x, g, t) - (q, g_t)\| \leq \varepsilon \qquad \forall t \in \mathbf{R},$$

that is, (μ_o, q) is a bifurcation point.

Proof : Suppose, by way of contradiction, that (μ_o, q) is not a bifurcation point, then there is an $\varepsilon > 0$ such that there is no $(\mu, x, g) \in [a, b] \times W \times H(F)$, with $x \neq q$, that satisfies

$$|\mu - \mu_o| + \|\pi_\mu (x, g, t) - (q, g_t)\| \le \varepsilon \qquad \forall t \in \mathbf{R}.$$

This implies that the set $\overline{B}\left(q, \frac{\varepsilon}{2}\right) \times H(F)$ is an isolating neighborhood of $\{q\} \times H(F)$ for every flow π_μ, where $\mu \in [\mu_o - \varepsilon, \mu_o + \varepsilon]$, so by theorem (3.4), the Conley index $h(\pi_\mu, \{q\} \times H(F))$ is independent of $\mu \in [\mu_o - \varepsilon, \mu_o + \varepsilon]$, that is,

$$h(\pi_{\mu_o - \varepsilon}, \{q\} \times H(F)) = h(\pi_{\mu_o + \varepsilon}, \{q\} \times H(F)).$$

But, since $\{q\} \times H(F)$ is an isolated invariant set for each flow π_μ, where $\mu \neq \mu_o$, then by lemma (4.5) we can find the same isolating neighborhood of $\{q\} \times H(F)$ for each flow π_μ where $\mu \in [a, \mu_o - \delta]$, $\delta < \varepsilon$. Hence by theorem (4.3),

$$h(\pi_a, \{q\} \times H(F)) = h(\pi_{\mu_o - \varepsilon}, \{q\} \times H(F)).$$

Similarly,

$$h(\pi_b, \{q\} \times H(F)) = h(\pi_{\mu_o + \varepsilon}, \{q\} \times H(F)).$$

Therefore, $h(\pi_a, \{q\} \times H(F)) = h(\pi_b, \{q\} \times H(F))$. This contradicts our assumption. Hence, (μ_o, q) is a bifurcation point, and this completes our proof.

To prove our next theorem, we need to state the following theorem. The proof will not be given here, but a proof is given in [2].

Theorem 4.7 (Separation Theorem). Let P and Q be disjoint closed subsets of a compact metric space K. Then either there exists a closed connected subset of K

meeting both P and Q, or there exist disjoint compact subsets A and B of K, such that $P \subset A$, $Q \subset B$, and $K = A \cup B$.

Definition : Define the set $S \subset [a, b] \times W$ by

$$S = cl\{(\mu, x) : x \neq q, \text{ and there is a bounded solution through } x \text{ for the problem (4.8) for some } g \in H(F)\}.$$

Theorem 4.8 Assume that there is a line of trivial solutions $[a, b] \times \{q\}$, and again that $\{q\} \times H(F)$ is an isolated invariant set for each flow π_μ, $\mu \neq \mu_o$, and that $h(\pi_a, \{q\} \times H(F)) \neq h(\pi_b, \{q\} \times H(F))$. Let C denote the component of S containing (μ_o, q), then either

(1) C is unbounded in $[a, b] \times W$, or

(2) C meets $\{a, b\} \times W$.

Proof : If the set W is bounded, then by C beeing unbounded we mean that it meets $[a, b] \times \partial W$. First note that S is non-empty, since the bifurcation point (μ_o, q) belongs to it. Suppose that C is bounded in $[a, b] \times W$ (if W is bounded, then by C beeing bounded in $[a, b] \times W$ we mean that it meets $[a, b] \times \partial W$), then we claim that C is compact. We show this by proving that C is also closed. So let $\{(\mu_n, x_n)\}$ be a sequence in C that converges to $(\overline{\mu}, \overline{x})$. We want to show that $(\overline{\mu}, \overline{x}) \in C$. Now the sequence $\{\mu_n\}$ is contained in $[a, b]$, which is closed, so its limit also belongs to $[a, b]$, that is, $\overline{\mu} \in [a, b]$. Let $(\mu, x) \in C \subset S$, then there exists a function $g \in H(F)$ such that the solution $u(t; x, \varphi(\mu, g))$ is bounded. Now suppose that $\{(\mu, u(t; x, \varphi(\mu, g))) : t \in \mathbf{R}\}$ is not contained completely in C, then the set $C \cup \{(\mu, u(t; x, \varphi(\mu, g))) : (\mu, u(t; x, \varphi(\mu, g))) \notin C\}$ is connected, and this contradicts the maximality of C. So we proved that for every $(\mu, x) \in C$ there exists a function $g \in H(F)$ such that $\{(\mu, u(t; x, \varphi(\mu, g))) : t \in \mathbf{R}\} \subset C$. Now from assumption (A2), we have that for any $g \in H(F)$, $\pi_{\overline{\mu}}(\overline{x}, g, t)$ is defined for $t \in I(\overline{x}, \varphi(\overline{\mu}, g))$. Also from theorem (4.2), we have $\pi_{\mu_n}(x_n, g, t) \rightarrow \pi_{\overline{\mu}}(\overline{x}, g, t)$ as $n \rightarrow \infty$, so

$$(\mu_n, u(t; x_n, \varphi(\mu_n, g))) \rightarrow (\overline{\mu}, u(t; \overline{x}, \varphi(\overline{\mu}, g))) \text{ as } n \rightarrow \infty. \qquad (*)$$

Now assume that $\{(\overline{\mu}, u(t; \overline{x}, \varphi(\overline{\mu}, g))) : t \in I(\overline{x}, \varphi(\overline{\mu}, g))\}$ is not contained completely in C, then the set $C \cup \{(\overline{\mu}, u(t; \overline{x}, \varphi(\overline{\mu}, g))) : (\overline{\mu}, u(t; \overline{x}, \varphi(\overline{\mu}, g))) \notin C\}$ is connected, since from (*) above, for every $t \in I(\overline{x}, \varphi(\overline{\mu}, g))$, any open set in $[a, b] \times W$ containing $(\overline{\mu}, u(t; \overline{x}, \varphi(\overline{\mu}, g)))$ must also contain points of the form $(\mu_n, u(t; x_n, \varphi(\mu_n, g)))$ for n large enough, and this contradicts the maximality of C. Therefore, we must have $(\overline{\mu}, u(t; \overline{x}, \varphi(\overline{\mu}, g))) \in C \; \forall t \in I(\overline{x}, \varphi(\overline{\mu}, g))$. Hence, in particular (at $t = 0$) we have that $(\overline{\mu}, \overline{x}) \in C$. Hence, C is compact.

Now assume that C also does not meet $\{a, b\} \times W$. Let U be a δ-neighborhood of C such that $dist(PC, \{a, b\}) > \delta$, where $P : [a, b] \times \mathbf{R}^n \rightarrow [a, b]$ is the projection mapping. Let $K = \overline{U} \cap S$. Then obviously K is closed, and it is also bounded, since $K \subset \overline{U}$, which means that K is compact. Now clearly $\partial U \cap C = \phi$ by construction and, therefore, $(\partial U \cap S) \cap C = \phi$. Also, there is no closed and connected set meeting both C and $\partial U \cap S$, otherwise, if C' is such a set, then $C \cup C'$ would be connected, and this contradicts the fact that C is a component. So by the separation theorem (4.7), there are two disjoint compact sets A and B of K, such that $C \subset A$, $\partial U \cap S \subset B$, and $K = A \cup B$.

Choose $\rho > 0$, such that $\rho < \min\{dist(A, \{a, b\} \times W), dist(A, B)\}$, and let A_ρ be a ρ-neighborhood of A. Now there is a maximal open interval $(\xi, \eta) \subset [a, b]$, such that $(\xi, \eta) \times \{q\} \subset A_\rho$ with $\mu_0 \in (\xi, \eta)$. We also may have $\mu \notin (\xi, \eta)$ with $(\mu, q) \in A_\rho$. Also $\partial A_\rho \cap K = \phi$ by construction, so the only solution on ∂A_ρ will be of the form (ξ, q), (η, q), or possibly other trivial solutions (μ, q) where $\mu \notin (\xi, \eta)$.

Now by hypothesis, $\{q\} \times H(F)$ is an isolated invariant set for each flow π_μ, where $\mu \neq \mu_o$, so by lemma (4.5) we can find a number $r_1 > 0$, such that $\overline{B}(q, r_1) \times H(F)$ is an isolating neighborhood of $\{q\} \times H(F)$ for each flow π_μ, with $\mu \in \left[a, \frac{\xi+\mu_o}{2}\right]$, and a number $r_2 > 0$ such that $\overline{B}(q, r_2) \times H(F)$ is an isolating neighborhood of $\{q\} \times H(F)$ for each flow π_μ, where $\mu \in \left[\frac{\mu_o+\eta}{2}, b\right]$. Let $r_o < \min\{r_1, r_2\}$, then $\overline{B}(q, r_o) \times H(F)$ is an isolating neighborhood of $\{q\} \times H(F)$ for each flow π_μ, where $\mu \in \left[a, \frac{\xi+\mu_o}{2}\right] \cup \left[\frac{\mu_o+\eta}{2}, b\right]$ and we can choose r_o so small that $\left[\frac{\xi+\mu_o}{2}, \frac{\mu_o+\eta}{2}\right] \times \overline{B}(q, r_o) \subset A_\rho$. Now let $O_\rho = A_\rho \times H(F)$, and let

$$O = O_\rho \cup ([a, b] \times B(q, r_o) \times H(F)).$$

Then O is open in $[a, b] \times W \times H(F)$, and if $(x, g) \in I_\mu = \{(x, g) \in \overline{O}_\mu : \pi_\mu(x, g, t) \in \overline{O}_\mu$ $\forall t \in \mathbf{R}\}$, then by construction we have that $(\mu, x, g) \notin \partial O$. By theorem (4.4), the Conley index $h(\pi_\mu, I_\mu)$ is independent of $\mu \in [a, b]$. But we also have

$$h(\pi_a, I_a) = h(\pi_a, \{q\} \times H(F)) \neq h(\pi_b, \{q\} \times H(F)) = h(\pi_b, I_b),$$

which is a contradiction and, hence, the theorem is proved.

Remark 4.9 Theorems (4.6) and (4.8) together prove the existence of a continuum (closed and connected set) that bifurcates from the line of trivial solutions. But this continuum lies in the set S, which means that C is a continuum of solutions to the initial value problems

$$u'(t) = \varphi(\mu, g)(u(t), t)$$

$$u(0) = x,$$

where $\varphi(\mu, g) \in H(f(\mu, ., .))$, so C is not a continuum of solutions to the original problem

$$u'(t) = f(\mu, u(t), t)$$

$$u(0) = x$$

as we would like it to be. We give some examples in chapter 6 where it is.

CHAPTER 5

Some Global Results

In this chapter we prove some global bifurcation results for our initial value problem

$$u'(t) = f(\mu, u(t), t), \tag{5.1}$$

where $f(\mu, ., .) \in C_c(W \times \mathbf{R}, \mathbf{R}^n)$ and $W \subset \mathbf{R}^n$ is open and connected. But now the parameter μ can be any real number, that is, $\mu \in \mathbf{R}$. We are going to assume that the function f satisfies the following conditions:

(B1) $f : \mathbf{R} \times W \times \mathbf{R} \to \mathbf{R}^n$ is bounded and uniformly continuous on sets of the form $[a, b] \times K \times \mathbf{R}$, where $[a, b] \subset \mathbf{R}$, K is compact in W, and for each $\mu \in \mathbf{R}$, $x \in W$ and $g \in H(f(\mu, ., .))$ the initial value problem

$$\begin{aligned} u'(t) &= g(\mu, u(t), t) \\ u(0) &= x \end{aligned} \tag{5.2}$$

has a unique solution

Note that (B1) is equivalent to saying that $H(f(\mu, ., .))$ is compact.

