Chaotic models in nonequilibrium statistical mechanics.

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CHAOTIC MODELS IN NONEQUILIBRIUM STATISTICAL MECHANICS

by

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A DISSERTATION

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ABSTRACT OF DISSERTATION
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This dissertation presents some results in dynamical systems relevant to nonequilibrium statistical mechanics.

In Chapter 2 we consider piecewise smooth, uniformly hyperbolic systems on a Riemannian manifold, allowing the angle between the unstable direction and the singularity manifolds to vanish. Under natural assumptions we prove that such systems exhibit exponential decay of correlations and satisfy a central limit theorem with respect to a mixing Sinai-Ruelle-Bowen measure (SRB-measure). These results have been shown previously for systems in which the angle between the singularity manifold and unstable direction is uniformly bounded away from zero.

In Chapter 3 we consider piecewise smooth, expanding maps of the unit interval with small holes. Assuming only that the size of the holes is small, we establish existence of a conditionally invariant, smooth measure for the transformation. Under the additional assumption that images of the holes do not overlap up to a certain iterate we also obtain uniqueness. Previous results assume that the transformation satisfies a Markov condition.

DEDICATION

In memory of my grandfathers

Jacobus Antonius van den Bedem

Antonius Franciscus Mulder

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CHAPTER 1

INTRODUCTION

1.1. Hyperbolic Dynamics and Statistical Mechanics

Modern statistical mechanics comprises two rather distinct areas: equilibrium and nonequilibrium statistical mechanics. Founded in the 1870s, equilibrium statistical mechanics is well-developed and has led to a deep understanding of many phenomena. Quite on the contrary, the development of nonequilibrium statistical mechanics was largely based on investigating the *approach to equilibrium* as introduced by Boltzmann and has progressed significantly slower. Over the past decade, a new point of view of nonequilibrium statistical mechanics has emerged. This new formalism studies *nonequilibrium steady states* by taking into account the underlying microscopic time evolution of the system, thus giving rise to a profound connection between dynamical systems and nonequilibrium statistical mechanics. In this chapter we explore this relationship and, in doing so, provide an informal physical motivation for the study of hyperbolic dynamical systems and decay of correlations. A splendid introduction to this new field of research is given in [**D**]. For an excellent survey of recent advancements, see [**R2**].

1.1.1. Chaotic Models in Nonequilibrium Statistical Mechanics. To maintain a system $\dot{x} = F(x)$ on a compact, smooth Riemannian manifold M out of equilibrium, we subject it to non-Hamiltonian forces. As the energy of the system increases due to these forces (it "heats up"), the need for the mathematical equivalent of a thermostat arises. We therefore replace the system of equations by $\dot{x} = F(x) + \Theta(x)$ with $\Theta(x)$ a thermostat dissipating excess energy. Systems of this kind are the subject of both ongoing research and hot debate in theoretical and computational physics;

see [**R2**] and the references therein. We avoid discussing physical objections to those models, but rather restrict ourselves to formally deriving their implications. Of particular interest to us is the *isokinetic thermostat* in which $\Theta(x)$ is chosen such that the total kinetic energy remains constant.

The thermostated system will attain a steady state, and a suitable representation of the system is needed, analogous to a canonical ensemble in equilibrium statistical mechanics. Due to phase space contraction, the interesting dynamics of these systems usually takes place on an compact, invariant set $\Lambda \subset M$, called an *attractor*, with smaller volume than the total phase space. We set some notation. Restricting ourselves to discrete time we introduce a transformation $T\colon M \to M$ of a compact, smooth, Riemannian manifold M endowed with an absolutely continuous probability measure μ_0. For $x \in M$ define the *stable manifold at* x to be $W^s(x) = \{y \in M\colon d(T^n x, T^n y) \to 0 \text{ as } n \to \infty\}$, and the *unstable manifold at* x to be $W^u(x) = \{y \in M\colon d(T^{-n} x, T^{-n} y) \to 0 \text{ as } n \to \infty\}$. We furthermore assume the existence of an attractor and rather heuristically define the *basin of attraction of* Λ to be the set of points approaching Λ under the forward dynamics.

To obtain a distribution representing a system in steady state Ruelle introduced the following as a principle (R) [**R1**]:

> The time averages of observables, on motions with initial data randomly sampled with the Liouville distribution μ_0, are described by a stationary probability distribution μ obtained by attributing a suitable probability density to the surface elements of the unstable manifolds of the points in the phase space.

