1997

An exact solution to the Einstein-Maxwell equations representing a nonspherical, highly charged object.

Govind K. Menon
University of Alabama at Birmingham

AN EXACT SOLUTION TO THE EINSTEIN-MAXWELL EQUATIONS REPRESENTING A NONSPHERICAL, HIGHLY CHARGED OBJECT

by

GOVIND K. MENON

A DISSERTATION

Submitted to the graduate faculty of The University of Alabama at Birmingham, in partial fulfillment of the requirements for the degree of Doctor of Philosophy

BIRMINGHAM, ALABAMA

1997

UMI Number: 9807893

UMI
300 North Zeeb Road
Ann Arbor, MI 48103

ABSTRACT OF DISSERTATION
GRADUATE SCHOOL, UNIVERSITY OF ALABAMA AT BIRMINGHAM

Degree Doctor of Philosophy Major Subject Physics

Name of Candidate Govind K. Menon

Committee Chair Dr. John H. Young

Title An Exact Solution to the Einstein-Maxwell Equations Representing a Nonspherical, Highly Charged Object

The Reissner-Nordström solution possesses a naked singularity when $e^2 > m^2$, where m is the mass and e is the net charge of the system. Also, the singularity at $r = 0$ is repulsive [i.e., no timelike geodesics (neutral particles) can reach the singularity]. These unusual properties of the Reissner-Nordström geometry are considered as an accident resulting from the highly symmetric nature of the space-time.

Here we wish to generalize the condition of spherical symmetry to axial symmetry and to probe into the issues of naked singularity and gravitational repulsion. To do this, we must construct a nonspherical solution to the Einstein-Maxwell set of equations in the event that $e^2 > m^2$.

The Erez-Rosen extension of the vacuum Schwarzschild solution to the non-spherical case gave one of the first physically significant solutions of the Einstein field equations. Nonvacuum extensions of the Erez-Rosen solution representing a non-spherical mass containing a very high net charge (i.e., when $e^2 > m^2$) will be discussed. The special case of spherical symmetry, as would be expected, results in the Reissner-Nordström solution.

The search for the physical singularities involves the calculation of a nontrivial scalar constructed from the Riemann curvature tensor. As it turns out, the resulting geometry does indeed possess a naked singularity. In addition, the space-time also entertains gravitational repulsion. However, unlike the Reissner-Nordström solution, it has been found that all timelike geodesics are not necessarily repelled from the origin.

ACKNOWLEDGEMENTS

I would like to offer my sincere thanks and gratitude to Dr. John H. Young, whose insightful guidance helped to make this dissertation possible. Also my thanks go to the other members of my dissertation committee. I am also thankful to Dr. Gilbert Weinstein, for overseeing two years of seminars on general relativity, which helped give me a greater understanding of the subject. In addition, acknowledgment goes to Mr. Michael Cervera with great appreciation for the splendid job done in typing this dissertation.

TABLE OF CONTENTS

Page

APPENDICES

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

General Relativity: A Primer

The general theory of relativity was proposed by Einstein in 1915 as a relativistically correct theory of gravitation. It is based upon the equivalence of the gravitational and inertial mass and upon the principle of general covariance. All fields in nature can be viewed as generating their own gravitational fields, and, in doing so, interacting gravitationally with other fields and particles. Such field interactions can be formulated by including not only the energy of fields as a gravitational source, but also the momentum and the flux of energy-momentum. One is immediately reminded of the energy-momentum tensor as a quantity that has the above-mentioned properties. This line of thinking led Einstein to propose gravitational field equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -8\pi \frac{G}{c^2} T_{\mu\nu}, \qquad (\mu, \nu = 0,1,2,3) \tag{1.1}$$

where $R_{\mu\nu}$ are the components of the Ricci curvature, R is the scalar curvature, and $T_{\mu\nu}$ is the components of the energy-momentum tensor of source fields present in an arbitrary coordinate system.

The gauge freedoms of the Einstein equation, unlike those of special relativistic theories, require no preferred set of inertial observers, but instead are infinite dimensional and are comprised of arbitrary diffeomorphisms; see, for example, Wald

[1]. The gravitational field itself is described by the metric tensor, $g_{\mu\nu}$; hence the familiar explanation of gravitation as a curvature in the fabric of space-time. The Ricci curvature is given by

$$R_{\mu\nu} = \Gamma^{\alpha}{}_{\mu\nu,\alpha} - \Gamma^{\alpha}{}_{\mu\alpha,\nu} + \Gamma^{\alpha}{}_{\beta\alpha}\Gamma^{\beta}{}_{\mu\nu} - \Gamma^{\alpha}{}_{\beta\nu}\Gamma^{\beta}{}_{\mu\alpha}. \tag{1.2}$$

The object on the right hand side of the above expression is defined in equation (1.4), and the comma denotes partial differentiation with respect to the appropriate coordinate. Unless otherwise specified, summation is to be understood over the full range of values of any repeated index.

The equivalence of gravitational and inertial mass led to the geodesic postulate which states that free particles follow geodesic trajectories. This can be viewed as a covariant generalization of special relativistic dynamics [2]. The geodesic is determined by the solution to

$$\frac{d^2 X^{\mu}}{d\tau^2} + \Gamma^{\mu}{}_{\alpha\beta}\frac{dX^{\alpha}}{d\tau}\frac{dX^{\beta}}{d\tau} = 0, \tag{1.3}$$

where τ is an affine parameter, X^{μ} is a curve representing the worldline of the particle, and $\Gamma^{\mu}{}_{\alpha\beta}$ is the Christoffel symbol of the connection, given by

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{g^{\mu\nu}}{2}\left\{g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu}\right\}. \tag{1.4}$$

Naively, one could view $\Gamma^{\mu}{}_{\alpha\beta}$ as a gravitational force. However, unlike special relativity, a free particle does not come with its share of conserved quantities. This is because, in general, a space-time does not necessarily admit Killing vector fields, and therefore, Noether's theorem would be inapplicable. Hence, solving the geodesic equation can be a nontrivial matter.

Of interest to us is the spherically symmetric solution to the Einstein-Maxwell system of equations, found independently by Reissner [3] and Nordström [4]. This was the first solution to the Einstein-Maxwell equations that demonstrated the gravitational effect of the electromagnetic field. The gravitational field is then described by the following metric

$$g=\left(1-\frac{2GM}{c^2r}+\frac{Gq^2}{4\pi\varepsilon_0c^4r^2}\right)c^2dt^2-\left(1-\frac{2GM}{c^2r}+\frac{Gq^2}{4\pi\varepsilon_0c^4r^2}\right)^{-1}dr^2-r^2d\Omega^2,$$

where r is the radial coordinate. Here M is the total mass in kilograms, q is the net charge in coulombs, and ε_0 is the permittivity of free space. In Heaviside-Lorentz units the above metric becomes

$$g=\left(1-\frac{2m}{r}+\frac{e^2}{r^2}\right)c^2dt^2-\left(1-\frac{2m}{r}+\frac{e^2}{r^2}\right)^{-1}dr^2-r^2d\Omega^2, \tag{1.5}$$

where m is related to the total mass M of the distribution by $mc^2=GM$, e is related to the total charge q of the source by $e^2c^4=4\pi Gq^2$, and $d\Omega^2$ is the standard Euclidean metric on a sphere of radius r. From this point on, we set $c=1=G$. Physically, this represents the gravitational field of a uniformly charged spherical object producing electromagnetic potential

$$A=\left(\frac{q}{r},0,0,0\right), \tag{1.6}$$

which is the familiar Coulomb potential.

The corresponding metric in the uncharged case was obtained earlier by Schwarzschild [5] and is given by

$$g = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 d\Omega^2 . \tag{1.7}$$

While the Newtonian analog of (1.5) and (1.7) are identical (since charge plays no role in Newtonian gravity), in general relativity the physical consequences are dramatically different. One crucial difference is the phenomenon of gravitational repulsion.

Objective

It is desirable to investigate the phenomenon of gravitational repulsion of a neutral particle by a charged mass. If the space-time admits a well-defined notion of a radial coordinate r, then repulsion in the radial coordinate is very much like in a Newtonian setting: $\ddot{r} > 0$ implies the force is repulsive. One needs a charged test particle to invoke such an event in Newtonian mechanics, which would be a simple Coulomb repulsion, not gravitational.

The origin of gravitational repulsion in this geometry is seen by studying the time-like radial geodesics. Parametrization by proper time gives

$$g_{00}(\dot{t})^2 + g_{11}(\dot{r})^2 = 1 . \tag{1.8}$$

Since $\frac{\partial}{\partial t}$ is a time-like Killing vector field, the energy of the particle is conserved; i.e.,

$$g_{00}(\dot{t}) = \varepsilon . \tag{1.9}$$

This gives

$$\dot{r}^2 + \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right) = \varepsilon^2 , \tag{1.10}$$

where ε is the energy per unit mass of the particle. Here the term in parenthesis is the effective potential to which the particle is subjected.

Equation (1.10) is very similar to the Kepler problem in classical mechanics, but the source of the effective potential is very different because angular momentum is the cause of such a term in classical mechanics. Therefore, the radial coordinate value of an infalling free particle must take a nonzero minimum, which leads to gravitational repulsion at this minimum value. Also, nonrelativistic particles (i.e., those with $\varepsilon < 1$) permit radial oscillations because the particle is trapped in the potential well. Figure 1 is a plot of the potential for $e^2 / m^2 > 1$.

One reaches a similar conclusion by taking the simpler approach of the pseudo-Newtonian gravitational potential as introduced by Peters [6] and extended to incorporate electromagnetic effects by Young [7]. Newtonian gravity is described completely by a potential, Φ, satisfying

$$\Delta\Phi = 4\pi\rho\,, \tag{1.11}$$

where Δ is the Laplacian and ρ is the mass density of all sources. Analogous to the electrostatic case, a gravitational energy density U_g can be defined as

$$U_g = \frac{-(\nabla\Phi)^2}{8\pi}. \tag{1.12a}$$

From the mass-energy equivalence we get

$$\rho_g = \frac{U_g}{c^2} = U_g. \tag{1.12b}$$

Including the electrostatic field energy as a source of gravitation we get

$$U_f = 2\pi(\nabla\varphi)^2, \tag{1.13}$$

where φ is the electrostatic potential [Jackson (8)]. As above, we get $\rho_f = U_f$. Setting

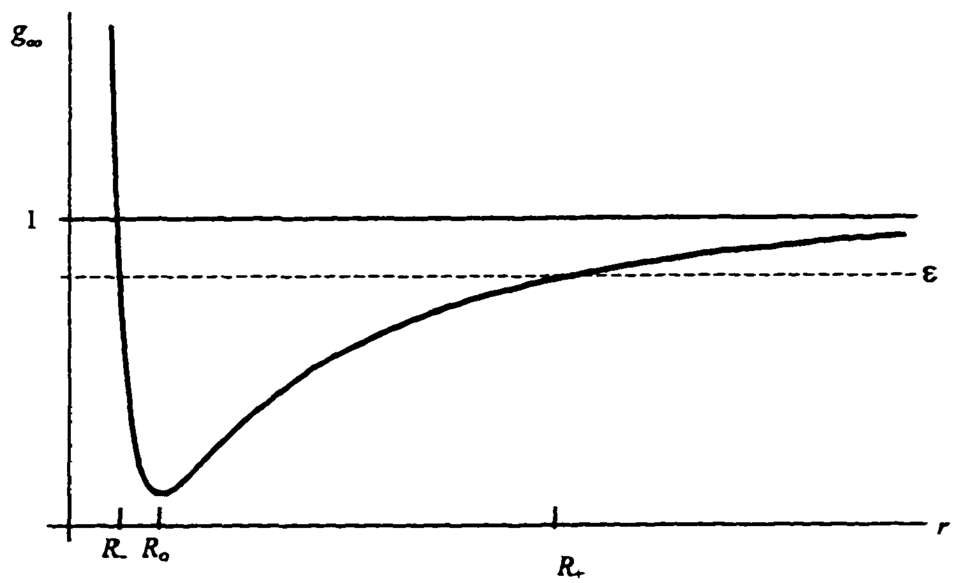

Figure 1. The effective potential in the Reissner-Nordström space-time for radial geodesics when $e^2 > m^2$.

$$\rho = \rho_g + \rho_f + \rho_m$$

in (1.11) we get the pseudo-Newtonian gravitational field equation

$$\Delta\Phi + \frac{(\nabla\Phi)^2}{2} - 2\pi(\nabla\varphi)^2 = \rho_m. \qquad (1.14)$$

In regions void of ρ_m (i.e., occupied by fields only),

$$\Delta\Phi + \frac{(\nabla\Phi)^2}{2} - 2\pi(\nabla\varphi)^2 = 0, \qquad \Delta\varphi = 0. \qquad (1.15)$$

For a spherically symmetric charged source, Φ is given by [7]

$$\Phi(r) = 2\ln\left[\exp\left(\frac{\sqrt{\pi}q}{r}\right) - \left(1 + \frac{\sqrt{\pi}M}{2q}\right)\sinh\left(\frac{\sqrt{\pi}q}{r}\right)\right], \qquad (1.16)$$

where M (in kilograms) and q (in Heaviside-Lorentz units) are the mass and charge of the source.

The similarities in the two very different approaches are clear from the plot of the pseudo-Newtonian potential (Figure 2). As before, the dynamical equation gives rise to gravitational repulsion[1] provided Q is greater in absolute value than M. Oscillatory behavior under a simple harmonic approximation has a period

$$T = \sqrt{2}\pi\left[\frac{R_0^2}{Q}\right], \qquad (1.17)$$

where the pseudo-Newtonian potential takes its minimum at R_0.

Various attempts have been made to understand the nature of gravitational repulsion [10-12]. The physical meaning of gravitational repulsion was investigated by Cohen and Gautreau [13], and later by Gron [14]. Yet another explanation was given

[1] For further details of the account see Young and Menon [9].