(B2) there is a function

$$\varphi : \mathbf{R} \times H(f(0, ., .)) \to \bigcup_{\mu \in \mathbf{R}} H(f(\mu, ., .))$$

continuous in the product topology, such that for all $\tau, \mu \in \mathbf{R}$, we have

$$\varphi(\mu, f_\tau(0, ., .)) = f_\tau(\mu, ., .).$$

Let $F = f(0, ., .)$, and for each $\mu \in \mathbf{R}$ consider the initial value problem

$$\begin{aligned} u'(t) &= \varphi(\mu, g)(u(t), t) \\ u(0) &= x, \end{aligned} \tag{5.3}$$

and let

$$D(\mu) = \{(x, g, t) \in W \times H(F) \times \mathbf{R} : t \in I(x, \varphi(\mu, g))\},$$

where $I(x, \varphi(\mu, g))$ is the maximal interval of existence of the solution to the problem (5.3) and define a flow $\pi_\mu : D(\mu) \to W \times H(F)$ by

$$\pi_\mu(x, g, t) = (u(t; x, \varphi(\mu, g)), g_t),$$

where $u(t; x, \varphi(\mu, g))$ is the solution to problem (5.3).

Remark 5.1 Theorems (4.2) and (4.3) in the previous chapter can be stated with the parameter μ no longer belonging to the interval $[a, b]$, but rather to the whole space of real numbers $\mathbf{R}$.

Now we are ready to prove our first result in this chapter, which is a global continuation theorem, similar to that in [26].

Theorem 5.2 Suppose that there is a bounded and connected open set $U \subset \mathbf{R}^n$, such that $\overline{U} \times H(F)$ is an isolating neighborhood for the flow π_0, and $h(\pi_0, I) \neq \overline{0}$, where

$$I = \left\{(x, g) \in \overline{U} \times H(F) : \pi_0(x, g, t) \in \overline{U} \times H(F) \;\; \forall t \in \mathbf{R}\right\}.$$

Let $S^+ = \left(\{0\} \times \overline{U}\right) \cup cl\{(\mu, x) \in [0, \infty) \times W$: there is a bounded solution through x for the problem (5.3) for some $g \in H(F)\}$, and let C^+ be the component of S^+ containing $\{0\} \times \overline{U}$. Also let $S^- = \left(\{0\} \times \overline{U}\right) \cup cl\{(\mu, x) \in (-\infty, 0] \times W$: there is a bounded solution through x for the problem (5.3) for some $g \in H(F)\}$, and let C^- be the component of S^- containing $\{0\} \times \overline{U}$. Then either C^+ meets $\{0\} \times \left(\mathbf{R}^n \backslash \overline{U}\right)$ or else it is unbounded in $[0, \infty) \times W$, and C^- either meets $\{0\} \times \left(\mathbf{R}^n \backslash \overline{U}\right)$ or else it is unbounded in $(-\infty, 0] \times W$.

Proof : Suppose, by way of contradiction, that C^+ is bounded in $[0, \infty) \times W$ and doesn't meet $\{0\} \times \left(\mathbf{R}^n \backslash \overline{U}\right)$. Let

$$\begin{aligned} S_o^+ &= cl\{(\mu, x) \in [0, \infty) \times W : \text{ there is a bounded solution through } x \\ &\qquad \text{for our problem for some } g \in H(F)\}, \\ C_o^+ &= \{(\mu, x) \in C^+ : \text{there is a bounded solution through } x \text{ for our problem}\}. \end{aligned}$$

Now $C_o^+ \subset C^+$, which implies that C_o^+ is bounded. Also C_o^+ is closed; the proof is similar to the proof in theorem (4.8). So let $\{(\mu_n, x_n)\} \subset C_o^+$ be a sequence that converges to a point $(\overline{\mu}, \overline{x})$. Now C_o^+ is bounded; this implies that there is a number $b > 0$ such that if $(\mu, x) \in C_o^+$ then $0 \leq \mu \leq b$, so the sequence $\{\mu_n\}$ is contained in the interval $[0, b]$, which implies that $\overline{\mu} \in [0, b]$ also. Now for any $g \in H(F)$, $\pi_{\overline{\mu}}(\overline{x}, g, t)$ is defined for $t \in I(x, \varphi(\overline{\mu}, g))$ (by assumption (B1)). From theorem (4.2) we have

$$\pi_{\mu_n}(x_n, g, t) \quad \rightarrow \quad \pi_{\overline{\mu}}(\overline{x}, g, t) \text{ as } n \rightarrow \infty \text{ for } t \in I(\overline{x}, \varphi(\overline{\mu}, g))$$

$$\Longrightarrow (\mu_n, u(t; x_n, \varphi(\mu_n, g))) \to (\overline{\mu}, u(t; \overline{x}, \varphi(\overline{\mu}, g))) \text{ as } n \to \infty \text{ for } t \in I(\overline{x}, \varphi(\overline{\mu}, g)).$$

Now, as argued in the proof of theorem (4.8), we must have that $(\overline{\mu}, u(t; \overline{x}, \varphi(\overline{\mu}, g))) \in C^+ \ \forall t \in I(\overline{x}, \varphi(\overline{\mu}, g)) = \mathbf{R}$, otherwise we would have a contradiction to the maximality of C^+; therefore, from the definition of C_o^+, $(\overline{\mu}, u(t; \overline{x}, \varphi(\overline{\mu}, g))) \in C_o^+ \ \forall t \in \mathbf{R}$. Hence, in particular $(\overline{\mu}, u(0; \overline{x}, \varphi(\overline{\mu}, g))) \in C_o^+$, that is, $(\overline{\mu}, \overline{x}) \in C_o^+$. Therefore, we proved that C_o^+ is closed and, hence, compact.

Let $\delta > 0$ be a positive real number, and let U_δ be a δ-neighborhood of C_o^+. Let

$$V = \{(\mu, x) \in U_\delta : \mu \geq 0\}.$$

Then V is open and bounded in $[0, \infty) \times \mathbf{R}^n$, since $V = U_\delta \cap ([0, \infty) \times \mathbf{R}^n)$, and $\partial V \cap C_o^+ = \phi$, where ∂V is the boundary of V in $[0, \infty) \times \mathbf{R}^n$. Let $K = S_o^+ \cap \overline{V}$, where, again, the closure is taken in $[0, \infty) \times \mathbf{R}^n$. Then K is compact in $[0, \infty) \times \mathbf{R}^n$, and $(\partial V \cap S_o^+) \cap C_o^+ = \phi$ by construction. Also there is no closed and connected subset of K that meets both C_o^+ and $\partial V \cap S_o^+$, since this again contradicts the maximality of C^+, so by the theorem (4.7), there are two disjoint compact sets A and B of K, such that

$$C_o^+ \subset A \qquad\qquad \partial V \cap S_o^+ \subset B \qquad\qquad K = A \cup B.$$

Let A_ρ be a ρ-neighborhood of A in $[0, \infty) \times W$, where $\rho < dist(A, B)$. Then $\partial A\rho \cap S_o^+ = \phi$. Let $O = A_\rho \times H(F)$. Then O is open in $[0, \infty) \times W \times H(F)$, and if $(x, g) \in I_\mu = \{(x, g) \in \overline{O}_\mu : \pi_\mu(x, g, t) \in \overline{O}_\mu \ \forall t \in \mathbf{R}\}$, then $(\mu, x, g) \notin \partial O$. Therefore by theorem (4.4), the Conley index $h(\pi_\mu, I_\mu)$ is independent of $\mu \in [0, \infty)$. But we assumed that C^+doesn't meet $\{0\} \times \left(\mathbf{R}^n \backslash \overline{U}\right)$, so we can choose δ and ρ so small that

$(A_\rho)_0 \subset U$. Therefore, $I_0 \subseteq I$. On the other hand, let $(x, g) \in I$. Then, by the definition of C_o^+, $(0, x) \in C_o^+ \subset A_\rho$. But $O = A_\rho \times H(F)$, hence $(0, x, g) \in O \Rightarrow (x, g) \in I_0$. Therefore, $I \subseteq I_0$. So $I_0 = I$ and, by hypothesis, we have $h(\pi_0, I_0) = h(\pi_0, I) \neq \overline{0}$. Therefore,

$$h(\pi_\mu, I_\mu) \neq \overline{0} \qquad\qquad \forall \mu \geq 0.$$

But $O = A_\rho \times H(F)$, and A_ρ is bounded in $[0, \infty) \times W$. Hence $(A_\rho)_\mu = \phi$ for all μ sufficiently large and, therefore, $O_\mu = \phi$. So

$$h(\pi_\mu, I_\mu) = \overline{0} \qquad\qquad \text{for all } \mu \text{ sufficiently large,}$$

which is a contradiction. Hence C^+ either meets $\{0\} \times \left(\mathbf{R}^n \backslash \overline{U}\right)$ or else it is unbounded. This completes our proof, since the proof for C^- is similar.

For our next result, we need to make the following remark.

Remark 5.3 The Conley index is, as we explained in chapter 3, a homotopy type of a pointed topological space. Since homotopy preserves connectedness, it makes sense to talk about a connected index or a disconnected index. An example of a disconnected index is $\overline{1}$.

From this point to the rest of this chapter we assume that there exists a line of trivial solutions, that is, there exists a point $q \in W$ such that

$$f(\mu, q, t) = 0 \ \forall \mu, t \in \mathbf{R}.$$

Theorem 5.4 Let μ_1 and μ_2 be two distinct real numbers such that $\mu_1 < \mu_2$, and suppose that $\{q\} \times H(F)$ is an isolated invariant set for each flow π_μ, where $\mu \in \mathbf{R} \backslash \{\mu_1, \mu_2\}$, and that

$$h\left(\pi_{\mu},\{q\}\times H\left(F\right)\right) \text{ is connected for } \mu<\mu_1, \text{ and}$$

$$h\left(\pi_{\mu},\{q\}\times H\left(F\right)\right) \text{ is disconnected for } \mu\in\left(\mu_1,\mu_2\right).$$

Let $S^+ = cl\{(\mu,x) : x \neq q, \mu \geq \mu_1$, and there is a bounded solution through x for the problem (5.3) for some $g \in H\left(F\right)\}$, and let C^+be the component of S^+ containing (μ_1, q). Assume that C^+ is non-empty. Then, either

(1) C^+ is unbounded in $[\mu_1,\infty)\times W$, or

(2) C^+ meets every point (μ, q), where $\mu\in[\mu_1,\mu_2]$.

Proof : Suppose that C^+ is bounded in $[\mu_1,\infty)\times W$ and doesn't meet the point $\left(\widetilde{\mu},q\right)$, where $\widetilde{\mu}\in(\mu_1,\mu_2]$. Then we can prove, as before, that C^+ is compact. Now by assumption $\{q\}\times H\left(F\right)$ is an isolated invariant set for each flow π_μ, where $\mu\neq\mu_1$, μ_2, also C^+ doesn't meet the point $\left(\widetilde{\mu},q\right)$. Assume that for every neighborhood of $\widetilde{\mu}$ in $\mathbf{R}$ there is a point α belonging to that neighborhood such that $(\alpha,q)\in C^+$. Let $\mu_n\in\left(\widetilde{\mu}-\frac{1}{n},\widetilde{\mu}+\frac{1}{n}\right)$ be such a point, that is, $(\mu_n,q)\in C^+$. Then we have a sequence $\{(\mu_n,q)\}$ in C^+ that converges to the point $\left(\widetilde{\mu},q\right)$, but C^+ is closed, which implies that $\left(\widetilde{\mu},q\right)\in C^+$. Therefore, we conclude that there exists a neighborhood (ζ,ξ) of $\widetilde{\mu}$ in $\mathbf{R}$ such that C^+ doesn't intersect the line $(\zeta,\xi)\times\{q\}$. Assume that $\mu_1<\zeta$, and let $\varepsilon=\xi-\zeta$, and $0<\delta<\varepsilon$, and let U be a δ-neighborhood of C^+ in $\mathbf{R}\times W$. We can choose δ so small that there is a point (μ,q) with $\mu\in(\zeta,\xi)$, such that $(\mu,q)\notin U$. Let $K=S^+\cap\overline{U}$. Then K is compact, and $C^+\cap(S^+\cap\partial U)=\phi$. So by the separation theorem, there are two disjoint compact sets A and B of K, such that

$$C^+ \subset A \qquad S^+ \cap \partial U \subset B \qquad K = A \cup B.$$

Let A_ρ be a ρ-neighborhood of A, where $\rho < dist\,(A, B) < \delta$. Let $O = A_\rho \times H\,(F)$. Then O is an open set in $\mathbf{R} \times W \times H\,(F)$, and it has the following properties :

(1) If $(x, g) \in I_\mu$, then $(\mu, x, g) \notin \partial O$.