In a laboratory situation, unstable manifolds are difficult to observe. Experimenters, including computational physicists, deem it natural to describe steady states by the long-time evolution of an absolutely continuous probability measure μ_0 under the flow $\Phi(x,t) = \Phi^t(x)$ introduced by $\dot{x} = F(x) + \Theta(x)$. Physically, this corresponds to putting test particles in the system and observing their long time distribution. More formally, we say that nonequilibrium steady states (in discrete time) are naturally described by weak limits of $\mu^n = \frac{1}{n}\sum_{k=0}^{n-1} T_*^k \mu_0$, where $T_*\mu_0(A) = \mu_0(T^{-1}A)$ for all measurable subsets $A \subset M$. These weak limit points are commonly denoted by μ^+.

On a theoretical level, we acknowledge that Principle (R) has been shown to hold only for very specific (hyperbolic) classes of systems. A closed subset $\Lambda \subset M$ is called *uniformly hyperbolic* if $T(\Lambda) = \Lambda$ and for each $x \in \Lambda$ the tangent space to M at x can be written as a direct sum of an expanding subspace E^u_x and a contracting subspace E^s_x, i.e. $T_x M = E^s_x \oplus E^u_x$, and the angle between E^s_x and E^u_x has a positive lower bound, uniformly on Λ. So, each tangent vector $u \in E^u_x$ (E^s_x) is expanded (contracted) by at least λ (λ^{-1}), with $\lambda > 1$. A diffeomorphism $T\colon M \to M$ is called *Anosov* if M is uniformly hyperbolic. A classic result in dynamical systems [**B**] states that Principle (R) follows as a theorem under these hyperbolicity assumptions, i.e. if T is a (C^2-) Anosov diffeomorphism and $\Lambda \subset M$ an attractor, then for any continuous observable $\psi\colon M \to \mathbb{R}$ one has

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \psi \circ T^k z = \int_\Lambda \psi \, d\mu^+ \qquad m\text{-a.e. in } W^s(\Lambda).$$

The measure μ^+ is a weak limit point of $\frac{1}{n}\sum_{k=0}^{n-1} T^k_* \mu_0$ and is exactly the suitable probability measure referred to in Principle (R). A measure with this property is also known as a Sinai-Ruelle-Bowen (SRB) measure. The measure m denotes Lebesgue measure, and the set $W^s(\Lambda) = \cup_{x\in\Lambda} W^s(x)$ is the basin of attraction.

To enhance our understanding of nonequilibrium statistical mechanics, it has been proposed [**GC**], [**R2**] to impose strong hyperbolicity assumptions on the underlying dynamics, such as to validate Principle (R) and then retain the consequences. An analogous approach in equilibrium statistical mechanics, with the *ergodic hypothesis* as a principle to obtain an equilibrium distribution, has proved successful. Despite the fact that these assumptions are physically quite unrealistic, this approach has led to some remarkable results, one of which is discussed in the next section.

1.1.2. Decay of Correlations in Statistical Mechanics. In this section we investigate how the new insights have been used to prove Ohm's law in a model for electrical conductivity in the periodic Lorentz gas. We present a simplified version of the system under consideration in [C4]. Principle (R) holds for this model in the sense that hyperbolicity and the existence of an SRB measure are indeed established.

The dynamical system corresponds to the motion of a single unit-charged particle subject to a weak external electrical field $\boldsymbol{E}$ between a finite number of fixed, disjoint, convex scatterers in a periodic domain of the plane, a so-called periodic Lorentz gas; see Figure 1. Between collisions the motion of the particle is governed by

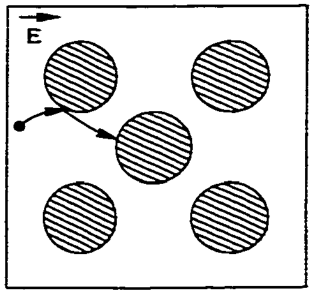


FIGURE 1. A two-dimensional periodic Lorentz gas.