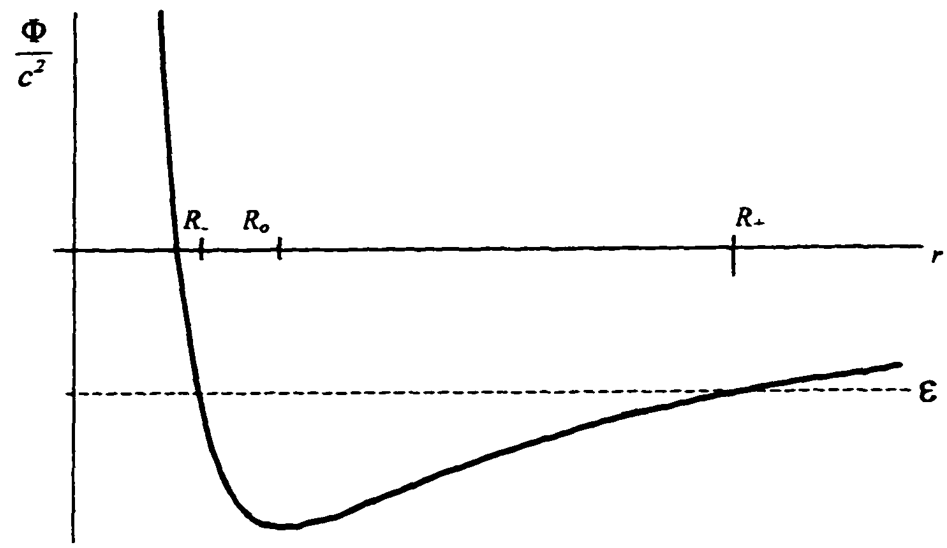

Figure 2. Plot of the pseudo-Newtonian potential versus *r*.

by Mahajan *et al.* [15] by reintroducing the concept of a gravitational force. Cohen and Gautreau compute the mass of a spacelike ball using Whittaker's theorem and come up with the expression

$$M_R = m - \frac{e^2}{R} \tag{1.18}$$

for the mass of the ball of radius R. Clearly, this quantity becomes negative for small values of R. This effective negative mass is what Cohen and Gautreau claim to be the cause of gravitational repulsion.

The spherically symmetric mass is completely devoid of structure. For the case of a structured source having quadrupole mass and charge moments, the coupled Einstein-Maxwell system of equations will be solved to give an exact metric of the Weyl class. The source structure can be represented in a very convenient way through multipole expansion in spheroidal coordinates. This method was initially carried out by Erez and Rosen [16] for the charge-free case. An extensive resurrection of interest in the Erez-Rosen geometry arose several years ago [17-24]. A major significance of the present work is that it extends the work of Erez and Rosen to include electromagnetic effects on gravitation. The space-time geometry represented is that of a mass and charge distribution with a quadrupole moment such that the absolute value of its charge to mass ratio is greater than unity.

What then are the properties of such a space-time? Does it entertain gravitational repulsion, or is repulsion due to the extreme symmetry of the Reissner-Nordström metric? Also of importance is whether or not the geometry admits a naked singularity.[2] In essence, a naked singularity is a physical singularity that is in the causal past of spatial infinity. Details of the procedure to investigate such issues will be taken up in the next chapter.

[2] The Reissner-Nordström metric admits a naked singularity when $e^2 / m^2 > 1$. This is in direct violation of the cosmic censorship conjecture proposed by Penrose [25]. Cosmologists view the naked singularity here as an accident because of its highly symmetric nature and do not regard it as a cosmologically reasonable space-time.

CHAPTER 2

A PARTICULAR SOLUTION TO THE STATIC, AXIALLY SYMMETRIC EINSTIEN-MAXWELL SYSTEM OF EQUATIONS

Weyl Geometry

It is intended to describe a static, axially symmetric space-time resulting from a charged object with a nontrivial quadrapole moment in the event that the charge, e parameter, of the source is greater than that of its mass, m. That this is possible in a curved space-time is not immediately obvious. In describing a field that satisfies a linear equation, one can think of multipole moments of a field in connection with the multipole expansion of that field. To make matters more complicated, the electromagnetic field is coupled to the gravitational field in a nontrivial way. However, in an asymptotically flat space-time, the coupling of the electromagnetic field to the gravitational field becomes weak as spatial infinity is approached, and individual pure multipoles can be identified there.

The general, static, axially symmetric Einstein-Maxwell space-time can be described by the Weyl [26] metric. In cylindrical coordinates (t,ρ,z,φ) this is given by

$$g = e^{2\psi}dt^2 - e^{2\gamma-2\psi}(d\rho^2 + dz^2) - \rho^2 e^{-2\psi}d\varphi^2, \tag{2.1}$$

where ψ and γ are functions of ρ and z only.

Since the Maxwell stress tensor is trace free ($T_\mu{}^\mu = 0$), the Einstein-Maxwell equations are [27]

$$R_{\mu\nu} = -\kappa\left[F_{\mu\alpha}F^{\alpha}{}_{\nu} + \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right], \tag{2.2}$$

where $\kappa = 8\pi$ and

$$F_{\mu\nu} \equiv A_{\nu,\mu} - A_{\mu,\nu}. \tag{2.3}$$

In a charge-free region the Maxwell field tensor must satisfy

$$\left(\sqrt{-g}F^{\mu\nu}\right)_{,\nu} = 0, \tag{2.4a}$$

and

$$F_{\mu\nu,\lambda} + F_{\lambda\mu,\nu} + F_{\nu\lambda,\mu} = 0. \tag{2.4b}$$

The electromagnetic potential considered here is of the form $A_{\mu} = (\Phi,0,0,0)$, where the electrostatic potential Φ, is assumed to be a function of ρ and z only. The above-mentioned relations yield the following system of coupled equations[1]

$$\nabla^2\psi = k^2 e^{-2\psi}\left[\Phi_{\rho}^2 + \Phi_z^2\right], \tag{2.5}$$

$$\nabla^2\Phi = 2\left[\psi_{\rho}\Phi_{\rho} + \psi_z\Phi_z\right], \tag{2.6}$$

with

$$\gamma_{\rho} = \rho\left[\psi_{\rho}^2 - \psi_z^2 - k^2 e^{-2\psi}\left(\Phi_{\rho}^2 - \Phi_z^2\right)\right], \tag{2.7}$$

and

$$\gamma_z = 2\rho\left[\psi_{\rho}\psi_z - k^2 e^{-2\psi}\Phi_{\rho}\Phi_z\right]. \tag{2.8}$$

Here $k^2 \equiv \frac{1}{2}\kappa$, and ∇^2 is the Laplacian in a cylindrical coordinate system, that is,

---

[1] It is convenient at this point to redesignate partial differentiation by omitting the comma, for example, $\Phi_{\rho} \equiv \dfrac{\partial\Phi}{\partial\rho}$.

$$\nabla^2 \equiv \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial z^2}.$$

For vacuum solutions to the Einstein equations (corresponding to $\Phi = 0$ everywhere), Erez and Rosen [16] constructed a method to yield exact solutions that correspond to a mass distribution with multipole moments and that contain the Schwarzschild solution as the spherically symmetric, or monopole, contribution to the field. Young and Bentley [28] have modified the Erez-Rosen space-time to describe the field due to a charged, structured mass. The resulting geometry, however, was restricted to cases in which the charge-to-mass parameter ratio was such that $e^2/m^2 < 1$. Here we make the necessary modifications to describe the reverse situation (i.e., $e^2/m^2 > 1$) for we might expect gravitational repulsion in this case.

The method employed by Young and Bentley [28] uses an intermediate function, $\overline{\psi}$, which, when assumed to be related to Φ through certain derivative conditions, reduces the system of equations, (2.5) – (2.8), to that of the vacuum form. They also show that the exact solutions to the problem of the static axially symmetric charge-mass distribution can then be obtained as a generalization of the Erez-Rosen method. Here we do the same, but we make appropriate changes so that the resulting geometry has $e^2/m^2 > 1$.

Introduction of an Auxiliary Function

The choice of an auxiliary function to simplify the Einstein-Maxwell system of equations has a considerable degree of freedom and has been discussed by several authors [29]. Here, the auxiliary function, $\overline{\psi}$, is related to ψ in terms of a parameter, α, as follows

$$e^{-\psi} = \alpha \cos\overline{\psi} + \sin\overline{\psi}. \qquad (2.9)$$

Equations (2.5) - (2.8) can be rewritten in terms of $\overline{\psi}$ as

$$\nabla^2\psi = \nabla\cdot\nabla\psi, \qquad (2.10a)$$

but

$$\nabla\psi = \nabla\overline{\psi}\frac{d\psi}{d\overline{\psi}}. \qquad (2.10b)$$

Therefore,

$$\nabla\cdot\nabla\psi = \nabla\cdot\left(\frac{d\psi}{d\overline{\psi}}\nabla\overline{\psi}\right)$$

$$= \left(\nabla\frac{d\psi}{d\overline{\psi}}\right)\cdot\nabla\overline{\psi} + \frac{d\psi}{d\overline{\psi}}\nabla^2\overline{\psi}. \qquad (2.10c)$$

Also,

$$\nabla\frac{d\psi}{d\overline{\psi}} = \frac{d}{d\overline{\psi}}(\nabla\psi)$$

$$= \frac{d}{d\overline{\psi}}\left(\frac{d\psi}{d\overline{\psi}}\nabla\overline{\psi}\right)$$

$$= \frac{d^2\psi}{d\overline{\psi}^2}\nabla\overline{\psi} + \frac{d\psi}{d\overline{\psi}}\frac{d}{d\overline{\psi}}(\nabla\overline{\psi}). \qquad (2.10d)$$

Since the last term in the above equation vanishes, (2.10c) gives

$$\nabla^2\psi = \frac{d^2\psi}{d\overline{\psi}^2}(\nabla\overline{\psi})^2 + \frac{d\psi}{d\overline{\psi}}\nabla^2\overline{\psi}. \qquad (2.10e)$$

From equation (2.9) we have

$$\psi = -ln\left[\alpha cos\overline{\psi} + sin\overline{\psi}\right]. \tag{2.11a}$$

Therefore,

$$\frac{d\psi}{d\overline{\psi}} = -\frac{\left[-\alpha\, sin\,\overline{\psi} + cos\,\overline{\psi}\right]}{\left[\alpha\, cos\,\overline{\psi} + sin\,\overline{\psi}\right]}, \tag{2.11b}$$

and so

$$\frac{d^2\psi}{d\overline{\psi}^2} = 1 + \frac{\left[\alpha\, sin\,\overline{\psi} - cos\,\overline{\psi}\right]^2}{\left[\alpha\, cos\,\overline{\psi} + sin\,\overline{\psi}\right]^2}. \tag{2.11c}$$

Substituting equation (2.11b) and (2.11c) into equation (2.10e), equation (2.5) can be written as

$$\left[\frac{\alpha\, sin\,\overline{\psi} - cos\,\overline{\psi}}{\alpha\, cos\,\overline{\psi} + sin\,\overline{\psi}}\right]\nabla^2\overline{\psi} + \left[1 + \frac{(\alpha\, sin\,\overline{\psi} - cos\,\overline{\psi})^2}{(\alpha\, cos\,\overline{\psi} + sin\,\overline{\psi})^2}\right]\left(\overline{\psi}_\rho^2 + \overline{\psi}_z^2\right)$$

$$= k^2\left(\alpha\, cos\,\overline{\psi} + sin\,\overline{\psi}\right)^2\left[\Phi_\rho^2 + \Phi_z^2\right]. \tag{2.12}$$

We now make the assumption that $\overline{\psi}$ and Φ can be related as

$$\begin{Bmatrix}\overline{\psi}_\rho \\ \overline{\psi}_z\end{Bmatrix} = -\frac{k(\alpha\, cos\,\overline{\psi} + sin\,\overline{\psi})^2}{\sqrt{1+\alpha^2}}\begin{Bmatrix}\Phi_\rho \\ \Phi_z\end{Bmatrix}. \tag{2.13}$$

Equation (2.12) now reduces to

$$\nabla^2\overline{\psi} = 0. \tag{2.14a}$$

A similar calculation using $\overline{\psi}$ and equation (2.13) can be used to replace equation (2.7) with

$$\gamma_\rho = -\rho\left(\overline{\psi}_\rho^2 - \overline{\psi}_z^2\right), \tag{2.14b}$$

and equation (2.8) can be written as

$$\gamma_z = -2\rho\overline{\psi}_\rho\overline{\psi}_z . \tag{2.14c}$$

Computational details for equation (2.14b) and (2.14c) can be found in Appendix A. Equation (2.14a) - (2.14c) are simply the vacuum equations of the Weyl-type field and can be used as replacements for the field equations (2.5) - (2.8). The above equations are much simpler and are effectively decoupled; they are, in fact, exactly of the vacuum form solved by Erez and Rosen [16].

Since $\overline{\psi}$ satisfies the Laplace equation, the possible solutions for $\overline{\psi}$ are well known. Having obtained the necessary solution for $\overline{\psi}$, getting Φ is only a matter of integrating equation (2.13). Integrating equation (2.14b) and (2.14c) gives γ. Before we proceed in the direction of the above-mentioned computations, we must check to see that equations (2.13) and (2.6) are consistent. To this end we note

$$\Phi_\rho = -\frac{\sqrt{1+\alpha^2}}{k} e^{2\psi}\overline{\psi}_\rho , \tag{2.15a}$$

and

$$\Phi_{\rho\rho} = -\frac{\sqrt{1+\alpha^2}}{k}\left[2e^{2\psi}\psi_\rho\overline{\psi}_\rho + e^{2\psi}\overline{\psi}_{\rho\rho}\right]$$

$$= 2\psi_\rho\Phi_\rho - \frac{\sqrt{1+\alpha^2}}{k} e^{2\psi}\overline{\psi}_{\rho\rho} . \tag{2.15b}$$

Also,

$$\frac{1}{\rho}\Phi_\rho = -\frac{\sqrt{1+\alpha^2}}{k}\frac{e^{2\psi}}{\rho}\overline{\psi}_\rho . \tag{2.15c}$$

Similarly,

$$\Phi_{zz} = 2\psi_z\Phi_z - \frac{\sqrt{1+\alpha^2}}{k}e^{2\psi}\overline{\psi}_{zz}. \tag{2.15b}$$

Since $\nabla^2\Phi \equiv \Phi_{\rho\rho} + \frac{1}{\rho}\Phi_\rho + \Phi_{zz}$, the above relations give

$$\nabla^2\Phi = 2\left[\psi_\rho\Phi_\rho + \psi_z\Phi_z\right] - \frac{\sqrt{1+\alpha^2}}{k}\left[\overline{\psi}_{\rho\rho} + \frac{1}{\rho}\overline{\psi}_\rho + \overline{\psi}_{zz}\right]e^{2\psi}$$

$$= 2\left[\psi_\rho\Phi_\rho + \psi_z\Phi_z\right] - \frac{\sqrt{1+\alpha^2}}{k}e^{2\psi}\nabla^2\overline{\psi}. \tag{2.15e}$$

Equation (2.15e) in conjunction with equation (2.14a) gives

$$\nabla^2\Phi = 2\left[\psi_\rho\Phi_\rho + \psi_z\Phi_z\right], \tag{2.15f}$$

which is equation (2.6). Therefore, we conclude that equation (2.13) does not give rise to any internal inconsistencies.