(2) There exists a point (μ, q, g), with $\mu \in (\zeta, \xi)$, that does not belong to O.

Moreover, there exists a number $\varepsilon_o < \delta$ such that $(\mu, q) \in A_\rho$ for every $\mu \in (\mu_1 - \varepsilon_o, \mu_1 + \varepsilon_o)$. Define

$$(A_\rho)_\mu = \{(x) \in W : (\mu, x) \in A_\rho\}.$$

By lemma (4.5), there exists a number $r_1 > 0$, such that $\overline{B}\,(q, r_1) \times H\,(F)$ is an isolating neighborhood of $\{q\} \times H\,(F)$ for each flow π_μ, where $\mu \in \left[\mu_1 + \frac{\varepsilon_o}{2}, \widetilde{\mu} - \varepsilon_1\right]$, $\varepsilon_1 > 0$ is small enough that $\widetilde{\mu} - \varepsilon_1 > \mu_1 + \delta$. Now let $\mu^* < \mu_1$, then similarly there exists a number $r_2 > 0$, such that $\overline{B}\,(q, r_2) \times H\,(F)$ is an isolating neighborhood of $\{q\} \times H\,(F)$ for each flow π_μ, where $\mu \in \left[\mu^*, \mu_1 - \frac{\varepsilon_o}{2}\right]$. Now choose a number $r < \min\{r_1, r_2\}$, such that:

(1) $\left(\mu_1 - \frac{\varepsilon_o}{2}, \mu_1 + \frac{\varepsilon_o}{2}\right) \times \overline{B}\,(q, r) \subset A_\rho$.

(2) $\left(\overline{A_\rho}\right)_\mu \cap \overline{B}\,(q, r) = \phi$ for some $\mu \in (\zeta, \xi)$.

Clearly, we can find such a number (see property (2) on O above). Now let

$$V = O \cup (\mathbf{R} \times B\,(q, r) \times H\,(F)).$$

Then, by construction, we know that, if $(x, g) \in I\mu = \{(x, g) \in \overline{V}_\mu : \pi_\mu\,(x, g, t) \in \overline{V}_\mu$

$\forall t \in \mathbf{R}\}$, then $(\mu, x, g) \notin \partial V$. So by theorem (4.4),

$$h(\pi_\mu, I_\mu) \text{ is independent of } \mu \in \mathbf{R}.$$

Now let $\eta_1 \in (\zeta, \xi)$ be such that $\left(\overline{A_\rho}\right)_{\eta_1} \cap \overline{B}(q,r) = \phi$, and let $I' = \{(x,g) \in \overline{O}_{\eta_1} : \pi_{\eta_1}(x,g,t) \in \overline{O} \ \forall t \in \mathbf{R}\}$, which could be empty, but $\overline{O}_{\eta_1} = \left(\overline{A_\rho}\right)_{\eta_1} \times H(F)$ and we have $\left(\overline{A_\rho}\right)_{\eta_1} \cap \overline{B}(q,r) = \phi$; therefore, $\overline{O}_{\eta_1} \cap \left(\overline{B}(q,r) \times H(F)\right) = \phi$, so I' and $\{q\} \times H(F)$ are disjoint isolated invariant sets with disjoint isolating neighborhoods $\overline{O}_{\eta_1}$ and $\overline{B}(q,r) \times H(F)$, respectively. Now

$$I_{\eta_1} = \{(x,g) \in \overline{V}_{\eta_1} : \pi_{\eta_1}(x,g,t) \in \overline{V}_{\eta_1} \forall t \in \mathbf{R}\}$$

$$= \{(x,g) \in \overline{O}_{\eta_1} \cup \left(\overline{B}(q,r) \times H(F)\right) : \pi_{\eta_1}(x,g,t) \in \overline{O}_{\eta_1} \cup \left(\overline{B}(q,r) \times H(F)\right) \forall t \in \mathbf{R}\}$$

$$\begin{aligned} &= \{(x,g) \in \overline{O}_{\eta_1} : \pi_{\eta_1}(x,g,t) \in \overline{O}_{\eta_1} \forall t \in \mathbf{R}\} \cup \\ \{(x,g) &\in \overline{B}(q,r) \times H(F) : \pi_{\eta_1}(x,g,t) \in \overline{B}(q,r) \times H(F) \ \forall t \in \mathbf{R}\} \\ &= I' \cup \{q\} \times H(F). \end{aligned}$$

Therefore, we have

$$\begin{aligned} h\left(\pi_{\eta_1}, I_{\eta_1}\right) &= h\left(\pi_{\eta_1}, I' \cup \{q\} \times H(F)\right) \\ &= h\left(\pi_{\eta_1}, I'\right) \vee h\left(\pi_{\eta_1}, \{q\} \times H(F)\right). \end{aligned}$$

But $h(\pi_{\eta_1}, \{q\} \times H(F))$ is disconnected by assumption since $\eta_1 > \mu_1$, so $h(\pi_{\eta_1}, I_{\eta_1})$ is also disconnected, no matter what $h\left(\pi_{\eta_1}, I'\right)$ is (see the definition in chapter 3). Now choose $\eta_2 < \mu_1$, such that $(A_\rho)_{\eta_2} = \phi$. Clearly we can find such a number since A_ρ is bounded. Then

$$h(\pi_{\eta_2}, I_{\eta_2}) = h(\pi_{\eta_2}, \{q\} \times H(F)),$$

which is connected by assumption. Hence

$$h(\pi_{\eta_1}, I_{\eta_1}) \neq h(\pi_{\eta_2}, I_{\eta_2}),$$

which is a contradiction, and this proves our theorem.

Corollary 5.5 Let μ_1 and μ_2 be two distinct real numbers, such that $\mu_1 > \mu_2$, and suppose that $\{q\} \times H(F)$ is an isolated invariant set for each flow π_μ, where $\mu \in \mathbf{R} \backslash \{\mu_1, \mu_2\}$, and that

$$h(\pi_\mu, \{q\} \times H(F)) \text{ is connected for } \mu > \mu_1, \text{ and}$$

$$h(\pi_\mu, \{q\} \times H(F)) \text{ is disconnected for } \mu \in (\mu_2, \mu_1).$$

Let $S^- = cl\{(\mu, x) : x \neq q,\ \mu \leq \mu_1$, and there is a bounded solution through x for the problem (5.3) for some $g \in H(F)\}$, and let C^- be the component of S^- containing (μ_1, q). Assume that C^- is non-empty. Then either

(1) C^- is unbounded in $(-\infty, \mu_1] \times W$, or

(2) C^- meets every point (μ, q), where $\mu \in [\mu_2, \mu_1]$.

The proof of this corollary is very similar to that of theorem (5.4) and is omitted.

Definitions :

(1) The point $q \in W$ (in the assumption above) is said to be an *attractor* for our problem (5.3) if there exists an open set $U_o \subset W$ such that $q \in U_o$ and for each $x \in U_o$:

(1) $u(t;x,\varphi(\mu,g)) \in U_o$ for all $t \geq 0$, $g \in H(F)$.

(2) $u(t;x,\varphi(\mu,g)) \to q$ as $t \to \infty$, that is , for every open neighborhood V of q there is a $\tau > 0$, such that $u(t;,x,\varphi(\mu,g)) \in V$ for all $t \geq \tau$.

(2) The point q is said to be a *repellor* if there is an open set $V_o \subset W$ such that $q \in V_o$ and for each $x \in V_o$:

(1) There exists a $\tau > 0$ such that $u(t;x,\varphi(\mu,g)) \notin V_o$ for all $t \geq \tau$, $g \in H(F)$ and $u(t;x,\varphi(\mu,g)) \in V_o$ for all $t < \tau$, $g \in H(F)$.

(2) $u(t;x,\varphi(\mu,g)) \to q$ as $t \to -\infty$.

Theorem 5.6 Assume that f is even in t, that is, $f(\mu,x,t) = f(\mu,x,-t)$. Let μ_1 and μ_2 be two distinct real numbers such that $\mu_1 < \mu_2$, and suppose that $\{q\} \times H(F)$ is an isolated invariant set for each flow π_μ, where $\mu \in \mathbf{R}\backslash\{\mu_1,\mu_2\}$, and that

$$q \text{ is an attractor for } \mu < \mu_1, \text{ and}$$

$$q \text{ is a repellor for } \mu \in (\mu_1,\mu_2).$$

Let $S^+ = cl\{(\mu,x) : x \neq q,\ \mu \geq \mu_1$, and there is a bounded solution through x for the problem (5.3) for some $g \in H(F)\}$, and let C^+ be the component of S^+ containing (μ_1,q). Assume that C^+ is non-empty. Then either

(1) C^+ is unbounded in $[\mu_1,\infty)\times W$, or

(2) C^+ meets every point (μ,q), where $\mu \in [\mu_1,\mu_2]$.

Proof : Let $u(t;x,f)$ be a solution to the problem (5.1), and define $y(t)$ by

$$y(t) = u(-t;x,f).$$

Let $\tau = -t$, then

$$\begin{aligned} \frac{dy}{dt} &= \frac{d}{dt}u(\tau; x, f) \\ &= \frac{d}{d\tau}u(\tau; x, f) \cdot \frac{d\tau}{dt} \\ &= -f(\mu, u(\tau; x, f), \tau) \\ &= -f(\mu, y(t), -t). \end{aligned}$$

Therefore, $y(t)$ satisfies the equation

$$y'(t) = -f(\mu, y(t), -t). \tag{5.4}$$

Now define a function $h : \mathbf{R} \times W \times \mathbf{R} \to \mathbf{R}^n$ by

$$h(\mu, x, t) = -f(\mu, x, -t)$$

and a function $\rho : \mathbf{R} \times H(h(0, ., .)) \to \bigcup_{\mu \in \mathbf{R}} H(h(\mu, ., .))$ by

$$\rho(\mu, h_\tau(0, ., .)) = h_\tau(\mu, ., .),$$

where $\tau \in \mathbf{R}$, h_τ is the τ-translate of h, and $h_\tau(\mu, x, t) = -f_\tau(\mu, x, -t)$. Then conditions B1 and B2 are satisfied on the new functions h and ρ. Now consider the initial value problem

$$\begin{aligned} y'(t) &= \rho(\mu, g)(y(t), t) \\ y(0) &= x, \end{aligned} \tag{5.5}$$

where $g \in H(h(0, ., .))$, and let

$$\tilde{D}(\mu) = \{(x, g, t) \in W \times H(h(0, ., .)) \times \mathbf{R} : t \in I(x, \rho(\mu, g))\},$$

where $I(x,\rho(\mu,g))$ is the maximal interval of existence of the solution to problem (5.5), and define a flow $\widetilde{\pi}:\widetilde{D}(\mu) \to W \times \mathbf{R}$ by

$$\widetilde{\pi}_\mu(x,g,t) = (y(t;x,\rho(\mu,g)), g_t),$$

where $y(t;x,\rho(\mu,g))$ is the solution to problem (5.5).