$$\begin{aligned} \dot{\boldsymbol{q}} &= \boldsymbol{p}/m \\ \dot{\boldsymbol{p}} &= \boldsymbol{E} - \xi\boldsymbol{p}, \end{aligned} \tag{1.1}$$

with $m = 1$, $\boldsymbol{q} = (q_1, q_2)$ the coordinates of the particle, and $\boldsymbol{p} = (p_1, p_2)$ the corresponding momenta. We will also write $\boldsymbol{X} = (\boldsymbol{q}, \boldsymbol{p})$. Initially the system is at rest, and at $t = 0$ a constant electrical field $\boldsymbol{E} = (E, 0)$ is turned on. The scalar *isokinetic* thermostat ξ is chosen such that $\boldsymbol{p}\cdot\boldsymbol{p}$ is constant. Setting $\boldsymbol{p}\cdot\dot{\boldsymbol{p}} = 0$ yields $\xi = \frac{\boldsymbol{E}\cdot\boldsymbol{p}}{\boldsymbol{p}\cdot\boldsymbol{p}}$, and choosing $\boldsymbol{p}\cdot\boldsymbol{p} = 1$ we finally obtain $\xi = \boldsymbol{E}\cdot\boldsymbol{p} = Ep_1$. The system (1.1) induces a flow ϕ_E^t on the compact phase space M. Define a class of measures on M by setting $\phi_{E*}^t\mu_0(A) = \mu_0(\phi_E^{-t}A)$ for all Borel sets A, where $d\mu_0 = d\boldsymbol{q}d\boldsymbol{p}$ is equilibrium (or Liouville) measure. The density $\rho_t = \rho_t(\boldsymbol{X})$ of the measure $\phi_{E*}^t\mu_0$ (with respect to μ_0) at time t satisfies the generalized Liouville (or continuity) equation $\frac{\partial\rho_t}{\partial t} + \nabla_{\boldsymbol{X}}\cdot[\dot{\boldsymbol{X}}\rho_t] = 0$. Since $\nabla_{\boldsymbol{X}}\cdot\dot{\boldsymbol{X}} = -2\xi$, it follows that the Liouville measure is not preserved whenever $E \neq 0$. The density ρ_t satisfies

$$\frac{\partial\rho_t}{\partial t} + \dot{\boldsymbol{X}}\cdot(\nabla_{\boldsymbol{X}}\rho_t) = -(\nabla_{\boldsymbol{X}}\cdot\dot{\boldsymbol{X}})|_{\phi_E^{-t}}\,\rho_t(\boldsymbol{X}) = 2E\,p_1(\phi_E^{-t})\,\rho_t(\boldsymbol{X}),$$

which can be written as $\frac{d}{dt}\rho_t = 2E\,p_1(\phi_E^{-t})\,\rho_t(\boldsymbol{X})$. Integrating both sides we obtain

$$\rho_t(\boldsymbol{X}) - \rho_0(\boldsymbol{X}) = 2E\int_0^t p_1(\phi_E^{-s})\,\rho_s(\boldsymbol{X})\,ds.$$

So, for any smooth function $\boldsymbol{F}\colon M \to \mathbb{R}^2$ we have $\mu_t(\boldsymbol{F}) = \int_M \boldsymbol{F}(\boldsymbol{X})\rho_t(\boldsymbol{X})\,d\boldsymbol{X} = \mu_0(\boldsymbol{F}) + 2E\int_M \boldsymbol{F}(\boldsymbol{X})\int_0^t p_1(\phi_E^{-s})\,\rho_s(\boldsymbol{X})\,ds$. Assuming integrability and changing coordinates $\boldsymbol{Y} = \phi_E^{-s}\boldsymbol{X}$, we can write

$$\mu_t(\boldsymbol{F}) = \mu_0(\boldsymbol{F}) + 2E\int_0^t \mu_0(p_1(\boldsymbol{F}\circ\phi_E^s))\,ds.$$

We will assume here that $\phi_{E*}^t\mu_0 \overset{\text{weakly}}{\to} \mu_E^+$ as $t \to \infty$. This is an assumption as suggested by Principle (R) from Section 1.1.1. As a result, we obtain

$$\mu_E^+(\boldsymbol{F}) = \mu_0(\boldsymbol{F}) + 2E\int_0^\infty \mu_0(p_1(\boldsymbol{F}\circ\phi_E^s))\,ds.$$

The microscopic current is defined as $\boldsymbol{p}$. In equilibrium $\mu_0(\boldsymbol{p}) = 0$. From a physical perspective this is obvious, and mathematically this can be seen from a symmetry argument. Indeed, μ_0 has a constant density, and $\boldsymbol{p}$ is an odd function under reversal of the velocity vector. We therefore obtain

$$\boldsymbol{J}(\boldsymbol{E}) = \mu_E^+(\boldsymbol{p}) = 2E\int_0^\infty \mu_0(p_1(\boldsymbol{p}\circ\phi_E^s))\,ds.$$