An expression for Φ can be given in terms of $\overline{\psi}$ by integrating equation (2.13) and is given by

$$\Phi = -\frac{\sqrt{1+\alpha^2}}{k}\int\frac{d(\tan\overline{\psi})}{(\alpha+\tan\overline{\psi})^2}$$

$$= \frac{\sqrt{1+\alpha^2}}{k}\left(\frac{1}{\alpha+\tan\overline{\psi}}\right) + C. \tag{2.16}$$

We fix the integration constant such that at spatial infinity $\Phi \to 0$. For an asymptotically flat space-time $g_{00} \to 1$ as one approaches spatial infinity. Clearly, from equation (2.9), this is possible when $\overline{\psi} = \pi/2$ at spatial infinity. Therefore, we must have

$$\Phi \to 0 \text{ as } \overline{\psi} \to \pi/2, \tag{2.17}$$

that is, the electromagnetic potential is given by

$$\Phi = \frac{\sqrt{1+\alpha^2}}{k}\frac{1}{\alpha + \tan\overline{\psi}}. \tag{2.18}$$

A Generalization to the Reissner-Nordström Solution

We now proceed to solve the field equations by first transforming to a spheroidal system of coordinates. The coordinate transformations are given by

$$\rho = \frac{m}{\alpha}\sqrt{\lambda^2+1}\sqrt{1-\mu^2}, \tag{2.19a}$$

$$z = \frac{m}{\alpha}\lambda\mu, \tag{2.19b}$$

where m is a constant yet to be determined. Since $\overline{\psi}$ satisfies Laplace's equation, $\overline{\psi}$ satisfies

$$\left[\left(\lambda^2+1\right)\overline{\psi}_\lambda\right]_\lambda + \left[\left(1-\mu^2\right)\overline{\psi}_\mu\right]_\mu = 0 \tag{2.20}$$

in the new coordinate system [30]. Equations (2.14b) and (2.14c), when written in the spheroidal coordinate system, become

$$\gamma_\lambda = \frac{\left(1-\mu^2\right)}{\left(\lambda^2+\mu^2\right)}\left[-\lambda\left(1+\lambda^2\right)\overline{\psi}_\lambda^2 + \lambda\left(1-\mu^2\right)\overline{\psi}_\mu^2 + 2\mu\left(1+\lambda^2\right)\overline{\psi}_\mu\overline{\psi}_\lambda\right] \tag{2.21}$$

and

$$\gamma_\mu = \frac{\left(1+\lambda^2\right)}{\left(\lambda^2+\mu^2\right)}\left[-\mu\left(1+\lambda^2\right)\overline{\psi}_\lambda^2 + \mu\left(1-\mu^2\right)\overline{\psi}_\mu^2 - 2\lambda\left(1-\mu^2\right)\overline{\psi}_\lambda\overline{\psi}_\mu\right]. \tag{2.22}$$

Details of the above computation can be found in Appendix B.

Equation (2.20) is separable in the usual manner, that is, by setting

$$\overline{\psi}(\lambda,\mu) = \Lambda(\lambda)M(\mu).$$

Choosing $\ell(\ell+1)$ as the separation constant, gives separated equations for Λ and M. They are

$$\left(\lambda^2+1\right)\frac{d^2\Lambda}{d\lambda^2}+2\lambda\frac{d\Lambda}{d\lambda}-\ell(\ell+1)\Lambda=0, \tag{2.23}$$

and

$$\left(1-\mu^2\right)\frac{d^2M}{d\mu^2}-2\mu\frac{dM}{d\mu}+\ell(\ell+1)M=0. \tag{2.24}$$

Equation (2.24) is simply the Legendre's equation. We take the Legendre polynomials as solutions to equation (2.24). Then $\overline{\psi}$ has the general solution

$$\overline{\psi}=\sum a_\ell \Lambda_\ell P_\ell(\mu), \tag{2.25}$$

where $\ell = 0,1,2,\ldots$, and a_ℓ's are just the coefficients of the expansion.

Proposition 1: For $\ell = 0$, $\Lambda = \tan^{-1}(\lambda)$ is a solution to (2.23).

Proof:

Let

$$\Lambda = \tan^{-1}(\lambda).$$

Then

$$\frac{d\Lambda}{d\lambda}=\frac{1}{1+\lambda^2}$$

and

$$\frac{d^2\Lambda}{d\lambda^2}=\frac{-2\lambda}{\left(1+\lambda^2\right)^2}.$$

Substituting the above in (2.23), we get

$$\left(\lambda^2+1\right)\frac{d^2\Lambda}{d\lambda^2}+2\lambda\frac{d\Lambda}{d\lambda}=\left(\lambda^2+1\right)\frac{(-2\lambda)}{\left(1+\lambda^2\right)^2}+2\lambda\frac{1}{\left(1+\lambda^2\right)}$$

$$=0.$$

Therefore, $\Lambda=\tan^{-1}(\lambda)$ is a solution to equation (2.23) when $\ell=0$.

We temporarily pause to analyze the space-time corresponding to the $\ell=0$ solution for $\overline{\psi}$. More precisely,

$$\begin{aligned}\overline{\psi}_0 &= \Lambda_0(\lambda)P_0(\mu)\\ &= \tan^{-1}(\lambda)\end{aligned} \tag{2.26}$$

is a valid solution for $\overline{\psi}$. Clearly $\overline{\psi}_0$ has no angular (μ) dependence, and hence Φ and γ are spherically symmetric. As a consequence of the Birkoff Theorem [31] the space-time corresponding to (2.26) must be isometric to the Reissner-Nordström solution. To better understand this isometry, we calculate the electromagnetic potential Φ, using (2.18) to give

$$\Phi(\overline{\psi}_0)=\frac{\sqrt{1+\alpha^2}}{k}\frac{1}{\alpha+\lambda}. \tag{2.27}$$

We now transform to yet another set of coordinates (r,θ), to rewrite Φ in a familiar form. The coordinate transformations are given by

$$\lambda\equiv\alpha\left(\frac{r}{m}-1\right),\quad \mu\equiv\cos\theta, \tag{2.28}$$

where $0\le r<\infty$, and $0\le\theta\le\pi$. In this coordinate system, the electromagnetic potential Φ takes the form

$$\Phi=\frac{q}{r}, \tag{2.29}$$

provided we set

$$\alpha = \frac{m}{\sqrt{e^2 - m^2}}, \tag{2.30}$$

where $e^2 = (kq)^2$ and m is the mass parameter of the space-time. Here q is the net charge of the mass distribution. Obviously, we must take e^2 to be greater than m^2. Since here $\Phi = q/r$, we are guaranteed to get the Reissner-Nordström metric as the unique solution. A quick computation of g_{00} will easily convince the reader of this fact. Here

$$\begin{aligned} g_{00} &= e^{2\psi} \\ &= \frac{1}{\left[\alpha \cos\overline{\psi}_0 + \sin\overline{\psi}_0\right]^2}. \end{aligned} \tag{2.31}$$

Also, $\tan\overline{\psi}_0 = \lambda$ implies

$$\sin\overline{\psi}_0 = \frac{\lambda}{\sqrt{1+\lambda^2}}, \text{ and } \cos\overline{\psi}_0 = \frac{1}{\sqrt{1+\lambda^2}}. \tag{2.32}$$

Therefore,

$$\begin{aligned} g_{00} &= \frac{1+\lambda^2}{(\alpha+\lambda)^2} \\ &= \frac{m^2}{r^2\alpha^2}\left[1+\alpha^2\left(\frac{r^2}{m^2}+1-\frac{2r}{m}\right)\right] \\ &= 1 - \frac{2m}{r} + \frac{e^2}{r^2}, \end{aligned} \tag{2.33}$$

which is as expected. Indeed, upon calculating the other terms of the metric we find

$$ds^2 = \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)dt^2 - \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \quad (2.34)$$

which is the Reissner-Nordström metric.

A generalization of the Reissner-Nordström solution can be obtained by taking for equation (2.25)

$$\overline{\psi} = \overline{\psi}_0 + a_\ell \Lambda_\ell P_\ell . \quad (2.35)$$

In the Erez-Rosen problem, the expansion coefficient, a_ℓ, can be related to the mass multipole moment of order ℓ by comparing the general form of g_{00} with its asymptotic form, $g_{00} \to \left(1 + 2\phi / c^2\right)$, where ϕ is the Newtonian gravitational potential. However, no such generalization can be made in a charged spacetime. When $e^2 / m^2 < 1$, Young and Bentley [28] argue that expectations of continuity in the family of solutions can be used to conclude the same result (i.e., looking at the geometry as $e \to 0$). This is not possible in our case since $e^2 / m^2 > 1$ and $e \to 0$ would imply $m \to 0$, and no multipole moments exist in this limit. However, it turns out that the $\ell = 1$ solution corresponds to the case when the electric dipole moment is nontrivial. Since what we have in mind are homogenous charge distributions (i.e., $\rho \geq 0$ or $\rho \leq 0$, where ρ is the electric charge density), we should not expect any meaningful $\ell = 1$ solution (because the net dipole moment of the electric potential can be made to vanish). The space-time of interest to us is obtained by taking a linear combination of the $\ell = 0$ and the $\ell = 2$ solutions for $\overline{\psi}$. Keep in mind that, either way, they are valid solutions to the Einstein-Maxwell system of equations.

Proposition 2: For $\ell = 2$, $\Lambda = \left(3\lambda^2 + 1\right)\cot^{-1}\lambda - 3\lambda$ is a solution to (2.23).

Proof:

Let

$$\Lambda = \left(3\lambda^2 + 1\right)\cot^{-1}\lambda - 3\lambda .$$

Then

$$\frac{d\Lambda}{d\lambda} = 6\lambda\cot^{-1}\lambda - \frac{\left(3\lambda^2 + 1\right)}{1+\lambda^2} - 3,$$

and

$$\frac{d^2\Lambda}{d\lambda^2} = 6\cot^{-1}\lambda - \frac{6\lambda}{1+\lambda^2} - \left[\frac{6\lambda\left(1+\lambda^2\right) - 2\lambda\left(3\lambda^2+1\right)}{\left(1+\lambda^2\right)^2}\right].$$

For $\ell = 2$, the left side of equation (2.23) is

$$\left(\lambda^2+1\right)\frac{d^2\Lambda}{d\lambda^2} + 2\lambda\frac{d\Lambda}{d\lambda} - 6\Lambda$$

$$= \left(\lambda^2+1\right)6\cot^{-1}\lambda - 6\lambda - 6\lambda + 2\lambda\frac{\left(3\lambda^2+1\right)}{1+\lambda^2} + 12\lambda^2\cot^{-1}\lambda$$

$$-2\lambda\frac{\left(3\lambda^2+1\right)}{1+\lambda^2} - 6\lambda - 6\left(3\lambda^2+1\right)\cot^{-1}\lambda + 18\lambda = 0,$$

which is the right side of equation (2.23).

Proposition 3: For $\Lambda = \left(3\lambda^2+1\right)\cot^{-1}\lambda - 3\lambda$, $\lim_{\lambda\to\infty}\Lambda = 0$, and $\Lambda \approx -\frac{1}{3\lambda^3}$ + higher order terms.

Proof:

For large λ

$$\cot^{-1}\lambda \approx \left(\frac{1}{\lambda} - \frac{1}{3\lambda^3} + \ldots\right).$$

Therefore,

$$\Lambda \approx (3\lambda^2+1)\left(\frac{1}{\lambda}-\frac{1}{3\lambda^3}\right)-3\lambda=(3\lambda^2+1)\frac{(3\lambda^3-\lambda)}{3\lambda^4}-3\lambda$$

$$=\frac{9\lambda^5}{3\lambda^4}-\frac{3\lambda^3}{3\lambda^4}+\frac{3\lambda^3}{3\lambda^4}-\frac{\lambda}{3\lambda^4}-3\lambda$$

$$=-\frac{1}{3\lambda^3}.$$

Therefore $\Lambda \approx -\frac{1}{3\lambda^3}$ + higher order terms. Clearly $\lim_{\lambda\to\infty}\Lambda=0$.

As per (2.35) and the following discussion, for $\overline{\psi}$ we choose the solution

$$\overline{\psi}=\tan^{-1}\lambda+\overline{q}\left[(3\lambda^2+1)\cot^{-1}\lambda-3\lambda\right]P_2(\mu), \tag{2.36}$$

where $\overline{q}=a_2$. Also $\lim_{\lambda\to\infty}\overline{\psi}=\pi/2$, which implies $\lim_{\lambda\to\infty}g_{00}=1$. Thus, $\overline{\psi}$ as defined in equation (2.36) has all the necessary properties consistent with asymptotic flatness.

The equation to be solved for γ involve derivatives of $\overline{\psi}$ which are readily obtained from (2.36) and are

$$\overline{\psi}_\lambda=\frac{1}{1+\lambda^2}+\overline{q}\left[6\lambda\cot^{-1}\lambda-\frac{(3\lambda^2+1)}{1+\lambda^2}-3\right]\left[\frac{1}{2}(3\mu^2-1)\right] \tag{2.39}$$

and

$$\overline{\psi}_\mu=3\overline{q}\left[(3\lambda^2+1)\cot^{-1}\lambda-3\lambda\right]\mu. \tag{2.40}$$

Substituting (2.39) and (2.40) into the equation for γ_μ, equation (2.22) becomes

$$\gamma_\mu=\frac{-\mu}{\lambda^2+\mu^2}\left[1+\frac{\overline{q}}{2}\left\{6\lambda\cot^{-1}\lambda-\frac{2(3\lambda^2+2)}{1+\lambda^2}\right\}(3\mu^2-1)(1+\lambda^2)\right]^2$$

$$+\frac{\mu}{\lambda^2+\mu^2}\left[9(1+\lambda^2)(1-\mu^2)\overline{q}^2\mu^2\left\{(3\lambda^2+1)\cot^{-1}\lambda-3\lambda\right\}^2\right]$$

$$+\frac{\mu}{\lambda^2+\mu^2}\left[-6\bar{q}\lambda\left(1-\mu^2\right)\left\{\left(3\lambda^2+1\right)\cot^{-1}\lambda-3\lambda\right\}\left\{1+\frac{\bar{q}}{2}\left[6\lambda\cot^{-1}\lambda\right.\right.\right.$$

$$\left.\left.\left.-\frac{2\left(3\lambda^2+2\right)}{1+\lambda^2}\right]\left(1+\lambda^2\right)\left(3\mu^2-1\right)\right\}\right]. \qquad (2.41)$$

The integration of γ_μ is obtained in a straightforward manner with the result

$$\gamma=-\frac{1}{2}\ln\left(\lambda^2+\mu^2\right)-\frac{\bar{q}}{2}\left(1+\lambda^2\right)\pi_1\left[3\mu^2-\left(3\lambda^2+1\right)\ln\left(\lambda^2+\mu^2\right)\right]$$

$$-\frac{\bar{q}^2}{8}\left(1+\lambda^2\right)^2\pi_1^2\left[9\frac{\mu^4}{2}-3\mu^2\left(3\lambda^2+2\right)+\left(1+6\lambda^2+9\lambda^4\right)\ln\left(\lambda^2+\mu^2\right)\right]$$

$$+\frac{9}{2}\bar{q}^2\left(1+\lambda^2\right)\pi_2^2\left[-\frac{\mu^4}{2}+\mu^2\left(1+\lambda^2\right)-\left(\lambda^2+\lambda^4\right)\ln\left(\lambda^2+\mu^2\right)\right]$$

$$-3\bar{q}\lambda\pi_2\left[\left(1+\lambda^2\right)\ln\left(\lambda^2+\mu^2\right)-\mu^2\right]-\frac{3}{2}\bar{q}^2\lambda\left(1+\lambda^2\right)\pi_1\pi_2\left[-3\frac{\mu^4}{2}+\mu^2\left(4+3\lambda^2\right)\right.$$

$$\left.-\left(1+4\lambda^2+3\lambda^4\right)\ln\left(\lambda^2+\mu^2\right)\right]+f(\lambda). \qquad (2.42)$$