Now, by assumption q is an attractor for $\mu < \mu_1$, where q is an equilibrium point for the problem (5.1), that is, $f(\mu,q,t) = 0 \ \forall \mu, t \in \mathbf{R}$. From the definition of y and h we see that q is still an equilibrium point for problem (5.4), but it becomes a repellor. Similarly, it becomes an attractor for problem (5.4) for $\mu \in (\mu_1,\mu_2)$. Now fix a $\mu \in (-\infty,\mu_1)$. For this value of μ we know that q is a repellor; therefore, there is an open set $V_o \subset W$ satisfying conditions (1) and (2) in the definition above. Consider the set $\overline{V_o} \times H(h(0,.,.))$. We can see clearly that this set is an isolating neighborhood of $\{q\} \times H(h(0,.,.))$ for the flow $\widetilde{\pi}_\mu$, and we have that for each $x \in V_o$ there exists a $\tau > 0$, such that $y(t;x,\rho(\mu,g)) \notin V_o$ for all $t \geq \tau$, $g \in H(h(0,.,.))$, and $y(t;x,\rho(\mu,g)) \in V_o$ for all $t < \tau$, $g \in H(h(0,.,.))$. Let $B \subset \overline{V_o} \times H(F)$ be an isolating block of $\{q\} \times H(h(0,.,.))$, then the exit set is non-empty. Also note that the set $V_o \times H(h(0,.,.))$ is connected and so is the isolating block B; therefore, the quotient space B/B^- is also connected, since it is the continuous image of B. So the Conley index $h\left(\widetilde{\pi}_\mu, \{q\} \times H(h(0,.,.))\right)$ is a connected quotient space with a distinguished point that belongs to it. Therefore

$$h\left(\widetilde{\pi}_\mu, \{q\} \times H(h(0,.,.))\right) \text{ is connected for } \mu < \mu_1.$$

Now fix a $\mu \in (\mu_1,\mu_2)$. Then, for this value of μ, we know that q is an attractor;

therefore, there is an open set $U_o \subset W$ satisfying conditions (1) and (2) in the definition above. Consider the set $\overline{U_o} \times H(h(0,.,.))$. Again, this set is an isolating neighborhood of $\{q\} \times H(h(0,.,.))$ for the flow $\widetilde{\pi}_\mu$, and for each $x \in U_o$ we have that $y(t; x, \rho(\mu, g)) \in U_o$ for all $t \geq 0$, $g \in H(h(0,.,.))$. Let $x_o \in \partial U_o$, and assume that there exists a $g \in H(h(0,.,.))$ such that the solution $u(t; x_o, \rho(\mu, g))$ stays in ∂U_o for all $t \in \mathbf{R}$. Now x_o belongs to ∂U_o, so there is a sequence $\{x_n\}$ of points in U that converges to x_o, but $u(t; x_n, \rho(\mu, g)) \to q$ as $t \to \infty$ and $u(t; x_o, \rho(\mu, g)) \in \partial U_o$ for all $t \in \mathbf{R}$, which contradicts the continuation of solutions with respect to initial conditions. Therefore, we can look at the set $\overline{U_o} \times H(h(0,.,.))$ as an isolating block and for the same reason we can see that solutions through any point on the boundary of U_o must go inside. This implies that the exit set is empty, so the Conley index $h\left(\widetilde{\pi}_\mu, \{q\} \times H(h(0,.,.))\right)$ is a connected quotient space with a distinguished point that does not belong to it. Therefore,

$$h\left(\widetilde{\pi}_\mu, \{q\} \times H(h(0,.,.))\right) \text{ is disconnected for } \mu \in (\mu_1, \mu_2).$$

Let $\widetilde{S}^+ = cl\{(\mu, x) : x \neq q, \mu \geq \mu_1$, and there is a bounded solution through x for problem (5.5) for some $g \in H(h(0,.,.))\}$, and let $\widetilde{C}^+$ be the component of $\widetilde{S}^+$ containing (μ_1, q). Clearly, $\widetilde{S}^+ = S^+$ and $\widetilde{C}^+ = C^+$. By assumption, C^+ is non-empty, which implies that $\widetilde{C}^+$ is also non-empty. Then by theorem (5.4), either $\widetilde{C}^+$ is unbounded in $\mathbf{R} \times W$, or it meets every point (μ, q) where $\mu \in [\mu_1, \mu_2]$. Therefore, the theorem is proved since $\widetilde{C}^+ = C^+$.

Corollary 5.7 Assume that f is even in t. Let μ_1 and μ_2 be two distinct numbers, such that $\mu_1 > \mu_2$, and suppose that $\{q\} \times H(F)$ is an isolated invariant set for each flow π_μ, where $\mu \in \mathbf{R} \backslash \{\mu_1, \mu_2\}$, and that

$$q \text{ is an attractor for } \mu > \mu_1, \text{ and}$$

$$q \text{ is a repellor for } \mu \in (\mu_2, \mu_1).$$

Let $S^- = \{(\mu, x) : x \neq q,\ \mu \leq \mu_1$, and there is a bounded solution through x for problem (5.3) for some $g \in H(F)\}$, and let C^- be the component of S^- containing (μ_1, q). Assume that C^- is non-empty. Then either

(1) C^- is unbounded in $(-\infty, \mu_1] \times W$, or

(2) C^- meets every point (μ, q), where $\mu \in [\mu_2, \mu_1]$.

CHAPTER 6

Examples and Applications

In this chapter, we discuss some examples to see how we can apply our results from chapters 4 and 5.

Example 6.1 Consider the ODE

$$u^{'}(t) = f(\mu, u(t), t), \tag{6.1}$$

where $f \in C_c([a,b] \times W \times \mathbf{R}, \mathbf{R}^n)$, $W \subset \mathbf{R}^n$ is open and connected, and f is T-periodic. Then f satisfies condition (A2) in chapter 4. And for every $\mu \in [a,b]$, we have

$$f(\mu, x, t+T) = f(\mu, x, t) \qquad\qquad \forall t \in \mathbf{R}, x \in W.$$

Therefore, for any translate f_τ of f, $\tau \in \mathbf{R}$, we have

$$f_\tau(\mu, x, t) = f(\mu, x, t+\tau) = f(\mu, x, t+\tau+T) = f_{\tau+T}(\mu, x, t) \ \ \forall t \in \mathbf{R}, x \in W,$$

which means that $f_\tau = f_{\tau+T}$ $\forall \tau \in$ $\mathbf{R}$. So the set of all translates of f is $A = \{f_t : t \in [0,T]\}$. This set is closed, as we now prove. Let $\{f_{\tau_n}\} \subset A$ be a sequence of translates that converges to $\tilde{f} \in C_c(\mathbf{R} \times W \times \mathbf{R}, \mathbf{R}^n)$. Then the sequence $\{\tau_n\} \subset \mathbf{R}$ is bounded, since it is contained in $[0,T]$, so it has a convergent subsequence which we relabel as $\{\tau_n\}$ that converges to a point $\tilde{\tau} \in [0,T]$. But $f_{\tau_n} \to \tilde{f}$ and f_{τ_n} is continuous for every n, so we have

$$\lim_{n\to\infty} f_{\tau_n}(\mu,x,t) = \lim_{n\to\infty} f(\mu,x,t+\tau_n) = f\left(\mu,x,t+\widetilde{\tau}\right) = f_{\widetilde{\tau}}(\mu,x,t).$$

Therefore, $\widetilde{f}(\mu,x,t) = f_{\widetilde{\tau}}(\mu,x,t)$ and, hence, $\widetilde{f}\in A$. Now, since A is closed, we must have $H(f(\mu,.,.)) = \overline{A} = A$, so any function in the hull of f is a translate of f. Also $H(f(\mu,.,.))$ is compact (see [6] for details). Now consider the initial value problem

$$u'(t) = f(\mu,u(t),t)$$

$$u(0) = x.$$

Let $y_1(t)$ be the solution to this problem, and let $\tau \in I(x,f)$, and

$g \in C_c([a,b] \times W \times \mathbf{R}, \mathbf{R}^n)$ be the τ-translate of f, that is, $g(\mu,x,t) = f(\mu,x,t+\tau)$ for every $\mu \in [a,b]$, $t \in \mathbf{R}$, and $x \in W$. Consider the initial value problem

$$u'(t) = g(\mu,u(t),t)$$

$$u(0) = y_1(\tau).$$

Let $y_2(t)$ be the solution to this problem, then $y_2(t) = y_1(t+\tau)$. This means that they have the same orbit, where by orbit of y_1, we mean the set $\{y_1(t) : t \in I(x,f)\}$, and the orbit of y_2 is the set $\{y_2(t) : t \in I(y_1(\tau),g)\} = \{y_1(t+\tau) : t \in I(y_1(\tau),g)\}$. So if assumption (A3) in chapter 4 is satisfied and our function f satisfies the assumptions in theorems (4.6),and (4.8), then the continuum of solutions is a continuum of solutions to the original problem

$$u'(t) = f(\mu,u(t),t).$$

See remark(4.9).

For our next example, an introduction to almost periodic functions is needed. We follow [6] closely.

Notation: α, β, ... will denote sequence $\{\alpha_n\}$, $\{\beta_n\}$, ... in $\mathbf{R}^n$. The notation $\beta \subset \alpha$ denotes β is a subsequence of α. If f and g are functions on $\mathbf{R}$, and α is a sequence in $\mathbf{R}$, then $T_\alpha f = g$ means $\lim_{n\to\infty} f(t+\alpha_n)$ exists and is equal to $g(t)$ for all $t \in \mathbf{R}$. The type of convergence (uniform, pointwise, etc.) will be specified when used. For example, we will write $T_\alpha f = g$ pointwise, $T_\alpha f = g$ uniformly, etc.

Definition: Let f be a continuous and complex valued function on $\mathbf{R}$, then f is *almost periodic* if for every α' there is an $\alpha \subset \alpha'$ such that $T_\alpha f$ exists uniformly.

If $AP = \{f : f \text{ is almost periodic}\}$, $|f| = \sup_t |f(t)|$ if $f \in AP$, then AP with this norm is a normed linear space, in fact a Banach space.

Lemma 6.2 (1) Every periodic function is almost periodic.

(2) Every almost periodic function is bounded.

Theorem 6.3 (properties of almost periodic functions)

(1) AP is an algebra (closed under addition, product, and scalar multiplication).

(2) If $f \in AP$, F is uniformly continuous on the range of f, then $F \circ f \in AP$.

(3) If $f \in AP$, $\inf_t |f(t)| > 0$, then $\frac{1}{f} \in AP$.

(4) If $f, g \in AP$, then $|f|, \min(f,g), \max(f,g) \in AP$.

(5) AP is closed under uniform limits on $\mathbf{R}$.

(6) $(AP, |.|)$ is a Banach space.

(7) If $f \in AP$, and $\frac{df}{dt}$ is uniformly continuous on $\mathbf{R}$, then$\frac{df}{dt} \in AP$.

The reader who is interested in the proofs of lemma (6.2) and theorem (6.3) can find them in [6].

Remark 6.4 The space AP contains all periodic functions. Theorem (6.3) shows that it contains all functions which are sums of periodic functions and, thus, all trigonometric polynomials. It also contains the uniform limits of trigonometric polynomials. One can actually show (but the proof is not trivial) that AP consists precisely of functions which are uniform limits of trigonometric polynomials.

Recall that, if f is periodic, then $H(f)$ consists of all translates of f. Now if $f \in AP$, then $H(f)$ may contain elements that are not translates of f. In fact, if $f(t) = \cos t + \cos\sqrt{2}t$, then $f \in AP$, since it is the sum of two periodic functions, but it is not periodic, since it takes the value 2 only at $t = 0$. Also $f(t) > -2$ for all t, and there is a sequence $\alpha' = \{\alpha_n'\} \subset \mathbf{R}$ such that $f\left(\alpha_n'\right) \to -2$. If $\alpha \subset \alpha'$ and $T_\alpha f = g$ uniformly, then $g(0) = \lim_{n\to\infty} f(\alpha_n) = -2$, and g cannot be a translate of f, but it belongs to $H(f)$ since it is the limit of a sequence of translates of f, namely $\{f_n\}$, where $f_n(t) = f(t + \alpha_n)$.

Theorem 6.5 $f \in AP$ if and only if $H(f)$ is compact in the topology of uniform convergence on $\mathbf{R}$. Furthermore, if $f \in AP$, then $H(g) = H(f)$ for all $g \in H(F)$.

Now we are ready for our next example.