This formula, known as the Kawasaki formula, gives the exact current response as a nonlinear function of the field. The inner integral on the right-hand side $C_{p_1,p_i}(t) = \mu_0(p_1(p_i\circ\phi_E^s))$ is a *time correlation function*. If the decay of correlations is sufficiently fast–for example for all smooth $f_{\boldsymbol{E}}, g_{\boldsymbol{E}}\colon M \to \mathbb{R}$ we have $\mu_0(f_{\boldsymbol{E}}(g_{\boldsymbol{E}}\circ\phi_E^s)) \le C_0(f,g)\theta^{\sqrt{t}}$ (*stretched exponential decay of correlations*) with $C_0(f,g)$ a constant uniformly in E and $0 < \theta < 1$–we can employ Dominated Convergence to write

$$\mu_0(f(g\circ\phi_0^s)) = \mu_0(\lim_{\boldsymbol{E}\downarrow 0} f_{\boldsymbol{E}}(g_{\boldsymbol{E}}\circ\phi_E^s)) = \lim_{\boldsymbol{E}\downarrow 0}\mu_0(f_{\boldsymbol{E}}(g_{\boldsymbol{E}}\circ\phi_E^s)) \le C_0(f,g)\theta^{\sqrt{t}}.$$

In this particular case we need fast decay of correlations with respect to the measure μ_0, which is not invariant under the dynamics. Thus, we obtain Ohm's Law $\boldsymbol{J}(\boldsymbol{E}) = \boldsymbol{D}\cdot\boldsymbol{E} + \boldsymbol{o}|\boldsymbol{E}|$, with the matrix -valued diffusion coefficient $\boldsymbol{D}$ given by the Green-Kubo formula

$$\boldsymbol{D} = 2\int_0^\infty \mu_0(\boldsymbol{p}\otimes(\boldsymbol{p}\circ\phi_0^s))\,ds, \tag{1.2}$$

where $\otimes$ denotes the tensor product. Decay of correlations is often referred to as a *statistical property* of a dynamical system (ϕ_0^s, μ_0), and it provides a measure of "randomness" of the system. Indeed, if in addition f and g satisfy $\mu_0(f) = \mu_0(g) = 0$ and μ_0 is invariant, we see that $\mu_0(f(g \circ \phi_0^s)) \to 0$ expresses a rate of mixing of the system. A different measure of chaoticity is provided by a dynamical central limit theorem (CLT), which states that the average of the dependent random variables $f(\phi_0^s(x))$ converges in distribution to the normal distribution.

1.1.3. History and Recent Developments. In the late 1960s mathematical physicists introduced *thermodynamic formalism*, the study of topological Markov chains endowed with an invariant probability measure, in an effort to understand the mathematics governing equilibrium statistical mechanics. These symbolic systems were shown to exhibit strong statistical properties.

In the early 1970s exponential decay of correlations (EDC) and a CLT were also shown to hold for smooth, uniformly hyperbolic dynamical systems; see e.g. [**B**]. Although of great mathematical interest, these results have only limited physical value. More realistic physical models exhibit nonuniform hyperbolic behavior and/or contain singularities–billiards and attractors being the most well-known examples of these. The methods employed in [**B**], a finite Markov partition of the manifold relating the dynamics to a topological Markov chain, do not easily generalize to systems with singularities. The singularities necessitate a countable Markov partition, corresponding to a topological Markov chain with infinitely many states.

Over the past two decades, investigating statistical properties for nonsmooth systems has grown into an active research area, with some of the forefront researchers in the field believing that singularities slowed the decay of correlations. Recent results show otherwise.

A CLT for billiards was shown by Bunimovich and Sinai [**BS**] in the early 1980s, but EDC proved much harder to obtain. In the early 1990s, stretched exponential decay for certain billiards was obtained by approximations of Markov partitions. In fact, this is the method employed in [**C4**]. An exponential upper bound on the decay of correlations was first obtained by Liverani [**L1**] for a class of two dimensional systems with singularities preserving a symplectic 2-form. Young [**Y**] recently developed a new technique to establish an exponential upper bound for more general class of dimensional systems preserving an SRB measure that is not necessarily smooth. In Chapter 2 we extend Young's result by relaxing some of the assumptions made in [**Y**].