Here $f(\lambda)$ is the integration constant, and π_1 and π_2 are given by

$$\pi_1=6\lambda\cot^{-1}\lambda-\frac{2\left(3\lambda^2+2\right)}{1+\lambda^2}$$

and

$$\pi_2=\left(3\lambda^2+1\right)\cot^{-1}\lambda-3\lambda.$$

Applying the condition[2] $\gamma|_{\mu=1}=0$, the constant of integration is easily determined and the final form for γ is

[2] As we shall see, the condition $\gamma|_{\mu=1}=0$ leads to the result $\lim_{\lambda\to\infty}\gamma=0$.

$$\gamma = \frac{1}{2}\ln\left(\frac{\lambda^2+1}{\lambda^2+\mu^2}\right) + \frac{\bar{q}}{2}(1+\lambda^2)\pi_1\left[3(1-\mu^2) + (3\lambda^2+1)\ln\left(\frac{\lambda^2+\mu^2}{\lambda^2+1}\right)\right]$$

$$+\frac{\bar{q}^2}{8}(1+\lambda^2)^2\pi_1^2\left[\frac{9}{2}(1-\mu^4) + 3(2+3\lambda^2)(\mu^2-1) + (9\lambda^4+1+6\lambda^2)\ln\left(\frac{\lambda^2+1}{\lambda^2+\mu^2}\right)\right]$$

$$+\frac{9}{2}\bar{q}^2(1+\lambda^2)\pi_2^2\left[\frac{1}{2}(1-\mu^4) + (\mu^2-1)(1+\lambda^2) + \lambda^2(1+\lambda^2)\ln\left(\frac{\lambda^2+1}{\lambda^2+\mu^2}\right)\right]$$

$$+3\bar{q}\lambda\pi_2\left[(\mu^2-1) + (1+\lambda^2)\ln\left(\frac{\lambda^2+1}{\lambda^2+\mu^2}\right)\right]$$

$$+\frac{3}{2}\bar{q}^2\lambda(1+\lambda^2)\pi_1\pi_2\left[\frac{3}{2}(\mu^4-1) + (3\lambda^2+4)(1-\mu^2)\right.$$

$$\left.+(4\lambda^2+1+3\lambda^4)\ln\left(\frac{\lambda^2+\mu^2}{\lambda^2+1}\right)\right], \qquad (2.43)$$

which is the expression for γ for the form of $\bar{\psi}$ chosen. Equation (2.43) has the desired asymptotic behavior, that is, $\lim_{\lambda\to\infty}\gamma = 0$, as can easily be checked.[3] The parameter $\bar{q}$ can be better understood by looking at the asymptotic behavior of the electromagnetic potential Φ. For large values of r,

$$\Phi \approx \frac{q}{r} + \frac{q\bar{q}}{3}\frac{(e^2-m^2)P_2(\mu)}{r^3}. \qquad (2.44)$$

At large distances from the source, we expect the expression for the electromagnetic

[3] From (2.41) we see that $\lim_{\lambda\to\infty}\gamma_\mu = 0$.

Also, $\gamma(\mu) - \gamma|_{\mu=1} = \int_1^\mu \gamma_{\mu'}d\mu'$.

Since $\gamma|_{\mu=1} = 0$, we have

$\lim_{\lambda\to\infty}\gamma(\mu) = \lim_{\lambda\to\infty}\int_1^\mu \gamma_{\mu'}d\mu' = \int_1^\mu \lim_{\lambda\to\infty}\gamma_{\mu'}d\mu' = 0$, since $\gamma_{\mu'}$ is continuous.

potential to agree with its special relativistic analog. Therefore,

$$\bar{q} = \frac{3Q}{q\left(e^2 - m^2\right)}, \tag{2.45}$$

where Q is the classical electric quadrupole moment and q is the net charge of the distribution. Computational details for (2.44) can be found in Appendix C.

To summarize, we have obtained a one-parameter family of solutions to the Einstein-Maxwell system of equations for a charged object with a nontrivial quadrupole moment. The question remains as to what the topology of such a geometry is. Physically speaking, we are looking for the singularities in the space-time. Unlike special relativity, where the ambient manifold is $\Re^4$, in general relativity, the topology can be nontrivial if the geometry is not globally hyperbolic [1]. The issues of physical singularities will be our primary concern in the next chapter.

CHAPTER 3

CALCULATION OF SCALAR CURVATURES AND INFINITIES

It is intended to calculate coordinate invariant quantities so as to separate the coordinate singularities from the physical singularities. We start by calculating the metric in the $(t, \lambda, \mu, \varphi)$ coordinate system. The Weyl metric can be rewritten as

$$g = e^{2\psi} c^2 dt^2 - \frac{m^2}{\alpha^2}\left[\frac{\lambda^2(1-\mu^2)}{\lambda^2+1} + \mu^2\right] e^{2\gamma - 2\psi} d\lambda^2$$

$$-\frac{m^2}{\alpha^2}\left[\frac{\mu^2(\lambda^2+1)}{(1-\mu^2)} + \lambda^2\right] e^{2\gamma-2\psi} d\mu^2 - \frac{m^2}{\alpha^2}(\lambda^2+1)(1-\mu^2) e^{-2\psi} d\varphi^2. \qquad (3.1)$$

Clearly the metric is singular when $\mu = 1$, but we do not expect this to be a physical singularity since the spherical coordinate is singular along the z axis ($\mu = 1$). From the expression for $e^{-\psi}$ in (2.9) and also from (3.1) we see the metric is singular when

$$\tan\overline{\psi} = -\alpha, \qquad (3.2)$$

since

$$g_{00} = e^{2\psi} = \left[\alpha \cos\overline{\psi} + \sin\overline{\psi}\right]^{-2}. \qquad (3.3)$$

Also, from (2.43) we see that γ may not be well defined when

$$\gamma = 0 = \mu. \qquad (3.4)$$

Physical singularities may be detected by computing scalars derived from the curvature tensor [1]. The scalar curvature R is zero everywhere in the Einstein-

Maxwell geometry. In light of (2.2), the simplest, nontrivial curvature scalar that can be constructed in the Einstein-Maxwell geometry is $R_{\mu\nu}R^{\mu\nu}$.

Since $R_{\mu\nu}$ is given in terms of $F_{\mu\nu}$ in equation (2.2), we begin by computing $F_{\mu\nu}$. Also, $A = \left(\Phi(\lambda,\mu),0,0,0\right)$, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$; therefore, the only non-vanishing terms of the Maxwell stress tensor are

$$F_{01} = \partial_0 A_1 - \partial_1 A_0 = -\frac{\partial\Phi}{\partial\lambda}, \tag{3.5}$$

$$F_{10} = -F_{10} = \frac{\partial\Phi}{\partial\lambda}, \tag{3.6}$$

$$F_{02} = \partial_0 A_2 - \partial_2 A_0 = -\frac{\partial\Phi}{\partial\mu}, \tag{3.7}$$

and

$$F_{20} = \frac{\partial\Phi}{\partial\mu}. \tag{3.8}$$

We also need $F_{\mu\nu}$ to compute $R_{\mu\nu}R^{\mu\nu}$. The nonvanishing independent terms of $F^{\mu\nu}\left(\equiv g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}\right)$ are

$$F^{01} = g^{00}g^{11}F_{01}$$

$$= \frac{\alpha^2 e^{-2\gamma}}{m^2\left[\frac{\lambda^2(1-\mu^2)}{(\lambda^2+1)}+\mu^2\right]}\frac{\partial\Phi}{\partial\lambda}, \tag{3.9}$$

and

$$F^{02} = g^{00}g^{22}F_{02}$$

$$= \frac{\alpha^2 e^{-2\gamma}}{m^2\left[\frac{\mu^2(\lambda^2+1)}{(1-\mu^2)}+\lambda^2\right]}\frac{\partial\Phi}{\partial\mu}. \tag{3.10}$$

The scalar

$$F_{\alpha\beta}F^{\alpha\beta} = F_{01}F^{01} + F_{02}F^{02} + F_{10}F^{10} + F_{20}F^{20}$$

$$= 2\left[F_{01}F^{01} + F_{02}F^{02}\right]. \tag{3.11}$$

The above can be rewritten as

$$F_{\alpha\beta}F^{\alpha\beta} = \frac{-2\alpha^2 e^{-2\gamma}}{m^2}\left[\frac{1}{\left(\frac{\lambda^2(1-\mu^2)}{\lambda^2+1}+\mu^2\right)}\left(\frac{\partial\Phi}{\partial\lambda}\right)^2\right.$$

$$\left.+\frac{1}{\left(\frac{\mu^2(\lambda^2+1)}{1-\mu^2}+\lambda^2\right)}\left(\frac{\partial\Phi}{\partial\mu}\right)^2\right]. \tag{3.12}$$

The nontrivial components of the Ricci tensor can now be calculated; they are

$$R_0^{\ 0} = -k\left[F_{0\alpha}F^{\alpha 0} + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\right]$$

$$= \frac{k}{2}\left[F_{01}F^{01} + F_{02}F^{02}\right], \tag{3.13}$$

$$R_1^{\ 1} = -k\left[F_{10}F^{01} + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\right], \tag{3.14}$$

$$R_1^{\ 2} = -k\left[F_{10}F^{02}\right], \tag{3.15}$$

$$R_2^{\ 1} = -k\left[F_{20}F^{01}\right], \tag{3.16}$$

$$R_2^{\ 2} = -k\left[F_{20}F^{02} + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\right], \tag{3.17}$$

and

$$R_3^{\ 3} = -\frac{k}{4}F_{\alpha\beta}F^{\alpha\beta}. \tag{3.18}$$

A similar calculation of $R^{\alpha}{}_{\beta}$ enables us to evaluate $R_{\mu\nu}R^{\mu\nu}$.

$$\begin{aligned} R_{\alpha\beta}R^{\alpha\beta} &= R_{\alpha}{}^{\beta}R^{\alpha}{}_{\beta} \\ &= \frac{k^2}{8}\left[F_{\alpha\beta}F^{\alpha\beta}\right]^2 + k^2\left[F_{10}F^{01} + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\right]^2 \\ &+ 2k^2\left[F_{10}F^{02}\right]\left[F^{10}F_{02}\right] + k^2\left[F_{20}F^{02} + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\right]^2. \end{aligned} \tag{3.19}$$

But

$$\begin{aligned} &\left(F_{10}F^{02}\right)\left(F^{10}F_{02}\right) \\ &= \frac{1}{c^4}\left(\frac{\partial\Phi}{\partial\lambda}\frac{\partial\Phi}{\partial\mu}\right)^2 \frac{e^{-4\gamma}\left(1-\mu^2\right)\left(\lambda^2+1\right)}{\left[\mu^2\left(\lambda^2+1\right)+\lambda^2\left(1-\mu^2\right)\right]\left[\lambda^2\left(1-\mu^2\right)+\mu^2\left(\lambda^2+1\right)\right]} \\ &\geq 0. \end{aligned} \tag{3.20}$$

From (3.20) it is quite clear that every term of the right side of (3.19) is greater than or equal to zero. Therefore, $R_{\alpha\beta}R^{\alpha\beta}$ is infinite if any of the terms in the right side of (3.19) are infinite, and in particular if $F_{\alpha\beta}F^{\alpha\beta}$ is infinite.

As mentioned before, the space-time metric is singular when $\mu = 1$; in particular, $g^{\varphi\varphi}(\mu = 1)$ is not defined. A quick inspection of equations (3.5) - (3.10) leads to the conclusion that F_{10}, F_{20}, F^{10} and F^{20} are finite when $\mu = 1$ provided

Φ_λ, and Φ_μ are finite. Also, from (3.19) we see that $R_{\alpha\beta}R^{\alpha\beta}$ is finite when $\mu = 1$ since $R_{\mu\nu}R^{\mu\nu}$ is made of well-behaved combinations of F_{10}, F_{20}, F^{10}, and F^{20} so long as the electromagnetic potential has well-behaved coordinate derivatives. Therefore, we can conclude that $\mu = 1$ is only a coordinate singularity. Points in space-time where (3.2) is satisfied are valid candidates for physical singularities. We now proceed to analyze the above-mentioned situation case by case.

The rest of the chapter involves taking limits, and we shall continually use simple results from "limit theory" to make the necessary conclusions. Here we simply state the required theorems without proof.[1]

Theorem 1. Let f and g be functions such that $\lim\limits_{x\to x_0} f = C$, and $\lim\limits_{x\to x_0} g = L$; then $\lim\limits_{x\to x_0} fg = CL$.

Theorem 2. Let f and g be functions such that $\lim\limits_{x\to x_0} f = \infty$, and $\lim\limits_{x\to x_0} g = L$; then $\lim\limits_{x\to x_0} fg = \infty$, when $L > 0$, and $\lim\limits_{x\to x_0} fg = -\infty$ when $L < 0$.

Theorem 3. If $\lim\limits_{x\to x_0} f / g$ results in the indeterminate form $0/0$ or ∞/∞, then $\lim\limits_{x\to x_0} f(x)/g(x) = \lim\limits_{x\to x_0} f'(x)/g'(x)$ provided the latter limit exists (or is infinite).

Let λ_0, μ_0 be such that $\tan\overline{\psi}(\lambda_0, \mu_0) = -\alpha$. The singularities in the Reissner-Nordström geometry are well known. Therefore, in all of the following calculations, we shall assume that $\overline{q} \neq 0$.

[1] The interested reader may refer to any standard texts in elementary real analysis: for example see. Bartle [32].

Case 1.

Let $\mu_0 \neq 0$ and $\mu_0 \neq 1$. Then γ is finite; also,[2]

$$\mu_0 \neq 0 \rightarrow \left.\frac{\partial\overline{\psi}}{\partial\mu}\right|_{\mu_0} \neq 0 . \tag{3.21}$$

From the second term in (3.12), and (3.21), we see that $F_{\alpha\beta}F^{\alpha\beta}(\lambda_0,\mu_0)$ is infinite, since

$$\tan\overline{\psi}(\lambda_0,\mu_0) = -\alpha \quad \Rightarrow \quad \alpha\cos\overline{\psi} + \sin\overline{\psi} = 0 , \tag{3.22}$$

and so from theorem 2 we have

$$\left.\left(\frac{\partial\Phi}{\partial\mu}\right)^2\right|_{(\lambda_0,\mu_0)} = \frac{\left(1+\alpha^2\right)}{k^2}\left(\frac{\partial\overline{\psi}}{\partial\mu}\right)^2 \left.\frac{1}{\left(\alpha\cos\overline{\psi} + \sin\overline{\psi}\right)^4}\right|_{(\lambda_0,\mu_0)} = \infty . \tag{3.23}$$

Therefore, $R_{\alpha\beta}R^{\alpha\beta}$ is infinite when $\mu_0 \neq 0$, and $\mu_0 \neq 1$, and $\tan\overline{\psi}(\lambda_0,\mu_0) = -\alpha$, and so (λ_o,μ_o) is a physical singularity.