Example 6.6 Let $a < 0$, and $b > 0$ be two real numbers, and consider the ODE

$$u'(t) = \mu u(t) + q(t)\, u^3(t) = f(\mu, u(t), t), \tag{6.2}$$

where $u(t) \in \mathbf{R}$, $\mu \in [a, b]$, and $q : \mathbf{R} \to \mathbf{R}$ is almost periodic. Then $f \in C_c([a, b] \times \mathbf{R} \times \mathbf{R}, \mathbf{R})$ is almost periodic. Also we have a line of trivial solutions $[a, b] \times \{0\}$, and $\{0\}$ is an isolated invariant set for each $\mu \neq 0$. Now let $F = f(b, ., .)$, then $\{0\} \times H(F)$ is an isolated invariant set for each flow π_μ, $\mu \neq 0$. Let $N_1 \times H(F)$ be an isolating neighborhood of $\{0\} \times H(F)$ for the flow π_b. Now consider the ODE

$$u'(t) = bu(t) + \lambda q(t)\, u^3(t) \qquad \lambda \in [0, 1], \tag{6.3}$$

and let $\overline{\pi}_\lambda$ be the skew-product flow associated with this equation. Then $N_1 \times H(F)$ is still an isolating neighborhood of $\{0\} \times H(F)$ for each flow $\overline{\pi}_\lambda$, $\lambda \in [0, 1]$. This implies that

$$h(\overline{\pi}_0, \{0\} \times H(F)) = h(\overline{\pi}_1, \{0\} \times H(F)). \tag{6.4}$$

Now the flow $\overline{\pi}_0$ is the skew-product flow associated with the initial value problem

$$u'(t) = bu(t) \tag{6.5}$$

$$u(0) = x,$$

which is defined by

$$\overline{\pi}_0(x, g, t) = (u(t; x, \varphi(b, g)), g_t) = (\beta(x, t), \gamma(g, t)),$$

where $g \in H(F)$, β is the flow $\beta(x,t) = u(t;x)$ on $\mathbf{R}$, where $u(t;x)$ is the solution to problem (6.5), and γ is the flow on $H(F)$ defined by $\gamma(g,t) = g_t$. It follows that $h(\overline{\pi}_0, \{0\} \times H(F)) = h(\beta \times \gamma, \{0\} \times H(F)) = h(\beta, \{0\}) \wedge h(\gamma, H(F))$. Therefore by (6.4) above, we have

$$h(\overline{\pi}_1, \{0\} \times H(F)) = h(\beta, \{0\}) \wedge h(\gamma, H(F)).$$

Now $b > 0$. This implies that $h(\beta, \{0\}) = \sum^1$. Now $h(\gamma, H(F))$ is of the form of a disjoint union of a compact connected space with a separate distinguished point (since $H(F)$ has an empty exit set under the flow γ), that is, $h(\gamma, H(F))$ is disconnected. But $h(\beta, \{0\})$ is connected, so $h(\overline{\pi}_1, \{0\} \times H(F))$ is itself connected by lemma (3.2). But $\overline{\pi}_1 = \pi_b$, and, hence,

$$h(\pi_b, \{0\} \times H(F)) \text{ is connected.} \tag{6.6}$$

On the other hand, let $N_2 \times H(F)$ be an isolating neighborhood of $\{0\} \times H(F)$ for the flow π_a and consider the ODE

$$u'(t) = au(t) + \lambda q(t) u^3(t) \qquad \lambda \in [0,1]. \tag{6.7}$$

Let $\tilde{\pi}_\lambda$ be the skew-product flow associated with this equation. Then also $N_2 \times H(F)$ is an isolating neighborhood of $\{0\} \times H(F)$ for each flow $\tilde{\pi}_\lambda$, $\lambda \in [0,1]$. This implies that

$$h(\tilde{\pi}_0, \{0\} \times H(F)) = h(\tilde{\pi}_1, \{0\} \times H(F)). \tag{6.8}$$

Now the flow $\tilde{\pi}_0$ is the skew-product flow associated with the initial value problem

$$u'(t) = au(t) \tag{6.9}$$

$$u(0) = x,$$

which is defined by

$$\widetilde{\pi}_0 (x, q, t) = (u (t; x, \varphi (a, g)), g_t) = (\alpha (x, t), \gamma (g, t)),$$

where α is the flow $\alpha (x; t) = u (t; x)$ on $\mathbf{R}$, $u (t; x)$ is the solution to problem (6.9), and γ is as above. So $h (\widetilde{\pi}_0, \{0\} \times H (F)) = h (\alpha \times \gamma, \{0\} \times H (F)) = h (\alpha, \{0\}) \wedge h (\gamma, H (F))$. Therefore, from (6.8) above we have

$$h (\widetilde{\pi}_1, \{0\} \times H (F)) = h (\alpha, \{0\}) \wedge h (\gamma, H (F)).$$

Now $a < 0$ implies that $h (\alpha, \{0\}) = \overline{1}$. So

$$h (\widetilde{\pi}_1, \{0\} \times H (F)) = h (\gamma, H (F)).$$

Therefore $h (\widetilde{\pi}_1, \{0\} \times H (F))$ is disconnected. But $\widetilde{\pi}_1 = \pi_a$ and, hence,

$$h (\pi_a, \{0\} \times H (F)) \text{ is disconnected.} \tag{6.10}$$

Comparing (6.6) and (6.10), we conclude that

$$h (\pi_a, \{0\} \times H (F)) \neq h (\pi_b, \{0\} \times H (F)).$$

Therefore, by Theorem (4.6) $(0, 0)$ is a bifurcation point, that is, for every $\varepsilon > 0$, there exists a $(\mu, x, g) \in [a, b] \times \mathbf{R}^n \times H (F)$ such that

$$\|u (t; x, \varphi (\mu, g))\| + |\mu| \leq \varepsilon,$$

where $u (t; x, \varphi (\mu, g))$ is the solution to the initial value problem

$$u' (t) = \varphi (\mu, g) (u (t), t)$$

$$u (0) = x.$$

Now if $\varphi(\mu, g)(u(t), t) = f(\mu, u(t), t+\tau)$ for some $\tau \in \mathbf{R}$, that is, $\varphi(\mu, g)$ is just a translate of $f(\mu, ., .)$. Then as in example (6.1) our solution is a solution to the original problem. But if $\varphi(\mu, g) \in H(f(\mu, ., .))$ is not a translate, then $\varphi(\mu, g)(u(t), t) = \mu u(t) + q^*(t) u^3(t)$, where $q^* \in H(q)$ is not a translate of q. Now $H(q) = H(q^*)$ (see theorem (6.5)), so $q \in H(q^*)$, hence there is a sequence $\{t_n\} \subset \mathbf{R}$ such that $q^*_{t_n} \to q$ uniformly. Setting $u_n(t) = u(t + t_n)$, and using the fact that $\|u(t)\| \leq \varepsilon - |\mu|$, we conclude that $\{u_n\}$ is a uniformly bounded and equicontinuous family of functions on the real line. It follows that there exists a subsequence $\{u_{n_k}\}$ which converges uniformly to a solution $v(t)$ of

$$u'(t) = u(t) + q(t) u^3(t)$$

$$u(0) = \lim_{n \to \infty} u_n(0)$$

and $\|v(t)\| \leq \varepsilon - |\mu| \Rightarrow \|v(t)\| + |\mu| \leq \varepsilon$. Therefore, assuming that the sequence $\{u_n(0)\}$ doesn't converge to zero, for every $\varepsilon > 0$, there exists a non-trivial solution $v(t)$ to the original problem that satisfies the above inequality. However, we cannot say the same thing about the continuum C (see theorem (4.8)), since not all the solutions on it are solution to the original problem.

Example 6.7 Let $a < 0$, and $b > 0$ be two real numbers and consider the second order ODE

$$x''(t) + \mu x'(t) + x(t) + q(t) x^3(t) = 0, \tag{6.11}$$

where $x(t) \in \mathbf{R}$, $\mu \in [a, b]$ and $q : \mathbf{R} \to \mathbf{R}$ is almost periodic. Equation (6.11) can be written as the following system of ODE

$$x'(t) = y(t)$$

$$y'(t) = -x(t) - \mu y(t) - q(t)\, x^3(t),$$

which, in matrix form, can be written as

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -\mu \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -q(t)\, x^3 \end{bmatrix}$$
$$= f(\mu, u(t), t)$$

where $u(t) = (x(t), y(t)) \in \mathbf{R}^2$. Then $f \in C_c([a, b] \times \mathbf{R}^2 \times \mathbf{R}, \mathbf{R}^2)$ is almost periodic, and we have a line of trivial solutions $[a, b] \times \{0\} \in [a, b] \times \mathbf{R}^2$ and, as in the previous example, $\{0\}$ is an isolated invariant set for each $\mu \neq 0$. We proceed as before to show that

$$h(\pi_\mu, \{0\} \times H(F)) = h(\beta_\mu, \{0\}) \wedge h(\gamma, H(F)),$$

where β_μ is the flow generated by

$$x'(t) = y(t)$$

$$y'(t) = -x(t) - \mu y(t),$$

and γ is as in example (6.6). But $h(\beta_b, \{0\}) = \sum^0$ (see theorem (3.3)) and, therefore, $h(\pi_b, \{0\} \times H(F)) = h(\gamma, H(F))$, which is disconnected. On the other hand, $h(\beta_a, \{0\}) = \sum^2$ (see theorem (3.3)), which is connected and, therefore, $h(\pi_a, \{0\} \times H(F))$

is connected (see lemma (3.2)). Hence, we have

$$h\left({}_{a}, \{0\} \times H(F)\right) \neq h\left(\pi_b, \{0\} \times H(F)\right),$$

and so $(0, 0)$ is a bifurcation point.

Now we give some examples on chapter 5.

Example 6.8 This is a very simple example that illustrates the different cases we discussed in theorems (5.4) and (5.6). Consider the ordinary differential equation

$$u' = -\mu u + u^3 = f(\mu, u), \tag{6.12}$$

where $u \in \mathbf{R}$. Note that the differential equation is autonomous, but we can still apply our result. Obviously, $f(\mu, 0) = 0$ for every $\mu \in \mathbf{R}$. If we linearize the differential equation near the equilibrium point 0, we have the following linear differential equation

$$u' = -\mu u$$

and, therefore, for $\mu < 0$, 0 is unstable, and we have $h(\beta_\mu, \{0\}) = \sum^1$, where β_μ is the flow associated with equation (6.12). On the other hand, for $\mu > 0$, 0 is asymptotically stable, and we have $h(\beta_\mu, \{0\}) = \bar{1}$. So we have

$$h(\beta_\mu, \{0\}) \text{ is connected for } \mu < 0, \text{ and}$$

$$h(\beta_\mu, \{0\}) \text{ is disconnected for } \mu > 0.$$

Also C^+ is non-empty, since for each $\mu > 0$, there are bounded solutions to the problem, $u = \mp\sqrt{\mu}$. So we can apply theorem (5.4) to conclude that C^+ is either unbounded in $[0,\infty) \times \mathbf{R}$, or it meets every point $(\mu, 0)$, where $\mu > 0$. Actually, if we solve the equation $f(\mu, u) = 0$, we find that we have two curves of equilibrium points, namely $u = 0$ and $u = \mp\sqrt{\mu}$, and our C^+ is the curve $u^2 = \mu$ and the area inclosed by it.

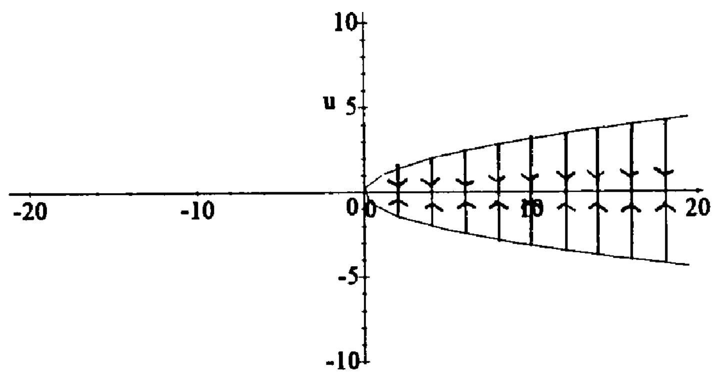

Figure 2. Graph of C^+ for example 6.8.

Now consider the differential equation

$$u' = \mu u - u^3 = f(\mu, u). \tag{6.13}$$

Here we apply theorem (5.6), since 0 is an attractor for $\mu < 0$ and a repellor for $\mu > 0$. The difference between the bifurcation for equations 6.12 and 6.13 is that, in the first case, the equilibrium point 0 is unstable for $\mu < 0$, and as the parameter μ passes through the bifurcation value $\mu = 0$, the equilibrium at the origin loses its instability, by giving it up to the two new unstable equilibria $u = \mp\sqrt{\mu}$, and becomes stable. In the second case the opposite happens where the origin loses its stability and becomes unstable.

Now we give another example where the continuum C^+ is bounded.