1.2. Scattering Dynamics–Expanding Maps with Small Holes

Pianigiani and Yorke [**PY**] give the following pictorial model of open systems, i.e. chaotic dynamical systems with holes in the phase space through which mass can escape. Imagine a Sinai billiard table (with dispersing boundary) in which the dynamic behavior of the ball is strongly chaotic. Let one or more holes be cut in the table, so that the ball can fall through, see Figure 2. One can also think of these holes as "pockets" at the corners of the table. Let the initial position of the ball be chosen at random with some smooth probability distribution. Denote by $P(t)$ the probability that the ball stays on the table for at least time t, and if it does, by $\rho(t)$ its (normalized) distribution on the table at time t. Some natural questions follow: At what rate does $P(t)$ converge to zero as $t \to \infty$? What is the limit probability distribution $\lim_{t\to\infty} \rho(t)$? And does it depend on the initial distribution $\rho(0)$? These are still open questions.

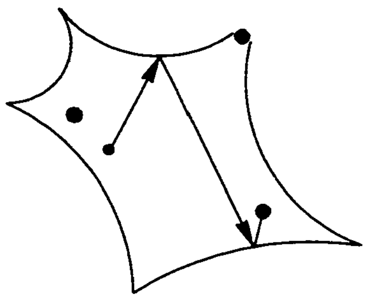


FIGURE 2. A Sinai billiard table with holes.

Over the past decade, the study of open systems has become an active research area in physics, known as chaotic scattering theory. A detailed account of results in this area is beyond the scope of this introduction, but we would like to briefly mention the following remarkable observation [**GN**]. Consider a two-dimensional periodic Lorentz gas $\{0 \le x \le L, -\infty < y < \infty\}$, with absorbing boundaries at $x = 0$ and $x = L$. Choose N_0 test particles with respect to an initial smooth distribution. Some of these particles will escape through the boundary after some time, and the number of particles present in the system at time t decreases exponentially $N_t \simeq N_0 \exp(-\gamma t)$. The parameter γ is known as the *escape rate*. Assume that there exists an invariant set in the phase space of this system, consisting of "trapped" trajectories. In physics literature this set is commonly denoted as the repeller Λ. Assume furthermore that the system has a positive Lyapunov exponent λ_1 with respect to an appropriate invariant measure on Λ and has positive Kolmogorov-Sinai entropy h_{KS}. Then the following *escape rate formula* is believed to hold true: $\gamma = \lambda_1 - h_{\text{KS}}$. From this equation the following relationship can be derived:

$$D = \lim_{L\to\infty} \frac{L^2}{\pi^2}(\lambda_1 - h_{\text{KS}}),$$

with D the diffusion coefficient of the system. This equation is yet another example of a profound relationship between macroscopic quantities (diffusion coefficient) and microscopic, dynamical quantities (right-hand side).

These developments renewed mathematical interest in open systems, and over recent years a number of publications in this area appeared ([**CMT1**], [**CMT2**], [**CMS**], [**LM**], [**PY**]). Pianigiani and Yorke [**PY**] studied one-dimensional, piecewise C^2, expanding (i.e., $s = \inf |T'(x)| > 1$)) transformations $T\colon \bar{A} \to \mathbb{R}$, where $A = \cup_{i=1}^{p} A_i$ and $A_i \subset I$ are disjoint, open intervals. It is assumed that $A \subset T(A)$ and that T satisfies a Markov condition in the sense that $A \cap T(\partial A) = \emptyset$. One of the first problems to consider is how to properly describe such systems. Indeed, we do not expect to find an equivalent invariant measure (equivalent to Lebesgue measure μ_0) since images of such measures will have exponentially decreasing norms as mass leaks out through the holes at a constant rate. However, if we renormalize the images of μ_0, we obtain a sequence of probability measures that may converge to some probability measure μ^+. The limit measure μ^+ is known as a *conditionally invariant measure* and is not invariant under the dynamics. Instead, the image of μ^+ is proportional to itself $\mu^+(T^{-1}E) = \lambda\mu^+(E)$ for measurable subsets $E \subset A$. The parameter λ is usually referred to as the *eigenvalue* of the measure and satisfies $\lambda < 1$.