Case 2.

Let $\mu_0 = 1$. Then, as before, γ is finite. Now the effective second term (using theorem 1) in (3.12) is

$$\frac{\left(1-\mu^2\right)}{\left[\mu^2\left(\lambda^2+1\right)+\lambda^2\left(1-\mu^2\right)\right]}\left(\frac{\partial\Phi}{\partial\mu}\right)^2$$

which is of the form $0/0$ at $(\lambda_0,1)$. By using theorem 3, the value of the above

---

[2] Suppose $\frac{\partial\overline{\psi}}{\partial\mu} = 0$. Then $\tan\left[\tan^{-1}\lambda + 0\right] = -\alpha$. This implies $\lambda = -\alpha$.
Therefore, $\left(3\alpha^2+1\right)Cot^{-1}(-\alpha) + 3\alpha = 0$. This is impossible since $\alpha > 0$.

expression as $\mu \to 1$ can be calculated.

$$\lim_{\mu \to 1} \frac{(1-\mu^2)}{\left[\mu^2(\lambda_0^2+1)+\lambda_0^2(1-\mu^2)\right]}\left(\frac{\partial \Phi}{\partial \mu}\right)^2$$

$$= \frac{(1+\alpha^2)}{k^2(\lambda_0^2+1)}\left(\frac{\partial \overline{\psi}}{\partial \mu}\right)\lim_{\mu \to 1} \frac{-2\mu}{4\left[\alpha \cos\overline{\psi} + \sin\overline{\psi}\right]^3\left(-\alpha \sin\overline{\psi} + \cos\overline{\psi}\right)}. \tag{3.24}$$

Therefore, from (3.22) we have

$$\left|\lim_{\mu \to 0} F_{\alpha\beta}F^{\alpha\beta}\right| \to \infty,$$

or equivalently

$$\lim_{\mu \to 1}\left|R_{\alpha\beta}R^{\alpha\beta}(\lambda_0,\mu)\right| = \infty,$$

that is, $(\lambda_0, 1)$ is a physical singularity if $\tan\overline{\psi}(\lambda_0,\mu_0) = -\alpha$.

Case 3.

Let $\mu_o = 0$ and $\lambda_0 \neq 0$; then $\gamma(\lambda_0, 0)$ is finite. Once again from (3.12) we have

$$\lim_{\mu \to 0}\left|F_{02}F^{02}\right|$$

$$= C\lim_{\mu \to 0} \frac{(1-\mu^2)\mu^2\left[(3\lambda_0^2+1)\cot^{-1}\lambda_0 - 3\lambda_0\right]^2}{\left[\mu^2(\lambda_0^2+1)+\lambda_0^2(1-\mu^2)\right]\left[\alpha \cos\overline{\psi} + \sin\overline{\psi}\right]^4}, \tag{3.25}$$

which is again of the form $0/0$ at $(\lambda_0 \neq 0, \mu_0 = 0)$ (where C is a nonzero positive constant). Again using L'Hopital's rule (theorem 3), we get

$$\lim_{\mu \to 0}\left|F_{02}F^{02}\right| = \infty. \tag{3.26}$$

Therefore, $\lim_{\mu \to 0} R_{\alpha\beta} R^{\alpha\beta} \to \infty$ when $tan(\overline{\psi}(\lambda_0, \mu_0)) = -\alpha$, where $\mu_0 = 0$ and $\lambda_0 \neq 0$.

Therefore, $(\lambda_0 \neq 0, \mu_0 = 0)$ such that $tan(\overline{\psi}(\lambda_0 \neq 0, 0)) = -\alpha$ is a physical singularity.

Case 4.

Let $\lambda_0 = 0$ and $\mu_0 = 0$ and $tan\,\overline{\psi}(\lambda_0, \mu_0) = tan\left(-\frac{\overline{q}\pi}{4}\right) = -\alpha$. From the expression for γ we find that

$$\lim_{\lambda \to 0} e^{-2\gamma} = h(\mu)\mu^{2+8\overline{q}+8\overline{q}^2}, \tag{3.27}$$

where $h(\mu)$ is nonvanishing, continuous, bounded function given by the expression

$$h(\mu) = e^{-2\left[(1-\mu^2)\left\{-6\overline{q}-\overline{q}^2\left(12+\frac{9\pi^2}{8}\right)\right\}-9\overline{q}^2(\mu^4-1)\left(1+\frac{\pi^2}{16}\right)\right]}. \tag{3.28}$$

From (3.12) we have

$$\lim_{\lambda \to 0} F_{01} F^{01} =$$

$$\lim_{\lambda \to 0} \frac{-2\alpha^2 e^{-2\gamma}(\lambda^2 + 1)}{m^2\left[\lambda^2(1-\mu^2) + \mu^2(\lambda^2+1)\right]} \frac{(\partial\overline{\psi}/\partial\lambda)^2}{\left[\alpha \cos\overline{\psi} + \sin\overline{\psi}\right]^4}. \tag{3.29}$$

Also, from (2.39) we have that

$$\lim_{\lambda \to 0} \frac{\partial\overline{\psi}}{\partial\lambda} = 1 - 2\overline{q}(3\mu^2 - 1) \tag{3.30}$$

is finite and well defined, and so from the theorem 1, (3.29), and (3.30) we have

$$\lim_{\lambda \to 0} F_{01} F^{01} = \frac{-2\alpha^2 h(\mu)\mu^{8\overline{q}+8\overline{q}^2}}{m^2\left[\alpha \cos\overline{\psi} + \sin\overline{\psi}\right]^4}\left[1 - 2\overline{q}(3\mu^2 - 1)\right]^2. \tag{3.31}$$

If $\overline{q} \neq -\frac{1}{2}$, then from (3.30) we have that $\frac{\partial\overline{\psi}}{\partial\lambda} \neq 0$. Therefore,

$$\left|\lim_{\mu\to 0}\lim_{\lambda\to 0} F_{01}F^{01}\right| = \infty \text{ if } 8\bar{q}^2 + 8\bar{q} \le 0. \tag{3.32}$$

that is,

$$\left|\lim_{\mu\to 0}\lim_{\lambda\to 0} F_{01}F^{01}\right| = \infty \text{ when } \bar{q} \in \left[-1,-\frac{1}{2}\right) \cup \left(-\frac{1}{2},0\right]. \tag{3.33}$$

Suppose $\bar{q} = -\frac{1}{2}$, then

$$\lim_{\lambda\to 0} F_{01}F^{01} = \frac{-2\alpha^2 h(\mu)\left\{1+\left(3\mu^2-1\right)\right\}^2}{m^2\mu^2\left(\alpha\cos\bar{\psi}+\sin\bar{\psi}\right)^4}. \tag{3.34}$$

The above expression is certainly of the form $0/0$ at $\mu_0 = 0$. Taking limits using theorem 3, we get

$$\left|\lim_{\mu\to 0}\lim_{\lambda\to 0} F_{01}F^{01}\right| = \infty, \tag{3.35}$$

or equivalently

$$\lim_{\mu\to 0}\lim_{\lambda\to 0} R_{\alpha\beta}R^{\alpha\beta} = \infty \text{ when } \bar{q} \in [-1,0]. \tag{3.36}$$

Otherwise, the above limit is still undetermined. After successive applications of theorem 3 and theorem 1 on (3.31), we get

$$\lim_{\mu\to 0}\lim_{\lambda\to 0} F_{01}F^{01} = \lim_{\mu\to 0}\frac{\mu^{8\bar{q}^2+8\bar{q}-8}}{\left(\textit{finite nonzero term}\right)}$$

$$= \infty \text{ if and only if } 8\bar{q}^2 + 8\bar{q} - 8 < 0, \tag{3.37}$$

or equivalently

$$\lim_{\mu\to 0}\lim_{\lambda\to 0} R_{\alpha\beta}R^{\alpha\beta} = \infty \text{ if } \bar{q} \in \left(\frac{-1-\sqrt{5}}{2},\frac{-1+\sqrt{5}}{2}\right). \tag{3.38}$$

In addition, $\lim_{\mu \to 0} \lim_{\lambda \to 0} F_{01}F^{01}$ is finite if $\bar{q} \notin \left(\frac{-1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2} \right)$. The next step is to see whether we can extract any more information from the term

$$F_{02}F^{02} = \frac{-2\alpha^2 e^{-2\gamma}(1-\mu^2)\mu^2}{m^2\left[\mu^2(\lambda^2+1)+\lambda^2(1-\mu^2)\right]} \frac{\left(\frac{1}{\mu}\frac{\partial \bar{\psi}}{\partial \mu}\right)^2}{\left[\alpha \cos\bar{\psi} + \sin\bar{\psi}\right]^4}. \tag{3.39}$$

Accordingly, we find that

$$\lim_{\mu \to 0} \lim_{\lambda \to 0} F_{02}F^{02} = (nonzero\ const) \lim_{\mu \to 0} \frac{\mu^{2+8\bar{q}+8\bar{q}^2}}{\left[\alpha \cos\bar{\psi} + \sin\bar{\psi}\right]^4} \tag{3.40}$$

$$= \infty \text{ if } 2+8\bar{q}+8\bar{q}^2 \le 0. \tag{3.41}$$

However, if $\bar{q} \notin (-1,0)$, $2+8\bar{q}+8\bar{q}^2 > 0$. That is, when $\bar{q} \notin (-1,0)$, the above limit is undetermined. Using theorem 3, we find

$$\lim_{\mu \to 0} \lim_{\lambda \to 0} F_{02}F^{02} = (nonzero\ const) \lim_{\mu \to 0} \frac{\mu^{8\bar{q}+8\bar{q}^2}}{\left[\alpha \cos\bar{\psi} + \sin\bar{\psi}\right]^3}$$

$$< (nonzero\ const) \lim_{\mu \to 0} \frac{\mu^{8\bar{q}+8\bar{q}^2}}{\left[\alpha \cos\bar{\psi} + \sin\bar{\psi}\right]^4}. \tag{3.42}$$

Therefore, from (3.31) it clear that we gain no new information. It turns out that the mixed term in (3.19) gives us no further information. Therefore,

$$\lim_{\mu \to 0} \lim_{\lambda \to 0} R_{\alpha\beta}R^{\alpha\beta} = \left\{ \begin{array}{c} \infty \text{ when } \bar{q} \in \left(\frac{-1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2} \right) \\ finite\ otherwise \end{array} \right\}, \tag{3.43}$$

when $\tan\left(\frac{\bar{q}\pi}{4}\right) = \alpha$.

Finally let $\bar{q}$ and α be such that $tan\left(\frac{\bar{q}\pi}{4}\right) \neq \alpha$. Therefore, while $tan\,\bar{\psi}(0,0) \neq -\alpha$, $\gamma(0,0)$ may not be finite.

Case a. Let $\bar{q} \neq -1/2$.

Then from (2.39) and (3.30)

$$\left.\frac{\partial\bar{\psi}}{\partial\lambda}\right|_{\lambda=0=\mu} = 1 - \frac{\bar{q}}{2}(0-4)$$

$$= 1 + 2\bar{q}$$

$$\neq 0\,. \tag{3.44}$$

Then

$$\lim_{\lambda\to 0} F_{01}F^{01} = \lim_{\lambda\to 0} \frac{-2\alpha^2 e^{-2\gamma}(\lambda^2+1)}{m^2\left[\lambda^2(1-\mu^2)+\mu^2(\lambda^2+1)\right]}\left(\frac{\partial\Phi}{\partial\lambda}\right)^2$$

$$= (nonzero\ constant)\frac{\mu^{2+8\bar{q}+8\bar{q}^2}}{\mu^2}. \tag{3.45}$$

Therefore,

$$\left|\lim_{\mu\to 0}\lim_{\lambda\to 0} F_{01}F^{01}\right| = \infty \text{ if and only if } \bar{q} \in \left(-1,-\frac{1}{2}\right) \cup \left(-\frac{1}{2},0\right). \tag{3.46}$$

Or equivalently

$$\lim_{\mu\to 0}\lim_{\lambda\to 0} R_{\mu\nu}R^{\mu\nu} = \infty \text{ if } \bar{q} \in \left(-1,-\frac{1}{2}\right) \cup \left(-\frac{1}{2},0\right) \tag{3.47}$$

and when $tan\,\bar{\psi}(0,0) \neq -\alpha$. The other terms in the expression for $R_{\alpha\beta}R^{\alpha\beta}$ do not give us any further information, and so

$$\lim_{\mu\to 0}\lim_{\lambda\to 0} R_{\mu\nu}R^{\mu\nu} = \infty \text{ if and only if } \bar{q} \in \left(-1,-\frac{1}{2}\right) \cup \left(-\frac{1}{2},0\right). \tag{3.48}$$

provided $\bar{q} = -\frac{1}{2}$.

Case b. Let $\bar{q} = -\frac{1}{2}$.

Then from (3.12) we have

$$\lim_{\mu\to 0}\lim_{\lambda\to 0} F_{01}F^{01} = (\text{nonzero constant}) \lim_{\mu\to 0} \frac{\left(1+\left(3\mu^2-1\right)\right)^2}{\mu^2}$$

$$= \lim_{\mu\to 0} 36\mu^2 - 6, \tag{3.49}$$

which is finite. Therefore, $\left|\lim_{\mu\to 0}\lim_{\lambda\to 0} F_{01}F^{01}\right|$ is finite when $\bar{q} = -1/2$. Similarly,

$$\lim_{\mu\to 0}\lim_{\lambda\to 0} F_{02}F^{02} = (\text{nonzero constant}) \lim_{\mu\to 0} \mu^{2+8\bar{q}+8\bar{q}^2}. \tag{3.50}$$

Since $2+8\bar{q}+8\bar{q}^2 = 0$ when $\bar{q} = -1/2$, we have that the above expression is finite.

It turns out that a similar calculation using the mixed term $\left(F_{01}F^{02}\right)\left(F^{10}F_{02}\right)$ in $R_{\mu\nu}R^{\mu\nu}$ gives us no further information. Therefore, from (3.48) we have

$$\lim_{\mu\to 0}\lim_{\lambda\to 0} R_{\mu\nu}R^{\mu\nu} = \infty \text{ if and only if } \bar{q} \in \left(-1,-\frac{1}{2}\right) \cup \left(-\frac{1}{2},0\right), \tag{3.51}$$

when $\tan\bar{\psi}(0,0) \neq -\alpha$.