Example 6.9 Consider the ODE

$$u' = -\mu^2 u + u - u^3 = f(\mu, u), \tag{6.14}$$

where $\mu \in \mathbf{R}$, and $u \in \mathbf{R}$. Clearly, $f(\mu, 0) = 0$ for every $\mu \in \mathbf{R}$. Again, if we look at the linearization near the point 0, we have the following equation

$$u' = \left(-\mu^2 + 1\right) u.$$

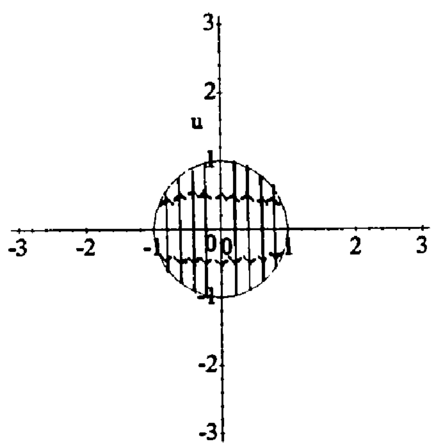

Figure 3. Graph of C^+ for example 6.9.

Therefore, if $|\mu| > 1$, the equilibrium point 0 is asymptotically stable (an attractor), and we have $h(\beta_\mu, \{0\}) = \bar{1}$, where β_μ is the flow associated with equation (6.14). But if $|\mu| < 1$, the equilibrium point 0 is unstable (a repellor), and $h(\beta_\mu, \{0\}) = \sum^1$. So we have

$$\{0\} \text{ is an attractor for } \mu < -1, \text{ and}$$

$$\{0\} \text{ is a repellor for } \mu \in (-1, 1).$$

Also, C^+ is non-empty, since for every $\mu \in (-1,1)$, we have bounded solutions for the problem, for example the equilibrium points $u = \mp\sqrt{1-\mu^2}$. Therefore, we can apply theorem (5.6) to conclude that C^+ is either unbounded in $[-1,\infty) \times \mathbf{R}$, or else it meets every point $(\mu, 0)$ where $\mu \in (-1,1)$. Now if we solve the equation $f(\mu, u) = 0$, we find out again that we have two curves of equilibrium points, namely $u = 0$ and $u = \mp\sqrt{1-\mu^2}$, so our C^+ is the circle $u^2 + \mu^2 = 1$ and the area inclosed by it. In this example, C^+ is bounded and meets every point $(\mu, 0)$, where $\mu \in (-1,1)$.

Let us now consider a more complicated example (see [1]):

Example 6.10 Let $\psi : [a,b] \times [0,\infty) \times \mathbf{R} \to \mathbf{R}$ be a function that is bounded and uniformly continuous on sets of the form $[a,b] \times K \times \mathbf{R}$, where $a < 0 < b$ and K is compact in $[0,\infty)$. Assume that for $\mu \neq 0$, and $v \in [0,\infty)$ in a small neighborhood of 0 (that might depend on μ)

$$\mu \operatorname{sign} \psi(\mu, v, t) < 0 \; \forall \, t \in \mathbf{R}.$$

Consider the two-dimensional system

$$\begin{aligned} x_1' &= \psi\left(\mu, x_1^2 + x_2^2, t\right) x_1 - x_2 \\ x_2' &= x_1 + \psi\left(\mu, x_1^2 + x_2^2, t\right) x_2. \end{aligned} \tag{6.15}$$

In polar coordinates the system can be written as

$$\begin{aligned} \theta' &= 1 \\ r' &= r\psi\left(\mu, r^2, t\right). \end{aligned}$$

The origin $(0,0)$ is an equilibrium point for every $\mu \in [a,b]/\{0\}$. For $\mu = a$ and r in a small neighborhood of 0, we have $\psi(a,r,t) > 0 \; \forall \; t \in \mathbf{R}$. Therefore, $r' > 0$ and all solutions spiral counter-clockwise away from the origin. Let $U_a \times H(F) \subset \mathbf{R}^2 \times H(F)$ be an isolating block of $\{(0,0)\} \times H(F)$, where U_a is connected. Then the exit set is non-empty and, therefore, the Conley index is in the form of a connected quotient space with a distinguished point that belongs to it; hence, $h(\pi_a, \{(0,0)\} \times H(F))$ is connected, where π_μ is the flow associated with equation (6.15). On the other hand, for $\mu = b$ and r in a small neighborhood of 0, we have $\psi(b,r,t) < 0 \; \forall \; t \in \mathbf{R}$; therefore, $r' < 0$, and all solutions spiral counter-clockwise towards the origin. Again let $U_b \times H(F) \subset \mathbf{R}^2 \times H(F)$ be an isolating block of $\{(0,0)\} \times H(F)$, then the exit set is empty, and the Conley index is of the form of a connected quotient space with a distinguished point that does not belong to it; hence, $h(\pi_b, \{0\} \times H(F))$ is disconnected; therefore, $h(\pi_a, \{0\} \times H(F)) \neq h(\pi_b, \{0\} \times H(F))$. Now we can apply theorem (4.6) to conclude that $(0,0,0)$ is a bifurcation point. As an example of the function ψ let $\psi(\mu,v,t) = v\cos^4 t - \mu$. Clearly if $\mu < 0$, then we have $\psi(\mu,v,t) > 0$ for every $v \in [0,\infty)$ and $t \in \mathbf{R}$, and if $\mu > 0$ then for $v < \mu$ we have $\psi(\mu,v,t) < 0$ for every $t \in \mathbf{R}$. Therefore for $\mu \neq 0$ and $v \in \left\{ \begin{array}{c} [0,\infty) \text{ for } \mu < 0 \\ [0,\mu) \text{ for } \mu > 0 \end{array} \right\}$ we have $\mu \operatorname{sign} \psi(\mu,v,t) < 0$ $\forall \; t \in \mathbf{R}$.

CHAPTER 7

Asymptotically Autonomous Differential Equations

In this chapter we consider parameter dependent non-autonomous ordinary differential equations that are asymptotically autonomous, in particular in the following setting:

$$u'(t) = f(\lambda, u(t)) + g(\lambda, u(t), t), \tag{7.1}$$

where $g_t \to 0$ as $|t| \to \infty$ in $H(g(\lambda, ., .))$. Here, $\lambda \in [a, b] \subset \mathbf{R}$, $u \in W$, which is an open and connected subset of $\mathbf{R}^n$ and $t \in \mathbf{R}$, also $f(\lambda, .) \in C_c(W, \mathbf{R}^n)$ and $g(\lambda, ., .) \in C_c(W \times \mathbf{R}, \mathbf{R}^n)$ are both bounded and uniformly continuous on sets of the form $[a, b] \times K$ and $[a, b] \times K \times \mathbf{R}$, respectively, where K is compact in W. We try to answer the following question: If bifurcation occurs for the limiting equation

$$u'(t) = f(\lambda, u(t)), \tag{7.2}$$

do we still have bifurcation for equation (7.1)?

Before we state our result, we need the following lemma, which will be given without proof [22]:

Lemma 7.1 Let π_1 and π_2 be local flows defined on locally compact metric spaces X_1 and X_2, respectively. Suppose $K_1 \subset X_1$ is a compact isolated invariant set for π_1 and

X_2 is compact, so that $K_1 \times X_2$ is a compact isolated invariant set for $\pi_1 \times \pi_2$. If $h(\pi_1, K_1) \neq \overline{0}$, then $h(\pi_1 \times \pi_2, K_1 \times X_2) \neq \overline{0}$.

Now consider the homotopy

$$u'(t) = f(\lambda, u(t)) + \mu g(\lambda, u(t), t), \tag{7.3}$$

where $\mu \in [0,1]$. Assume the following:

(C1) For each $x \in W$, $\lambda \in [a,b]$, $\mu \in [0,1]$, and $g^* \in H(g(\lambda,.,.))$, the initial value problem

$$\begin{aligned} u'(t) &= f(\lambda, u(t)) + \mu g^*(\lambda, u(t), t) \\ u(0) &= x \end{aligned} \tag{7.4}$$

has a unique solution.

(C2) There exists a point $q \in W$ such that

$$f(\lambda, q) = g(\lambda, q, t) = 0 \text{ for every } \lambda \in [a,b], t \in \mathbf{R},$$

$h(\beta_a, \{q\})$ is disconnected (connected), and $h(\beta_b, \{q\})$ is connected (disconnected), where, for each $\lambda \in [a,b]$ β_λ is the flow generated by equation (7.2), that is,

$$\beta_\lambda(x,t) = u(t;x),$$

where $u(t;x)$ is the solution to the initial value problem

$$\begin{aligned} u'(t) &= f(\lambda, u(t)) \\ u(0) &= x. \end{aligned}$$

Now for a fixed $\lambda \in [a, b]$, define the family of flows $\pi_{\lambda,\mu}$ on $W \times H(g(\lambda, ., .))$ by

$$\pi_{\lambda,\mu}(x, g^*, t) = (u(t; x, f + \mu g^*), g_t^*),$$

where $\mu \in [0, 1]$, $g^* \in H(g(\lambda, ., .))$, $u(t; x, f + \mu g^*)$ is the solution to the initial value problem (7.4). This defines a family of skew-product flows on the space $W \times H(g(\lambda, ., .))$.

Theorem 7.2 Suppose (C1) and (C2) hold and:

(1) there exists $\lambda_o \in [a, b]$ such that for each $\lambda \in [a, b] \setminus \{\lambda_o\}$ the set $\{q\} \times H(g(\lambda, ., .))$ is an isolated invariant set for each flow $\pi_{\lambda,\mu}$, $\mu \in [0, 1]$,

(2) for each $\lambda \in [a, b] \setminus \{\lambda_o\}$ there exists an isolating neighborhood $\overline{U_\lambda} \subset W$ of $\{q\}$ for the flow β_λ, and

$$h(\beta_\lambda, \{q\}) \neq \overline{0} \text{ for every } \lambda \in [a, b] \setminus \{\lambda_o\},$$

(3) if $u(t)$ is a solution of equation (7.4) for some $\lambda \in [a, b] \setminus \{\lambda_o\}$, $\mu \in [0, 1]$ and $g^* \in H(g(\lambda, ., .))$, such that $u(t) \in \overline{U_\lambda}$ for all $t \in \mathbf{R}$, then $u(t) \in U_\lambda$ for all $t \in \mathbf{R}$.

(4) Let $S_o = cl\{(\lambda, x) : x \neq q$ and there is a bounded solution through x for the problem (7.2)$\}$, and let C_o be the component of S_o containing (λ_o, q). Assume that C_o contains a point (λ^*, p), where $\lambda^* \neq \lambda_o$ and $p \neq q$ is an isolated equilibrium point for equation (7.2) and that there exists an isolating neighborhood $\overline{V_{\lambda^*}} \subset W$ ($\overline{V_{\lambda^*}} \cap \overline{U_{\lambda^*}} = \phi$) of $\{p\}$ for the flow β_{λ^*} satisfying: if $u(t)$ is a solution of equation (7.4) for $\lambda = \lambda^*$, $\mu \in [0, 1]$ and $g^* \in H(g(\lambda^*, ., .))$ such that $u(t) \in \overline{V_{\lambda^*}}$ for all $t \in \mathbf{R}$, then $u(t) \in V_{\lambda^*}$ for all $t \in \mathbf{R}$.

Then (λ_o, q) is a bifurcation point for the problem (7.1).

Remark 7.3 If we look at equation (7.2), we notice that under (C1), (C2) and the assumptions of theorem (7.2), all the conditions of theorem (4.6) are satisfied. In fact $\{q\}$ is isolated for each flow β_λ, $\lambda \in [a,b] \backslash \{\lambda_o\}$. Also, from (C2), $h(\beta_a, \{q\})$ is disconnected (connected), and $h(\beta_b, \{q\})$ is connected (disconnected). So $h(\beta_a, \{q\}) \neq h(\beta_b, \{q\})$. Therefore, (λ_o, q) is a bifurcation point for the limiting equation (7.2), and that is how we know that C_o (in the statement of the above theorem) exists.

Now we prove the theorem.

Proof: For a fixed $\lambda \in [a,b] \backslash \{\lambda_o\}$, let $N_\lambda = \overline{U_\lambda} \times H(g(\lambda,.,.))$. We claim that N_λ is an isolating neighborhood of $\{q\} \times H(g(\lambda,.,.))$ for each flow $\pi_{\lambda,\mu}$, $\mu \in [0,1]$. To prove our claim, suppose that the solution $u(t) = u(t; x, f + \mu g^*) \in \overline{U_\lambda}$ for all $t \in \mathbf{R}$, then we consider two possibilities:

(1) If $\mu = 0$ or $g^* = 0$, then u is a solution to $u'(t) = f(\lambda, u(t))$ and $u(t) \in \overline{U_\lambda}$ for all $t \in \mathbf{R}$. But from hypothesis (2) $\overline{U_\lambda}$ is an isolating neighborhood for the flow β_λ and therefore $u(t) \in U_\lambda$ for all $t \in \mathbf{R}$.

(2) If $0 < \mu \leq 1$ and $g^* \neq 0$, then by hypothesis (3) $u(t) \in U_\lambda$ for all $t \in \mathbf{R}$.

Therefore N_λ is an isolating neighborhood of $\{q\} \times H(g(\lambda,.,.))$ for each flow $\pi_{\lambda,\mu}$, where $\mu \in [0,1]$. So by theorem (4.3) we have

$$h(\pi_{\lambda,0}, \{q\} \times H(g(\lambda,.,.))) = h(\pi_{\lambda,1}, \{q\} \times H(g(\lambda,.,.))) \;\; \forall \lambda \in [a,b] \backslash \{\lambda_o\}, \quad (7.5)$$

where $\pi_{\lambda,1}$ is the flow associated with

$$u'(t) = f(\lambda, u(t)) + g^*(\lambda, u(t), t). \quad (7.6)$$

Now $\pi_{\lambda,0}$ is the skew-product flow associated with the limiting equation (7.2). Let γ_λ denote the flow on $H(g(\lambda,.,.))$ defined by $\gamma_\lambda(g^*,t) = g_t^*$ for $g^* \in H(g(\lambda,.,.))$ and $t \in \mathbf{R}$. Now

$$\pi_{\lambda,0}(x, g^*, t) = (u(t; x, f), g_t^*) = (\beta_\lambda(x,t), \gamma_\lambda(g^*,t)).$$

Hence $\pi_{\lambda,0} = \beta_\lambda \times \gamma_\lambda$. It follows that

$$\begin{aligned} h(\pi_{\lambda,0}, \{q\} \times H(g(\lambda,.,.))) &= h(\beta_\lambda \times \gamma_\lambda, \{q\} \times H(g(\lambda,.,.))) \\ &= h(\beta_\lambda, \{q\}) \wedge h(\gamma_\lambda, H(g(\lambda,.,.)))\ \forall \lambda \in [a,b] \backslash \{\lambda_o\}. \end{aligned}$$

So from (7.5) above, we have

$$\begin{aligned} h(\pi_{a,1}, \{q\} \times H(g(a,.,.))) &= h(\beta_a, \{q\}) \wedge h(\gamma_a, H(g(a,.,.))) \\ h(\pi_{b,1}, \{q\} \times H(g(b,.,.))) &= h(\beta_b, \{q\}) \wedge h(\gamma_b, H(g(b,.,.))). \end{aligned}$$

Clearly, $h(\pi_{a,1}, \{q\} \times H(g(a,.,.))) \neq h(\pi_{b,1}, \{q\} \times H(g(b,.,.)))$, since the first one is disconnected (the smash product of two disconnected spaces is disconnected), and the second one is connected by lemma (3.2). Therefore by theorem (4.6) (λ_o, q) is a bifurcation point for equation (7.6). Now, $g_t \to 0$ in $H(g(\lambda,.,.))$ as $|t| \to \infty$, so $g^*(\lambda,.,.)$ is either a translate of $g(\lambda,.,.)$ or else $g^*(\lambda,.,.) = 0$. Suppose that $u(t)$ is a bounded solution of equation (7.6) with $g^*(\lambda,.,.) = g_\tau(\lambda,.,.)$, then $y(t) = x(t-\tau)$ is a solution to equation (7.1). Then we can assume that in equation (7.6) either $g^*(\lambda,.,.) = g(\lambda,.,.)$ or $g^*(\lambda,.,.) = 0$. Now (λ_o, q) is a bifurcation point for equation (7.6), so let $S = cl\{(\lambda, x) : x \neq q$ and there is a bounded solution through x for problem (7.6) for some $g^* \in H(g(\lambda,.,.))\} = \{(\lambda, x) : x \neq q$ and there is a bounded solution through x for problem (7.1) or (7.2)}, and let C be the component of S containing (λ_o, q). We want to prove that the point (λ_o, q) is a bifurcation point for equation (7.1), so we need to prove that the new bifurcation continuum C is different from C_o, that is, $C \neq C_o$. Suppose by way of contradiction that $C = C_o$, which means that the bifurcation continuum for equation (7.1) is the same one that we get from the limiting equation (7.2). Now, by assumption, C_o contains a point (λ^*, p) where $\lambda^* \neq \lambda_o$ and $p \neq q$ is an isolated equilibrium point for equation (7.2), but $0 \in H(g(\lambda^*,.,.))$ and so (λ^*, p) is an isolated equilibrium point for equation (7.6) with $g^* = 0$, also $(\lambda^*, p) \in C$ since $C = C_o$. Now, as was proved or assumed earlier, $\overline{U_{\lambda^*}} \times H(g(\lambda^*,.,.))$ and $\overline{V_{\lambda^*}} \times H(g(\lambda^*,.,.))$ are isolating neighborhoods for each flow $\pi_{\lambda^*,\mu}$, $\mu \in [0,1]$. Also they are disjoint. It can be proved similarly that the set $M_{\lambda^*} = \left(\overline{U_{\lambda^*}} \cup \overline{V_{\lambda^*}}\right) \times H(g(\lambda^*,.,.))$ is an isolating neighborhood for each flow $\pi_{\lambda^*,\mu}$, $\mu \in [0,1]$. Let $I(\mu)$ denote the maximal invariant set in M_{λ^*} for the flow $\pi_{\lambda^*,\mu}$. By theorem (4.3) we get

$$h(\pi_{\lambda^*,1}, I(1)) = h(\pi_{\lambda^*,0}, I(0)).$$

But $I(0) = (\{q\} \cup \{p\}) \times H(g(\lambda^*,.,.))$, so

$$\begin{aligned} h(\pi_{\lambda^*,0}, I(0)) &= h(\pi_{\lambda^*,0}, (\{q\} \cup \{p\}) \times H(g(\lambda^*,.,.))) \\ &= h(\pi_{\lambda^*,0}, \{q\} \times H(g(\lambda^*,.,.))) \vee h(\pi_{\lambda^*,0}, \{p\} \times H(g(\lambda^*,.,.))) \\ &= (h(\beta_{\lambda^*}, \{q\}) \wedge h(\gamma_{\lambda^*}, H(g(\lambda^*,.,.)))) \vee h(\pi_{\lambda^*,0}, \{p\} \times H(g(\lambda^*,.,.))). \end{aligned}$$

But by assumption $h(\beta_{\lambda^*}, \{q\}) \neq \overline{0}$, therefore by lemma (7.1) $h(\pi_{\lambda^*,0}, \{q\} \times H(g(\lambda^*,.,.))) \neq$

$\bar{0}$, and so $h(\pi_{\lambda^*,0}, I(0)) \neq \bar{0}$. Hence,

$$h(\pi_{\lambda^*,1}, I(1)) \neq \bar{0}.$$

Now $C = C_o$, so all solutions on C are solutions of the limiting equation (7.2) and the point p is an isolated equilibrium point under the flow $\pi_{\lambda^*,1}$ with $g^* = 0$. Therefore, $I(1) = (\{q\} \cup \{p\}) \times \{0\}$. Let $s > 0$and $K(s) = \{g_t : |t| \geq s\} \cup \{0\}$. Then for each $\lambda \in [a,b]$, $K(s)$ is an isolating neighborhood of $\{0\}$ for the flow γ_λ on $H(g(\lambda,.,.))$. We claim that for all $\mu \in [0,1]$ and all $s > 0$ the set $\overline{U_{\lambda^*}} \times K(s)$ is an isolating neighborhood of $\{q\} \times \{0\}$ for the flow $\pi_{\lambda^*,\mu}$. This is so since, if $\pi_{\lambda^*,\mu}(x, g^*, t) \in \overline{U_{\lambda^*}} \times K(s)$ for all $t \in \mathbf{R}$, then $g_t^* \in K(s)$ for all $t \in \mathbf{R}$, which can only be if $g^* = 0$. Also $u(t; x, f + \mu g^*)$ will satisfy $u'(t) = f(\lambda^*, u(t))$, and $u(t) \in \overline{U_{\lambda^*}}$ for all $t \in \mathbf{R}$, so by hypothesis, $u(t) \in U_{\lambda^*}$ for all $t \in \mathbf{R}$. Therefore, our claim is proved, and we can prove similarly that for all $\mu \in [0,1]$ and $s > 0$ the set $\overline{V_{\lambda^*}} \times K(s)$ is an isolating neighborhood of $\{p\} \times \{0\}$ for the flow $\pi_{\lambda^*,\mu}$. Hence, for all $\mu \in [0,1]$ and all $s > 0$, the set $\left(\overline{U_{\lambda^*}} \cup \overline{V_{\lambda^*}}\right) \times K(s)$ is an isolating neighborhood of $(\{q\} \cup \{p\}) \times \{0\}$ for the flow $\pi_{\lambda^*,\mu}$. By theorem (4.3) we get

$$\begin{aligned} h(\pi_{\lambda^*,1}, (\{q\} \cup \{p\}) \times \{0\}) &= h(\pi_{\lambda^*,0}, (\{q\} \cup \{p\} \times \{0\})) \\ &= h(\beta_{\lambda^*}, \{q\} \cup \{p\}) \wedge h(\gamma_{\lambda^*}, \{0\}) \\ &= h(\beta_{\lambda^*}, \{q\} \cup \{p\}) \wedge \bar{0} = \bar{0}. \end{aligned}$$

Thus

$$h(\pi_{\lambda^*,1}, I(1)) = \bar{0}$$

which is a contradiction. Therefore, $C \neq C_o$ and (λ_o, q) is a bifurcation point for equation (7.1). This completes our proof.

Next we are going to give a similar result to theorem (7.2) on small perturbations:

Corollary 7.4 Suppose that (C1), (C2) and conditions (1)-(3) in the statement of theorem (7.2) are satisfied. Let $\varepsilon > 0$ and consider the differential equation

$$u'(t) = f(\lambda, u(t)) + \varepsilon g(\lambda, u(t), t). \tag{7.7}$$

Let S_o and C_o be as in theorem (7.2) and assume that C_o contains a point (λ^*, p), where $\lambda^* \neq \lambda_o$ and $p \neq q$ is an isolated equilibrium point for equation (7.2). Then there is an $\varepsilon_o > 0$ such that (λ_o, q) is a bifurcation point for equation (7.7) whenever $|\varepsilon| \leq \varepsilon_o$.

Proof: By theorem (7.2) it suffices to show that, given an isolating neighborhood $\overline{V_{\lambda^*}} \subset W$ of $\{p\}$ for the flow β_{λ^*}, there is an $\varepsilon_o > 0$ such that, for $|\varepsilon| \leq \varepsilon_o$, if $u(t)$ is a solution to

$$u'(t) = f(\lambda, u(t)) + \varepsilon\mu g^*(\lambda, u(t), t)$$

for $\mu \in [0,1]$ and $g^* \in H(g(\lambda^*, ., .))$, such that $u(t) \in \overline{V_{\lambda^*}}$ for all $t \in \mathbf{R}$, then $u(t) \in V_{\lambda^*}$ for all $t \in \mathbf{R}$. Suppose that there is no such ε_o. Then there is a sequence $\{\varepsilon_n\}$, converging to zero, such that for each n there is a solution $u_n = u_n(t)$ to

$$u'(t) = f(\lambda, u(t)) + \varepsilon_n\mu_n g_n(\lambda, u(t), t),$$

where $\mu_n \in [0,1]$ and $g_n \in H(g(\lambda^*, ., .))$, such that $u_n(t) \in \overline{V_{\lambda^*}}$ for all $t \in \mathbf{R}$ and $u_n(t_n) \in \partial V_{\lambda^*}$ for some $t_n \in \mathbf{R}$. For each n let $y_n(t) = u_n(t + t_n)$. Then y_n solves

$$u'(t) = f(\lambda, u(t)) + \varepsilon_n\mu_n (g_n)_{t_n}(\lambda, u(t), t),$$

with $y_n(t) \in \overline{V_{\lambda^*}}$ for all $t \in \mathbf{R}$ and $y_n(0) \in \partial V_{\lambda^*}$. But $\varepsilon_n \mu_n (g_n)_{t_n} \to 0$ since the sequence $\{\mu_n\}$ converges to some $\overline{\mu} \in [0,1]$ and $(g_n)_{t_n} \to 0$ by assumption. Therefore, it follows that there is a subsequence of $\{y_n\}$ that converges to a solution $y(t)$ to

$$u'(t) = f(\lambda, u(t)),$$

with $y(t) \in \overline{V_{\lambda^*}}$ for all $t \in \mathbf{R}$ and $y(0) \in \partial V_{\lambda^*}$. This contradicts the assumption that p is an isolated equilibrium point for equation (7.2), and this contradiction proves the corollary.