CHAPTER 4

DISCUSSION AND CONCLUSIONS

Naked Singularities

In the previous chapters we have calculated an exact solution to the Einstein-Maxwell system of equations. In this chapter, we will conclude our discussion after examining a few physical properties of this geometry.

We begin by briefly summarizing the results of the last chapter.

1. $|\mu| = 1$ is only a coordinate singularity.

2. All points (λ_0, μ_0) that satisfy the condition $\tan\overline{\psi}(\lambda_0, \mu_0) = -\alpha$ are true physical singularities if the condition $\lambda_0 = 0 = \mu_0$ is not satisfied.

3. At $\lambda_0 = 0 = \mu_0$, if $\tan(\overline{q}\pi/4) = \alpha$, then $\lim_{\mu \to 0} \lim_{\lambda \to 0} R_{\alpha\beta} R^{\alpha\beta} = \infty$ if and only if

$$\overline{q} \in \left(\left(-1-\sqrt{5}\right)/2, \left(-1+\sqrt{5}\right)/2\right).$$

Therefore, $(0,0)$ is a physical singularity, under the above conditions.

4. At $\lambda_0 = 0 = \mu_0$, if $\tan(\overline{q}\pi/4) \neq \alpha$, then $\lim_{\mu \to 0} \lim_{\lambda \to 0} R_{\alpha\beta} R^{\alpha\beta} = \infty$ if and only if

$$\overline{q} \in (-1, -1/2) \cup (-1/2, 0).$$

As before, $(0,0)$ is a physical singularity when the above conditions are met. However, this does not mean that all the singularities are accounted for. Although, any unaccounted singularity may occur only at $\lambda_0 = 0 = \mu_0$, we have not exhausted the cases as to when the singularities may occur; for example, there could be infinities in $\lim_{\lambda \to 0} \lim_{\mu \to 0} R_{\alpha\beta} R^{\alpha\beta}$. A more comprehensive analysis is required on this issue to determine the shape of the mass distribution that leads to this geometry.

From the form of the metric it is quite clear that t is everywhere timelike. Much like the Reissner-Nordström solution when $e^2 > m^2$, the singularity at $r = 0$ is naked along $\mu = \pm 1/\sqrt{3}$. This can be seen as follows. Along $\mu = \pm 1/\sqrt{3}$, we have that $\overline{\psi} = \tan^{-1} \lambda$. Therefore, the metric is singular if and only if $r = 0$ (i.e., at $\lambda = -\alpha$) when $\mu = \pm 1/\sqrt{3}$. It is intended to construct a future pointing causal curve originating from the singularity which approaches spatial infinity as $t \to \infty$. Clearly, then, the singularity at r = 0 is a naked singularity. Since

$$\lim_{\lambda \to \infty} \left[\frac{\lambda^2 (1 - \mu^2)}{\lambda^2 + 1} + \mu^2 \right] = 1, \tag{4.1}$$

there exists a constant C_1 such that for $0 \leq \lambda < \infty$

$$C_1 > \frac{\lambda^2 (1 - \mu^2)}{\lambda^2 + 1} + \mu^2. \tag{4.2}$$

For exactly the same reason, there exists a constant C_2 such that for $0 \leq \lambda < \infty$

$$C_2 > e^{2\gamma}. \tag{4.3}$$

Also,

$$e^{-4\psi} = \left[\alpha \cos \overline{\psi} + \sin \overline{\psi} \right]^4 \leq \left(|\alpha| + 1 \right)^4. \tag{4.4}$$

Let Γ be curve defined by

$$\Gamma(t) = \left[\sqrt{C_1 C_2 (|\alpha| + 1)^4}\, t, t, Cos^{-1} \frac{1}{\sqrt{3}}, \varphi_0 \right], \tag{4.5}$$

then from (4.2)-(4.4) we have

$$g(\dot{\Gamma}(t), \dot{\Gamma}(t)) = \left[C_1 C_2 (|\alpha| + 1)^4 - \left[\frac{\lambda^2 (1 - \mu^2)}{\lambda^2 + \mu^2} + \mu^2 \right] e^{2\gamma - 4\psi} \right] e^{2\psi}$$

$$\geq 0. \tag{4.6}$$

Therefore, Γ has the following properties: a) Γ is timelike, b) $\Gamma(0)$ is located at the physical singularity $r = 0$, and c) the particle Γ approaches spatial infinity as $t \to \infty$. Hence we conclude that the singularity at $\left(r = 0, \mu = 1/\sqrt{3}\right)$ is a naked singularity. This geometry does not describe a black hole. Although the 1-paramater solution we have obtained does not make the property of "naked singularities" stable about the Reissner-Nordström solution, nonetheless, this implies that the naked singularity in the Reissner-Nordström solution when $e^2 > m^2$ is not accidentally due to spherical symmetry. On account of the Penrose Cosmic Censorship Conjecture, our result lends itself to perhaps another conjecture that, in nature (in an asymptotically predictable space-time), the condition of $e^2 > m^2$ is not realized!

Gravitational Repulsion

As mentioned in the Introduction, the phenomenon of gravitational repulsion is well known [11-15]. Here we look for geodesic repulsion, since after all, the solution we have is a generalization of the Reissner-Nordström solution for the specific case of $e^2 > m^2$.

The simplest geodesic one can study in this geometry is the geodesic along the symmetry axis (i.e., along $\theta = 0$ in the (t, r, θ, φ) coordinate system). It is then necessary to check that there exists a geodesic along the symmetry axis, that is, we are looking for a geodesic of type

$$X(\tau) = \left(t(\tau), r(\tau), 0, \varphi_0\right), \tag{4.7}$$

where τ is an affine parameter and φ_0 is a constant value of the coordinate. The geodesic equation is

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0. \tag{4.8}$$

Since for X, $\frac{d^2\theta}{d\tau^2} = 0$, we need to check whether

$$2\Gamma^2{}_{01}\dot{t}\dot{r} + \Gamma^2{}_{00}\dot{t}^2 + \Gamma^2{}_{11}\dot{r}^2 = 0. \tag{4.9}$$

Similarly, we need to check whether

$$2\Gamma^3{}_{01}\dot{t}\dot{r} + \Gamma^3{}_{00}\dot{t}^2 + \Gamma^3{}_{11}\dot{r}^2 = 0, \tag{4.10}$$

since $\frac{d^2\varphi_0}{d\tau^2} = 0$. To check (4.9) and (4.10), we compute the necessary connection coefficients

$$\Gamma^2{}_{01} = \frac{1}{2}g^{22}\left\{\partial_0 g_{12} + \partial_1 g_{02} - \partial_2 g_{01}\right\}$$

$$= 0, \tag{4.11}$$

$$\Gamma^2{}_{00} = \frac{1}{2}g^{22}\left\{\partial_0 g_{02} + \partial_0 g_{02} - \partial_2 g_{00}\right\}$$

$$= -\frac{1}{2}g^{22}\partial_2 g_{00}, \tag{4.12}$$

$$\Gamma^2{}_{11} = \frac{1}{2}g^{22}\left\{\partial_1 g_{12} + \partial_1 g_{12} - \partial_2 g_{11}\right\}$$

$$= -\frac{1}{2} g^{22} \partial_2 g_{11} \,, \tag{4.13}$$

$$\Gamma^3{}_{01} = \frac{1}{2} g^{33} \left\{ \partial_0 g_{13} + \partial_1 g_{03} - \partial_3 g_{01} \right\}$$

$$= 0 \,, \tag{4.14}$$

$$\Gamma^3{}_{00} = \frac{1}{2} g^{33} \left\{ \partial_0 g_{00} + \partial_0 g_{03} - \partial_3 g_{00} \right\}$$

$$= 0 \,, \tag{4.15}$$

and

$$\Gamma^3{}_{11} = \frac{1}{2} g^{33} \left\{ \partial_1 g_{13} + \partial_1 g_{13} - \partial_3 g_{11} \right\}$$

$$= 0 \,. \tag{4.16}$$

From (4.14)-(4.16) we get

$$2\Gamma^3{}_{01} \dot{t} \dot{r} + \Gamma^3{}_{00} \dot{t}^2 + \Gamma^3{}_{11} \dot{r}^2 = 0 + 0 + 0$$

$$= 0 \,,$$

which verifies (4.10). Using (4.11)-(4.13), we get

$$2\Gamma^2{}_{01} \dot{t} \dot{r} + \Gamma^2{}_{00} \dot{t}^2 + \Gamma^2{}_{11} \dot{r}^2 = -\frac{1}{2} g^{22} \left(\partial_\theta g_{00} \right) \dot{t}^2 - \frac{1}{2} g^{22} \left(\partial_\theta g_{11} \right) \dot{r}^2 \,. \tag{4.17}$$

Clearly,

$$\partial_\theta \left(g_{00} \right) = \partial_\theta e^{2\psi}$$

$$= \frac{\partial}{\partial\theta} \frac{1}{\left[\alpha \cos\overline{\psi} + \sin\overline{\psi} \right]^2}$$

$$= \frac{2\left(-\alpha \sin\overline{\psi} + \cos\overline{\psi} \right)}{\left[\alpha \cos\overline{\psi} + \sin\overline{\psi} \right]} \frac{\partial\overline{\psi}}{\partial\theta} \,.$$

Since $\left.\frac{\partial\psi}{\partial\theta}\right|_{\theta=0} = 0$, we have

$$\partial_\theta\left(g_{00}\right)\Big|_{\theta=0} = 0, \qquad (4.18a)$$

and similarly

$$\partial_\theta\left(g_{11}\right)\Big|_{\theta=0} = 0. \qquad (4.18b)$$

Therefore, (4.17) gives

$$2\Gamma^2{}_{01}\dot{t}\dot{r} + \Gamma^2{}_{00}\dot{t}^2 + \Gamma^2{}_{11}\dot{r}^2\Big|_{X(\tau)} = 0,$$

since

$$\theta(X(\tau)) = 0.$$

This verifies (4.9). This readily proves that X is a geodesic along $\theta = 0$ if $\dot{X}(0) = \dot{t}(0)\partial_t + \dot{r}(0)\partial_r$. We will now proceed to look at geodesic with the above-mentioned property.

The affine parameter τ can be chosen such that

$$g_{00}\dot{t}^2 + g_{11}\dot{r}^2 = 1. \qquad (4.19)$$

But since $\gamma = 0$ along $\theta = 0$, we have

$$g_{00}\dot{t}^2 - \left(g_{00}\right)^{-1}\dot{r}^2 = 1. \qquad (4.20)$$

Clearly ∂_t is Killing; therefore,

$$g\left(\partial_t, \dot{X}\right) = constant \equiv \varepsilon,$$

that is,

$$g_{00}\dot{t} = \varepsilon. \qquad (4.21)$$

By (4.21), equation (4.20) becomes

$$\dot{r}^2 = \varepsilon^2 - g_{00}. \tag{4.22}$$

The turning points for the particle are where

$$\varepsilon^2 - g_{00} = 0. \tag{4.23}$$

The number of turning points of the neutral particle X depends on the values α, and $\bar{q}$ of the space-time, and ε (the energy of the particle).

Since $g_{00} > 0$, for sufficiently low energies, a neutral particle never reaches $r = 0$ (Figure 3-Figure 6). If $\tan(\bar{q}\pi / 4) = \alpha$, then $g_{00}\big|_{r=0} = \infty$; therefore, from (4.22) and (4.23), no timelike geodesics can reach $r = 0$. From Figures 3-6, it is clear that the particle may be repelled even before it reaches the origin ($r = 0$). Since $g_{00} \to 1$ from below as $r \to \infty$, the particle may oscillate if $\varepsilon^2 < 1$, just as in the case of the Reissner-Nordström solution when $e^2 > m^2$. However, here the particle could pass through $r = 0$ and approach $r \to \infty$ along $\theta = \pi$ provided the necessary conditions are met (Figure 5, when ε is greater than the maximum value attained by g_{∞}). The radial geodesic along the symmetry axis here is far more exotic than in the Reissner-Nordström geometry. This is perhaps due to the fact that the exact shape of the mass distribution depends on the values of α and $\bar{q}$.

Yet another curious property of this geometry is that the gravitational field depends on $\bar{q} \propto Q / q$ (this follows from (2.45)). In the Reissner-Nordström geometry, the gravitational field is independent of the sign of the net charge of the space-time. This is also true here. If the charge distribution in our case is

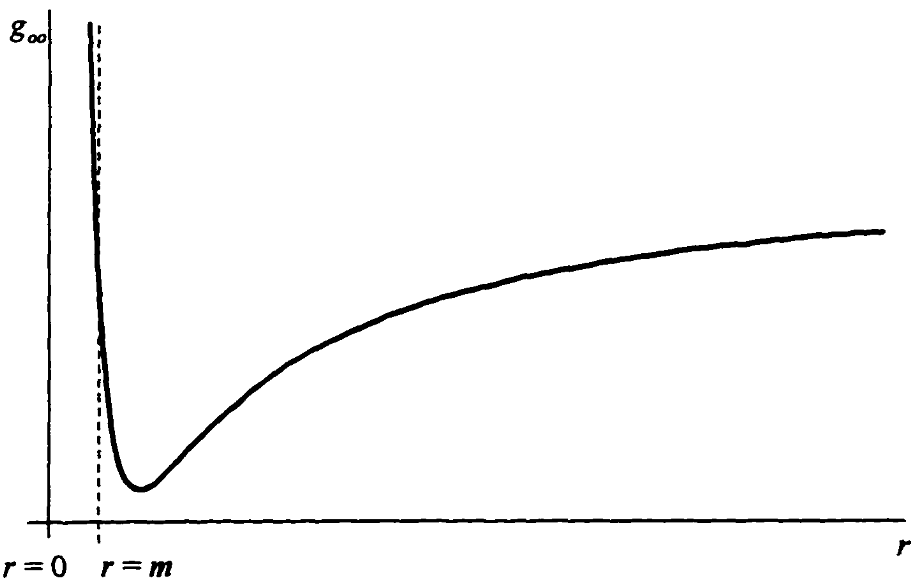

Figure 3. Graph of g_{oo} along the symmetry axis when $\bar{q} = 0$ and $\alpha = 1$.