Now we are ready to discuss the following example that can be applied to corollary (7.4):

Example 7.5 Consider the following ODE

$$u'' + u' + \lambda u - u^3 + \varepsilon \lambda e^{-t^2} \sin t \; u^7 = 0,$$

where $u, t \in \mathbf{R}$ and $\lambda \in [-1, 1]$. This equation can be written as the following system

$$\begin{aligned} x' &= y \\ y' &= -\lambda x - y + x^3 - \varepsilon \lambda e^{-t^2} \sin t \; x^7. \end{aligned} \tag{7.8}$$

So the functions f and g in corollary (7.4) are

$$f(\lambda, x, y) = \left(y, -\lambda x - y + x^3\right), \; g(\lambda, x, y, t) = \left(0, -\lambda e^{-t^2} \sin t \; x^7\right).$$

Clearly, $f(\lambda, ., .) \in C_c(\mathbf{R}^2, \mathbf{R}^2)$and $g(\lambda, ., ., .) \in C_c\left(\mathbf{R}^2 \in \mathbf{R} \to \mathbf{R}^2\right)$, and they are both bounded and uniformly continuous on sets of the form $[-1, 1] \times K$ and $[-1, 1] \times K \times \mathbf{R}$, respectively, where K is compact in $\mathbf{R}^2$. Also $g_t \to 0$ as $|t| \to \infty$ in $H(g(\lambda,.,.,.))$. Now $f(\lambda,0,0) = g(\lambda,0,0,t) = 0$ for every $\lambda \in [-1,1]$ and $t \in \mathbf{R}$. Linearizing equation (7.8) near the equilibrium point $(0,0)$ gives us the following linear system, which in matrix form can be written as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\lambda & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

and we have the eigenvalues $\frac{-1\mp\sqrt{1-4\lambda}}{2}$. So for $\lambda = -1$, the eigenvalues are $\frac{-1-\sqrt{5}}{2} < 0$ and $\frac{-1+\sqrt{5}}{2} > 0$. Therefore, $h(\beta_{-1}, \{(0,0)\}) = \sum^1$. For $\lambda = 1$, the eigenvalues are $\frac{-1\mp\sqrt{3}i}{2}$ and, therefore, $h(\beta_1, \{(0,0\})) = \sum^0 = \overline{1}$. Also for each $\lambda \in (0, \frac{1}{4}]$ we have two negative real eigenvalues, and for each $\lambda \in (\frac{1}{4}, 1]$ we have two complex eigenvalues with negative real parts, and for each $\lambda \in [-1,0)$ we have two real eigenvalues, one of them is negative and the other is positive. Thus

$$\begin{aligned} h(\beta_\lambda, \{(0,0\})) &= \overline{1} \text{ for } \lambda \in (0,1] \\ h(\beta_\lambda, \{(0,0)\}) &= \Sigma^1 \text{ for } \lambda \in [-1,0), \end{aligned}$$

where β_λ is the flow associated with the equation

$$\begin{aligned} x' &= y \\ y' &= -\lambda x - y + x^3. \end{aligned} \tag{7.9}$$

Also for each $\lambda \in [-1,1] \backslash \{0\}$, the set $\{(0,0)\} \times H(g(\lambda,.,.,.))$ is an isolated invariant set for each flow $\pi_{\lambda,\mu}$, $\mu \in [0,1]$ associated with the equation

$$\begin{aligned} x' &= y \\ y' &= -\lambda x - y + x^3 - \varepsilon\mu\lambda e^{-t^2} \sin t \; x^7. \end{aligned} \tag{7.10}$$

Now for $\lambda \in [-1,1] \setminus \{0\}$, let $\overline{U_\lambda} \subset \mathbf{R}^2$ be an isolating neighborhood of $(0,0)$ for the flow β_λ and let $(x(t), y(t))$ be a solution of (7.10) for some $\mu \in [0,1]$ such that $(x(t), y(t)) \in \overline{U_\lambda}$ for all $t \in \mathbf{R}$, then $(x(t), y(t)) \in U_\lambda$ for all $t \in \mathbf{R}$ since $|g(\lambda, x(t), y(t), t)| \leq |x^7(t)|$, which is very small near $(0,0)$. If we look again at equation (7.9) we see that for each $\lambda \in (0,1]$ there are two non-trivial equilibrium points, namely $\left(\sqrt{\lambda}, 0\right)$ and $\left(-\sqrt{\lambda}, 0\right)$. Let $\lambda^* \in (0,1]$ and linearize equation (7.9) for $\lambda = \lambda^*$ near the equilibrium point $\left(\sqrt{\lambda^*}, 0\right)$ to get the linear system

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2\lambda^* & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

and we have the eigenvalues $\frac{-1 \mp \sqrt{1+8\lambda^*}}{2}$, so the equilibrium point is hyperbolic, since one of the eigenvalues is negative and the other is positive. Hence, $\left(\sqrt{\lambda^*}, 0\right)$ is an isolated equilibrium point for equation (7.9). Now all the assumptions in corollary (7.4) are satisfied; therefore, there is an $\varepsilon_o > 0$ such that the point $(0,0,0)$ is a bifurcation point for our problem whenever $|\varepsilon| \leq \varepsilon_o$.

CHAPTER 8

Conclusion and Future Work

This chapter concludes by summarizing our results and suggesting directions for future work.

1 Conclusion

This dissertation was concerned with the study of the structure of the solution set to parameter dependent non-autonomous ordinary differential equations. Our results can be summarized in three parts:

(1) In the first part we studied the problem

$$u'(t) = f(\mu, u(t), t),$$

where $\mu \in [a, b]$. We proved in theorem (4.6) that, under some assumptions on the function f, if there is a change in the Conley index $h(\pi_\mu, \{q\} \times H(F))$, where $f(\mu, q, t) = 0$ for every $\mu, t \in \mathbf{R}$, as we cross a special value $\mu_o \in [a, b]$, then there is a bifurcation point. Moreover in theorem (4.8) we proved the existence of a continuum bifurcating from the bifurcation point that either meets $[a, b] \times \partial W$, or meets $\{a, b\} \times W$.

(2) In the second part we considered the same problem, but with $\mu \in \mathbf{R}$, and we studied the global behavior of the solution set. First, we proved a global continuation theorem

(theorem (5.2)). Then in theorem (5.4) we proved that, if we have two distinct real numbers, μ_1 and μ_2, such that the Conley index $h(\pi_\mu, \{q\} \times H(F))$ is connected for $\mu < \mu_1$ and disconnected for $\mu \in (\mu_1, \mu_2)$, then the continuum C^+ bifurcating from (μ_1, q) is either unbounded in $[\mu_1, \infty) \times W$, or it meets every point (μ, q), where $\mu \in [\mu_1, \mu_2]$. Then in theorem (5.6) we proved a similar result on the continuum C^+ under the assumption that q is an attractor for $\mu < \mu_1$ and a repellor for $\mu \in (\mu_1, \mu_2)$.

(3) In the third part, we studied parameter dependent non-autonomous ordinary differential equations that are asymptotically autonomous, in particular the following problem

$$u'(t) = f(\mu, u(t)) + g(\mu, u(t), t),$$

where $g_t \to 0$ as $|t| \to \infty$ in $H(g(\mu, ., .))$. We proved in theorem (7.2) that, under some assumptions on the functions f and g, if there is a bifurcation point (μ_o, q) for the limiting equation

$$u'(t) = f(\mu, u(t)),$$

then (μ_o, q) is also a bifurcation point for the original problem. We then proved a similar result (corollary (7.4)) on small perturbations.

We also gave some examples throughout this dissertation to illustrate the use of our results.

2 Future Work

One possible direction for future work is to consider multiparameter bifurcation problems, that is, to study equations that involve more that one parameter. Another possible direction is to study the existence of continua bifurcating from infinity. We say that (μ_o, ∞) is a bifurcation point provided that for all $\varepsilon > 0$ there is a value μ of the parameter and a bounded solution $u = u_\mu(t)$ satisfying

$$|\mu - \mu_o| < \varepsilon \text{ and } \|u_\mu\| < \frac{1}{\varepsilon}.$$

LIST OF REFERENCES

1. J. C. ALEXANDER AND J. A. YORKE, Global bifurcation of periodic orbits, *American Journal of Mathematics* **100** (1978), 263-292.

2. R. F. BROWN, "A Topological Introduction to Nonlinear Analysis," Birkhauser, Boston, 1993.

3. E.CODDINGTON AND N. LEVINSON, "Theory of Ordinary Differential Equations," McGraw-Hill, New York, 1955.

4. C. CONLEY, "Isolated Invariant Sets and the Morse Index," CBMS, vol. 38, American Mathematical Society, Providence, RI, 1978.

5. J. GUCKENHEIMER AND P. HOLMES, "Nonlinear Oscillations, Dynamical Systems, and Bifurcation of Vector Fields," Springer-Verlag, Berlin, 1983.

6. J. HALE, "Ordinary Differential Equations," Krieger, Florida, 1969.

7. J. HALE AND H. KOCAK, "Dynamics and Bifurcation," Springer-Verlag, Berlin, 1991.

8. J. B. KELLER AND S. ANTMAN, (editors), "Bifurcation Theory and Nonlinear Eigenvalue Problems," Benjamin, New York, 1969.

9. J. L. KELLEY, "General Topology," Springer-Verlag, Berlin, 1955.

10. M. A. KRASNOSELSKI, "Topological Methods in the Theory of Nonlinear Integral Equations," Macmillan, New York, 1965.

11. J. LERAY AND J. SCHAUDER, Topologie et equations fonctionelles, *Ann. Sci. École Norm.* Sup. **51** (1934), 45-78.

12. M. MEMORY AND J. R. WARD, The Conley index and the method of averaging, *Journal of Mathematical Analysis and Applications* **158** (1991), 509-518.

13. R. K. MILLER AND G. R. SELL, Topological dynamics and its relations to integral equations and non-autonomous systems, in Dynamical systems: An International Symposium, vol. 1, Academic Press, New York (1976), 223-249.

14. J. R. MUNKRES, "Topology, A First Course," Prentice-Hall, New Jersey, 1975.

15. P. H. RABINOWITZ, Some global results for nonlinear eigenvalue problems, *Journal of Functional Analysis* **7** (1971), 487-513.

16. P. H. RABINOWITZ, On bifurcation from infinity, *Journal of Differential Equations* **14** (1973), 462-475.

17. P. H. RABINOWITZ, (editor), "Applications to Bifurcation Theory," Academic Press, New York, 1977.

18. K. P. RYBAKOWSKI, "The Homotopy Index and Partial Differential Equations," Springer Verlag, Berlin, 1987.

19. G. R. SELL, "Topological Dynamics and Ordinary Differential Equations," Van Nostrand Reinhold, London, 1971.

20. J. SMOLLER, "Shock Waves and Reaction-Diffusion Equations," Grundlehren der Math. Wissenschaften, **248**, Springer-Verlag, New York, 1983.

21. J. R. WARD, Conley index and non-autonomous ordinary differential equations, *Results in Mathematics* **14** (1988), 191-209.

22. J. R. WARD, A topological method for bounded solutions of non-autonomous ordinary differential equations, *Transactions of the American Mathematical Society* **333** (1992), 709-720.

23. J. R. WARD, Homotopy and bounded solutions of ordinary differential equations, *Journal of Differential Equations* **107** (1994), 428-445.

24. J. R. WARD, Homotopy index and asymptotically autonomous differential equations, *Journal of Differential Equations* **122** (1995), 267-281.

25. J. R. WARD, Bifurcation continua in infinite dimensional dynamical systems and applications to differential equations, *Journal of Differential Equations*, to appear.

26. J. R. WARD, A global continuation theorem and bifurcation from infinity for infinite dimensional dynamical systems, *Proc. of the Royal Society of Edinburgh*, to appear.

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