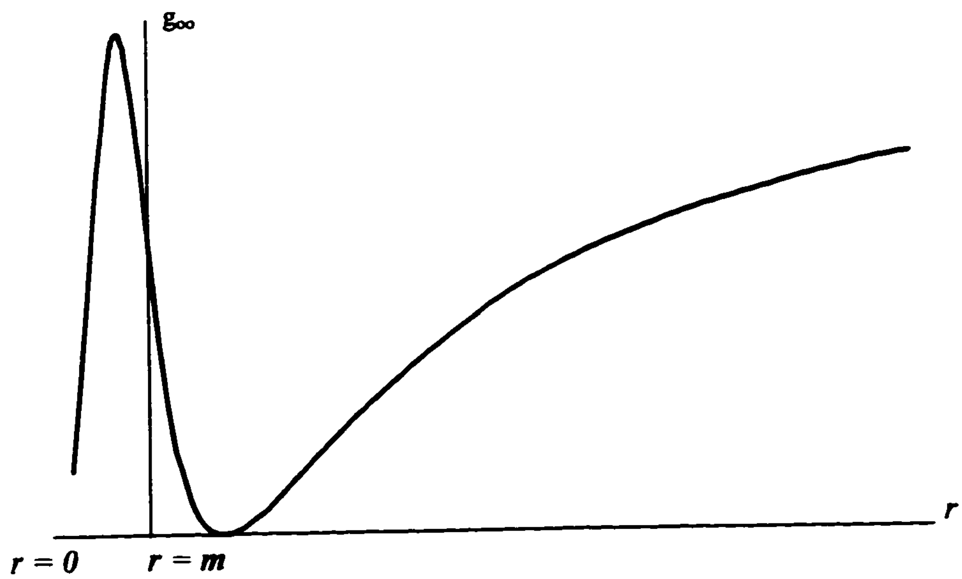

Figure 4. Graph of g_{oo} along the symmetry axis when $\bar{q} = 0.1$ and $\alpha = 1$.

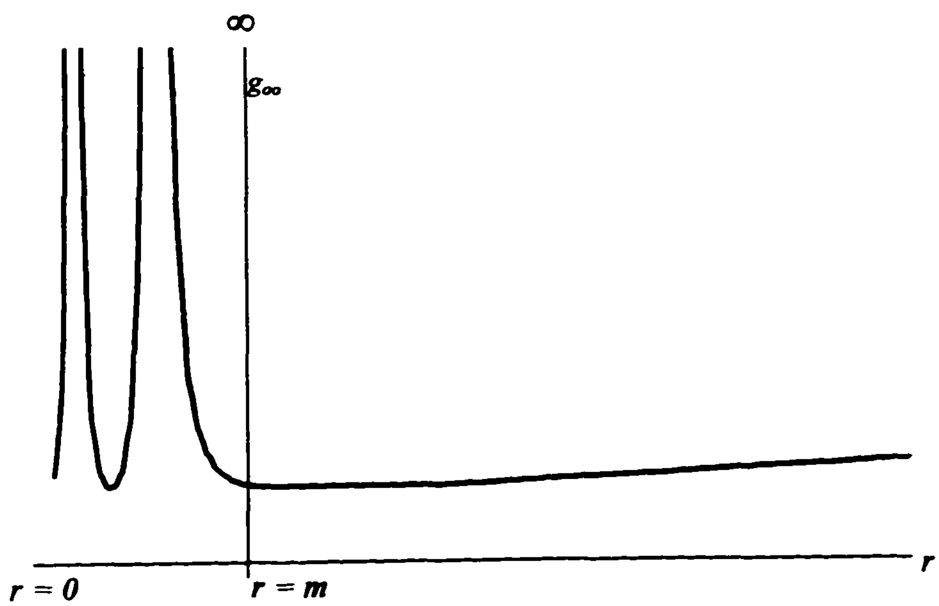

Figure 5. Graph of g_{∞} along the symmetry axis when $\bar{q} = 0.6$ and $\alpha = 1$.

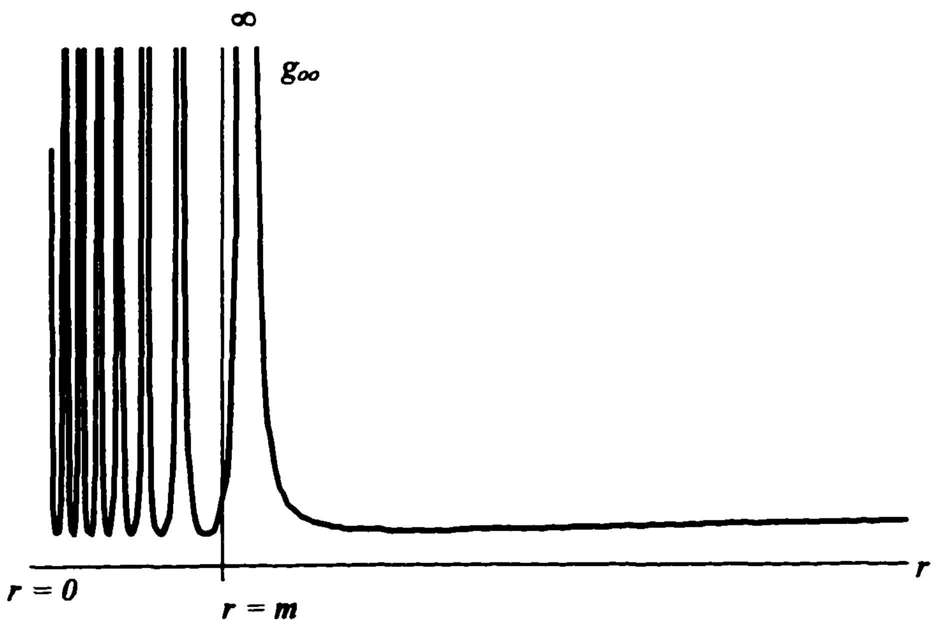

Figure 6. Graph of g_{oo} along the symmetry axis when $\bar{q} = 2$ and $\alpha = 1$.

homogeneous (i.e., $\rho \geq 0$ or $\rho \leq 0$, where ρ is the electric charge density), then $\bar{q}$, hence the gravitational field is independent of the charge distribution since

$$\frac{Q}{q} \propto \frac{Q_{zz}}{q} \propto \frac{\int \rho\left(r^2 - z^2\right)dv}{\int \rho dv} = \frac{\int -\rho\left(r^2 - z^2\right)dv}{\int -\rho dv}. \tag{4.24}$$

To summarize, we have obtained a family of solutions to the Einstein-Maxwell system of equations. The solutions correspond to that of a charged mass distribution with a nontrivial quadrupole moment. The resulting geometry is such that its charge to mass ratio in absolute value is greater than one. As expected, the solution corresponding to a trivial quadrupole moment $(\bar{q} = 0)$ is isometric to the Reissner-Nordström solution.

Upon calculation of the scalar $R_{\mu\nu}R^{\mu\nu}$, we find that, for all values of $\bar{q}$, the geometry admits a naked singularity. Furthermore, for sufficiently low energies, neutral particles are repelled from the singularity at $r = 0$.

REFERENCES

1. R. M. Wald, *General Relativity* (The University of Chicago Press, Chicago, 1984).

2. S. Weinberg, *Gravitation and Cosmology* (Wiley and Sons, Inc., New York, 1972).

3. H. Reissner, Ann. Phys. 50, 106 (1916).

4. G. Nordström, Proc. Kon. Ned. Akad. Wet. 20, 1238 (1918).

5. K. Schwarzchild, Sitzber. Deut. Akad. Wiss. Berlin, Kl. Math.-Phys. Tech., 189 (1916).

6. P. C. Peters, Am. J. Phys. 49(6), 564 (1981).

7. J. H. Young, Am. J. Phys. 59(6), 565 (1991).

8. J. D. Jackson, *Classical Electrodynamics* (Wiley and Sons, Inc., New York, 1975).

9. J. H. Young, and G. M. Menon, "Radial motion of a particle in the pseudo-Newtonian gravitational field of a highly charged mass," (to be submitted).

10. O. Gron, Phys. Lett. 94A(9), 424 (1983).

11. F. de Felice, Astron. & Astrophys. 34, 15 (1974).

12. F. de Felice, M. Calvani, and L. Nobili, Il Nuovo Cimento 26B(1)(1975).

13. J. M. Cohen, and R. Gautreau, Phys. Rev. 19(8), 2273 (1979).

14. O. Gron, General Rel. Grav. 20(2), 123 (1988).

15. S. M. Mahajan, A. Qadir, and P. M. Valanju, Il Nouvo Cimento 65B(2), 404 (1981).

16. G. Erez, and N. Rosen, Bull. Res. Council of Israel 8F, 47 (1959).

17. R. Geroch, J. Math. Phys. 11, 2580 (1970).

18. H. Quevedo, Phys. Rev. D 39, 2904 (1989).

19. V. Manko, Gen. Rel. Grav. 21, 1193 (1989).

20. H. Quevedo, and L. Parkes, Gen. Rel. Grav. 21,1047 (1989).

21. H. Quevedo, and B. Mashhoon, Phys. Lett. A 148, 149 (1990).

22. H. Quevedo, and L. Parkes, Gen. Rel. Grav. 23, 495 (1991).

23. H. Bondi, and W. Rindler, Gen. Rel. Grav. 23, 487 (1991).

24. J. Hernandez-Pastora, and J. Martin, Gen., Rel. Grav. 26, 877 (1994).

25. R. Penrose, *Singularities and Time Asymmetry*, in General Relativity, an Einstein Centenary Survey, ed. S. W. Hawking and W. Israel (Cambridge: Cambridge University Press, 1979).

26. H. Weyl, Ann. Phys. (Leipz.) 54, 117 (1918); 59, 185 (1919).

27. R. Adler, M. Bazin, and M. Schiffer, *Introduction to General Relativity*, (McGraw-Hill, New York, 1975).

28. J. H. Young, and S. L. Bentley, Phys. Rev. 11(12), 3388 (1975).

29. B. K. Harrison, J. Math. Phys. 9, 1744 (1968); R. M. Misra, D. B. Pandey, D. C. Srivastava, and S. N. Tripathi, Phys. Rev.D 7, 1587 (1973).

30. M. Spiegel, *Schaum's Outline Series: Mathematical Handbook of Formulas and Tables* (McGraw-Hill, New York, 1968).

31. S. W. Hawking, and G. F. R. Ellis, *The large scale structure of space-time* (Cambridge University Press, Cambridge, 1973).

32. R. G. Bartle, *The elements of real analysis* (Wiley and Sons, Inc., New York, 1964).

APPENDIX A

PROOF OF EQUATIONS (2.14b) AND (2.14c)

Here we give a proof of (2.14b) and (2.14c). We first note that since

$$\psi = -\ln(\alpha \cos\overline{\psi} + \sin\overline{\psi}), \tag{A1}$$

we have

$$\begin{Bmatrix} \psi_\rho \\ \psi_z \end{Bmatrix} = -\frac{(-\alpha \sin\overline{\psi} + \cos\overline{\psi})}{(\alpha \cos\overline{\psi} + \sin\overline{\psi})} \begin{Bmatrix} \overline{\psi}_\rho \\ \overline{\psi}_z \end{Bmatrix}. \tag{A2}$$

Using the above relation and substituting (2.13) into (2.7) we have

$$\gamma_\rho = \rho\left[\left(\overline{\psi}_\rho^2 - \overline{\psi}_z^2\right)\left(\frac{-\alpha \sin\overline{\psi} + \cos\overline{\psi}}{\alpha \cos\overline{\psi} + \sin\overline{\psi}}\right)^2 - \frac{(1+\alpha^2)}{(\alpha \cos\overline{\psi} + \sin\overline{\psi})^2}\left(\overline{\psi}_\rho^2 - \overline{\psi}_z^2\right)\right]$$

$$= -\rho\left(\overline{\psi}_\rho^2 - \overline{\psi}_z^2\right)\left[\frac{\alpha^2 \sin^2\overline{\psi} + \cos^2\overline{\psi} - 2\alpha \sin\overline{\psi}\cos\overline{\psi} - 1 - \alpha^2}{(\alpha \cos\overline{\psi} + \sin\overline{\psi})^2}\right]$$

$$= -\rho\left(\overline{\psi}_\rho^2 - \overline{\psi}_z^2\right), \tag{A3}$$

which is (2.14b).

Similarly, from (2.8) we get

$$\gamma_z = 2\rho\left[\frac{(-\alpha \sin\overline{\psi} + \cos\overline{\psi})^2}{(\alpha \cos\overline{\psi} + \sin\overline{\psi})^2}\overline{\psi}_\rho\overline{\psi}_z - \frac{(1+\alpha^2)}{(\alpha \cos\overline{\psi} + \sin\overline{\psi})^2}\overline{\psi}_\rho\overline{\psi}_z\right]$$

$$= -2\rho\overline{\psi}_\rho\overline{\psi}_z\left[\frac{-\alpha^2 \cos^2\overline{\psi} - \sin^2\overline{\psi} - 2\alpha \cos\overline{\psi} \sin\alpha}{(\alpha \cos\overline{\psi} + \sin\overline{\psi})^2}\right]$$

$$= -2\rho\overline{\psi}_\rho\overline{\psi}_z, \tag{A4}$$

which is (2.14c).

APPENDIX B

PROOF OF EQUATION (2.21)

Here we compute the expression satisfied by the quantities γ_λ and γ_μ. The coordinate transformations of interest are

$$\rho = \sqrt{1+\lambda^2}\sqrt{1-\mu^2}, \quad \text{and} \quad z = \lambda\mu, \tag{B1}$$

where $-\alpha \le \lambda < \infty$ and $-1 \le \mu \le 1$. The coefficient m/α has been dropped in (B1) since (2.14b) and (2.14c) are invariant under transformations of the type $\{\rho' \quad z'\} = \chi\,\{\rho \quad z\}$, where χ is an arbitrary constant. Here, the Jacobian is given by

$$\begin{pmatrix} \partial_\lambda \\ \partial_\mu \end{pmatrix} = \begin{pmatrix} \dfrac{\partial\rho}{\partial\lambda} & \dfrac{\partial z}{\partial\lambda} \\ \dfrac{\partial\rho}{\partial\mu} & \dfrac{\partial z}{\partial\mu} \end{pmatrix} \begin{pmatrix} \partial_\rho \\ \partial_z \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{\lambda\sqrt{1-\mu^2}}{\sqrt{1+\lambda^2}} & \mu \\ \dfrac{-\mu\sqrt{1+\lambda^2}}{\sqrt{1-\mu^2}} & \lambda \end{pmatrix} \begin{pmatrix} \partial_\rho \\ \partial_z \end{pmatrix}. \tag{B2}$$

Hence we can relate the partial derivatives of γ with respect to the (ρ, z) coordinate system to the same in the (λ, μ) coordinate system as follows:

$$\begin{pmatrix} \gamma_\lambda \\ \gamma_\mu \end{pmatrix} = \begin{pmatrix} \dfrac{\lambda\sqrt{1-\mu^2}}{\sqrt{1+\lambda^2}} & \mu \\ \dfrac{-\mu\sqrt{1+\lambda^2}}{\sqrt{1-\mu^2}} & \lambda \end{pmatrix}^{-1} \begin{pmatrix} \gamma_\rho \\ \gamma_z \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{\lambda}{\Delta} & \dfrac{-\mu}{\Delta} \\ \dfrac{\mu\sqrt{1+\lambda^2}}{\Delta\sqrt{1-\mu^2}} & \dfrac{\lambda\sqrt{1-\mu^2}}{\Delta\sqrt{1+\lambda^2}} \end{pmatrix} \begin{pmatrix} \gamma_\rho \\ \gamma_z \end{pmatrix}, \tag{B3}$$

where

$$\Delta = \frac{\lambda^2 + \mu^2}{\rho}.$$

Similarly,

$$\begin{pmatrix} \overline{\psi}_\lambda \\ \overline{\psi}_\mu \end{pmatrix} = \begin{pmatrix} \frac{\lambda}{\Delta} & \frac{-\mu}{\Delta} \\ \frac{\mu\sqrt{1+\lambda^2}}{\Delta\sqrt{1-\mu^2}} & \frac{\lambda\sqrt{1-\mu^2}}{\Delta\sqrt{1+\lambda^2}} \end{pmatrix} \begin{pmatrix} \overline{\psi}_\rho \\ \overline{\psi}_z \end{pmatrix}. \tag{B4}$$

Solving for γ_λ from (B3), we get

$$\gamma_\lambda = \left[\frac{\lambda\sqrt{1-\mu^2}}{\sqrt{1+\lambda^2}} \gamma_\rho + \mu\gamma_z \right]. \tag{B5}$$

Substituting (2.14b) and (2.14c) in the above results in

$$\gamma_\lambda = \rho \left[\frac{\lambda\sqrt{1-\mu^2}}{\sqrt{1+\lambda^2}} \left(\overline{\psi}_z^2 - \overline{\psi}_\rho^2 \right) - 2\mu \left(\overline{\psi}_\rho \overline{\psi}_z \right) \right]. \tag{B6}$$

The expression for $\overline{\psi}_z$ and $\overline{\psi}_\rho$ in terms of $\overline{\psi}_\lambda$ and $\overline{\psi}_\mu$ can be obtained from (B4). A resubstitution of this result into (B6) gives

$$\gamma_\lambda = \frac{\left(1-\mu^2\right)}{\left(\lambda^2+\mu^2\right)} \left[-\lambda\left(1+\lambda^2\right)\overline{\psi}_\lambda^2 + \lambda\left(1-\mu^2\right)\overline{\psi}_\mu^2 + 2\mu\left(1+\lambda^2\right)\overline{\psi}_\mu \overline{\psi}_\lambda \right], \tag{B7}$$

which is (2.21). A similar calculation gives (2.22).

APPENDIX C

THE ASSYMPTOTIC BEHAVIOR OF THE ELECTROSTATIC POTENTIAL

It is intended here to calculate the asymptotic form of Φ. From (2.18) we have

$$\Phi = -\frac{\sqrt{1+\alpha^2}}{k}\frac{1}{\alpha+\tan\overline{\psi}}$$

Let $x = \frac{1}{r}$. Then

$$\Phi \approx \Phi(x=0) + \left.\frac{\partial\Phi}{\partial x}\right|_{x=0}\frac{1}{r} + \frac{1}{2}\left.\frac{\partial^2\Phi}{\partial x^2}\right|_{x=0}\frac{1}{r^2} + \frac{1}{3!}\left.\frac{\partial^3\Phi}{\partial x^3}\right|_{x=0}\frac{1}{r^3} + \ldots \tag{C1}$$

But

$$\Phi(x=0) = \Phi(r=\infty) = 0\,, \tag{C2}$$

so it remains to calculate the partial derivatives of Φ. Let

$$\tilde{\Phi} = \frac{1}{\alpha+\tan\overline{\psi}}\,, \tag{C3}$$

then

$$\left.\frac{\partial\tilde{\Phi}}{\partial x}\right|_{x=0} = \left.\frac{d\tilde{\Phi}}{d\overline{\psi}}\right|_{\overline{\psi}=\pi/2}\cdot\left.\left(\frac{d\overline{\psi}}{d\lambda}\frac{d\lambda}{dx}\right)\right|_{x=0}. \tag{C4}$$

Clearly,

$$\left.\frac{d\tilde{\Phi}}{d\overline{\psi}}\right|_{\overline{\psi}=\pi/2} = -1\,, \tag{C5a}$$

$$\left.\frac{d^2\tilde{\Phi}}{d\overline{\psi}^2}\right|_{\overline{\psi}=\pi/2} = -2\alpha\,, \tag{C5b}$$

and

$$\left.\frac{d^3\tilde{\Phi}}{d\overline{\psi}^3}\right|_{\overline{\psi}=\pi/2} = -\left(2+6\alpha^2\right). \tag{C5c}$$

Also,

$$\frac{\partial\overline{\psi}}{\partial\lambda}=\frac{1}{1+\lambda^2}+\overline{q}P_2(\mu)\left[6\lambda\cot^{-1}\lambda-\frac{(3\lambda^2+1)}{1+\lambda^2}-3\right], \tag{C6}$$

and

$$\frac{d\lambda}{dx}=\frac{d}{dx}\left(\alpha\left(\frac{1}{mx^{-1}}-1\right)\right)=-\alpha m\left(\frac{\lambda}{\alpha}+1\right)^2. \tag{C7}$$

To leading order

$$\frac{\partial\overline{\psi}}{\partial\lambda}\frac{d\lambda}{dx}=-\frac{m}{\alpha}\left[\frac{1}{1+1/\lambda^2}+\overline{q}\left\{6\lambda^2-2-3\lambda^2-\frac{3\lambda^4}{1+\lambda^2}-\frac{\lambda^2}{1+\lambda^2}\right\}P_2\right]. \tag{C.8}$$

Therefore,

$$\lim_{x\to 0}\frac{\partial\overline{\psi}}{\partial x}=-\frac{m}{\alpha}+\overline{q}P_2(\mu)\lim_{\lambda\to\infty}\left[\frac{3\lambda^2}{1+\lambda^2}-2-\frac{\lambda^2}{1+\lambda^2}\right]$$

$$=-\frac{m}{\alpha}+\overline{q}P_2(\mu)\{3-2-1\}$$

$$=-\frac{m}{\alpha}. \tag{C9}$$

For large λ

$$\overline{\psi}\approx\tan^{-1}\lambda-\frac{\overline{q}P_2}{3\lambda^3}, \tag{C10}$$

that is,

$$\frac{\partial\overline{\psi}}{\partial\lambda}\approx\frac{1}{1+\lambda^2}+\frac{\overline{q}P_2}{\lambda^4}. \tag{C11}$$

Then

$$\frac{\partial\overline{\psi}}{\partial x}=-\alpha m\left(\frac{\lambda^2}{\alpha^2}+1+\frac{2\lambda}{\alpha}\right)\left(\frac{1}{1+\lambda^2}+\frac{\overline{q}P_2}{\lambda^4}\right). \tag{C12}$$

Also,

$$\frac{\partial^2\overline{\psi}}{\partial x^2}=\frac{d\lambda}{dx}\frac{\partial}{\partial\lambda}\left(\frac{\partial\overline{\psi}}{\partial x}\right)$$

$$\approx\alpha^2m^2\left(\frac{\lambda^2}{\alpha^2}+1+\frac{2\lambda}{\alpha}\right)\frac{\partial}{\partial\lambda}\left[\left(\frac{\lambda^2}{\alpha^2}+1+\frac{2\lambda}{\alpha}\right)\left(\frac{1}{1+\lambda^2}+\frac{\overline{q}P_2}{\lambda^4}\right)\right]$$

$$=\alpha^2m^2\left(\frac{\lambda^2}{\alpha^2}+1+\frac{2\lambda}{\alpha}\right)\left[\frac{1}{\alpha^2}\frac{2\lambda}{\left(1+\lambda^2\right)^2}-\frac{2\overline{q}P_2}{\alpha^2}\frac{1}{\lambda^3}\right.$$

$$\left.-\frac{2\lambda}{\left(1+\lambda^2\right)^2}-\frac{4\overline{q}P_2}{\lambda^5}+\frac{2}{\alpha}\frac{\left(1-\lambda^2\right)}{\left(1+\lambda^2\right)^2}-\frac{6\overline{q}P_2}{\alpha\lambda^4}\right],$$

that is,

$$\left.\frac{\partial^2\overline{\psi}}{\partial x^2}\right|_{x=0}=\alpha^2m^2\left.\frac{\lambda^2}{\alpha^2}\frac{2}{\alpha}\frac{(-1)\lambda^2}{\left(1+\lambda^2\right)^2}\right|_{\lambda\to\infty}$$

$$=-\frac{2m^2}{\alpha}. \qquad \text{(C13)}$$

Taking another derivative, we find

$$\frac{\partial^3\overline{\psi}}{\partial x^3}=\frac{d\lambda}{dx}\frac{\partial}{\partial\lambda}\left(\frac{\partial^2\overline{\psi}}{\partial x^2}\right). \qquad \text{(C14)}$$

But

$$\frac{1}{\alpha^2m^2}\frac{\partial}{\partial\lambda}\left(\frac{\partial^2\overline{\psi}}{\partial x^2}\right)\approx\frac{\partial}{\partial\lambda}\left[\left(\frac{\lambda^2}{\alpha^2}+1+\frac{2\lambda}{\alpha}\right)\left[\frac{1}{\alpha^2}\frac{2\lambda}{\left(1+\lambda^2\right)^2}\right.\right.$$

$$\left.\left.-\frac{2\overline{q}P_2}{\alpha^2\lambda^3}-\frac{2\lambda}{\left(1+\lambda^2\right)^2}+\frac{2}{\alpha}\frac{\left(1-\lambda^2\right)}{\left(1+\lambda^2\right)^2}\right]\right]$$

$$\approx -\frac{2}{\alpha^4}\frac{\lambda^6}{\left(1+\lambda^2\right)^4}+\frac{2\bar{q}P_2}{\alpha^4\lambda^2}+\frac{2}{\alpha^2}\frac{\lambda^6}{\left(1+\lambda^2\right)^4}+\frac{4}{\alpha^2}\frac{\lambda^6}{\left(1+\lambda^2\right)^4}, \tag{C15}$$

that is,

$$\frac{\partial^3\bar{\psi}}{\partial x^2} \approx -\alpha^3 m^3 \frac{\lambda^2}{\alpha^2}\left[\frac{-2}{\alpha^4}\frac{\lambda^6}{\left(1+\lambda^2\right)^4}+\frac{2\bar{q}P_2}{\alpha^4\lambda^2}\right.$$

$$\left.+\frac{2}{\alpha^2}\frac{\lambda^6}{\left(1+\lambda^2\right)^4}+\frac{4}{\alpha^2}\frac{\lambda^6}{\left(1+\lambda^2\right)^4}\right]. \tag{C16}$$

Taking limits, we have

$$\left.\frac{\partial^3\psi}{\partial x^3}\right|_{x=0} = -\alpha m^3\left[-\frac{2}{\alpha^4}+\frac{2\bar{q}P_2}{\alpha^4}+\frac{2}{\alpha^2}+\frac{4}{\alpha^2}\right]$$

$$=\frac{2m^3}{\alpha^3}-\frac{2m^3\bar{q}P_2}{\alpha^3}-\frac{6m^3}{\alpha}. \tag{C17}$$

From (C5a) and (C9) we have

$$\left.\frac{\partial\tilde{\Phi}}{\partial x}\right|_{x=0}=\left.\frac{d\tilde{\Phi}}{d\bar{\psi}}\right|_{\bar{\psi}=\pi/2}\left.\frac{\partial\bar{\psi}}{\partial x}\right|_{x=0}$$

$$=(-1)\left(-\frac{m}{\alpha}\right)=\frac{m}{\alpha}. \tag{C18}$$

Also,

$$\frac{\partial^2\tilde{\Phi}}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial\tilde{\Phi}}{\partial x}\right)=\frac{d^2\tilde{\Phi}}{d\bar{\psi}^2}\left(\frac{\partial\bar{\psi}}{\partial x}\right)^2+\frac{d\tilde{\Phi}}{d\bar{\psi}}\frac{\partial^2\bar{\psi}}{\partial x^2}. \tag{C19}$$

From (C5a), (C9), (C5b), and (C13) we get

$$\left.\frac{\partial^2\tilde{\Phi}}{\partial x^2}\right|_{x=0}=(-2\alpha)\left(\frac{m^2}{\alpha^2}\right)+(-1)\left(-\frac{2m^2}{\alpha}\right)=0. \tag{C20}$$

Finally,

$$\frac{\partial^3\tilde{\Phi}}{\partial x^3}=\frac{\partial}{\partial x}\left(\frac{\partial^2\Phi}{\partial x^2}\right)$$

$$=\frac{d^3\tilde{\Phi}}{d\bar{\psi}^3}\left(\frac{\partial\bar{\psi}}{\partial x}\right)^3+3\frac{d^2\tilde{\Phi}}{d\bar{\psi}^2}\frac{\partial\bar{\psi}}{\partial x}\frac{\partial^2\bar{\psi}}{\partial x^2}+\frac{d\tilde{\Phi}}{d\bar{\psi}}\frac{\partial^3\bar{\psi}}{\partial x^3}. \qquad \text{(C21)}$$

Therefore from (C5c), (C9), (C5b), (C13), (C5a), and (C17), we have

$$\left.\frac{\partial^2\tilde{\Phi}}{\partial x^3}\right|_{x=0}=\frac{2m^3}{\alpha^3}+\frac{6m^3}{\alpha}-\frac{8m^3}{\alpha}-\frac{4m^3}{\alpha}-\frac{2m^3}{\alpha^3}+\frac{2m^3\bar{q}P_2}{\alpha^2}+\frac{6m^3}{\alpha}$$

$$=\frac{2m^3\bar{q}P_2}{\alpha^3}. \qquad \text{(C22)}$$

Now we are in a position to find the asymptotic behavior of Φ from (C3), (C1), (C18), (C19) and (C21), resulting in

$$\Phi\approx\frac{\sqrt{1+\alpha^2}}{k}\left[0+\frac{m}{\alpha}\frac{1}{r}+0+\frac{2m^3\bar{q}P_2(\mu)}{6\alpha^3}\frac{1}{r^3}\right]$$

$$=\frac{q}{r}+\frac{q\bar{q}}{3}\left(e^2-m^2\right)P_2(\mu)\frac{1}{r^3}. \qquad \text{(C23)}$$

GRADUATE SCHOOL
UNIVERSITY OF ALABAMA AT BIRMINGHAM
DISSERTATION APPROVAL FORM

Name of Candidate Govind K. Menon

Major Subject Physics

Title of Dissertation An Exact Solution to the Einstein-Maxwell Equations Representing a Nonspherical, Highly Charged Object

Dissertation Committee:

John H. Young, **Chair** John H. Young

Joseph G. Harrison Joseph G. Harrison

H. Tiit Tohver H. Tiit Tohver

Peter V. O'Neil Peter V. O'Neil

Mubenga N. Nkashama M. NKashama

Director of Graduate Program [illegible]

Dean, UAB Graduate School Joan F. Lorden

Date 10/1/97