Pianigiani and Yorke show that an absolutely continuous conditionally invariant probability measure μ^+ exists for these expanding transformations with holes. Moreover, if T satisfies a transitivity condition, then μ^+ is unique in the class of probability measures with a continuous density bounded away from 0. If, furthermore, all iterates of T are transitive, it is shown that any measure ν with a continuous density bounded away from 0 converges to μ^+ in the following sense. Define for any measurable set $E \subset A$ the conditional probability $\nu_n(E) = \nu_0(T^{-n}A)^{-1}\nu_0(T^{-n}E)$. Then ν_n converges weakly to the unique invariant measure μ^+.

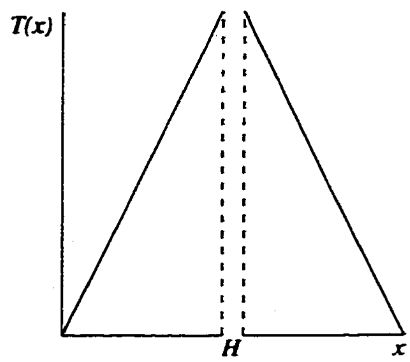


FIGURE 3. An expanding map T with hole H.

The subject of Chapter 3 is the study of rather arbitrary piecewise C^2 maps T with $s = \inf |T'| > 1$ on an interval in which finitely many small, open holes are punched. The approach taken in Chapter 3 differs from Pianigiani and Yorke's in that we start with a piecewise C^2, expanding transformation $\widehat{T}$ of the unit interval I. We remove a finite number of small, open intervals from I to obtain a transformation T with holes that is not necessarily Markov. We show existence of an absolutely continuous, conditionally invariant probability measure μ^+. If, in addition, we assume that $\widehat{T}$ is mixing, μ^+ is equivalent to Lebesgue measure and unique in the class of probability measures with a density of bounded variation. Convergence of the conditional probabilities ν_n as described above is believed to be within reach.

CHAPTER 2

STATISTICAL PROPERTIES OF HYPERBOLIC SYSTEMS WITH TANGENTIAL SINGULARITIES

2.1. Introduction

Recent developments in nonequilibrium statistical mechanics have sparked the interest in the investigation of statistical properties (in particular, EDC) of nonuniformly hyperbolic dynamical systems. In Chapter 1 we have seen that the first result in this direction, an exponential upper bound for the decay of correlations for a class of two-dimensional systems with singularities preserving a symplectic 2-form, was obtained by Liverani [**L1**]. Young [**Y**] only recently proved the existence of an exponential upper bound for more general two-dimensional systems preserving an SRB measure that is not necessarily smooth. Both Liverani and Young assume that the singularity manifolds $\mathcal{S}$ and the unstable cones are uniformly transversal.

In some physical models, however, the angle between the singularity manifolds and the unstable direction vanishes. This is the case in, for example, Wojtkowski's "falling balls"-model [**W**]. Consider the vertical half line $\{q : q \geq 0\}$ with $n \geq 2$ point masses $m_1 \geq m_2 \geq \ldots \geq m_n$ at positions $0 \leq q_1 \leq q_2 \leq \ldots \leq q_n$ freely falling due to a constant gravitational force. The masses collide elastically with each other, and m_1 collides elastically with the floor $q = 0$. In some parts of the phase space of the arising Hamiltonian flow with collisions, the singularity manifolds and the unstable direction are tangential whenever the system comprises $n \geq 3$ balls.

This motivates us to generalize Young's result by allowing the angle between the unstable direction and $\mathcal{S}$ to vanish. We will consider uniformly hyperbolic, piecewise smooth maps (the exact definitions can be found in Section 2.2) together with an SRB-measure (the existence of such a measure will be proved). We allow for tangencies of the singularity manifolds and the unstable directions, but exclude coincidence of the two on a set of positive (induced) volume. Under natural conditions, we prove EDC for a class of Hölder continuous functions and show that a central limit theorem (CLT) holds. We do so by showing that our systems meet the conditions Young uses in [Y] to obtain EDC and CLT. We have included some examples.

2.2. Setting

Let M denote a compact, smooth Riemannian manifold, possibly with boundary. For simplicity we restrict ourselves to two dimensions. Assume that M is endowed with Riemannian metric d and Lebesgue measure m. The induced metric and measure on submanifolds γ of M are denoted by d_γ and m_γ, respectively. Throughout, we denote the tangent space to a (sub)manifold S at the point x by $\mathcal{T}_x S$. Let $\mathcal{S}_i \subset M$, $i = 1, \ldots, N$ denote a finite collection of compact, smooth curves, where $\partial M \subset \cup_i \mathcal{S}_i$. We may assume that for every $\mathcal{S}_i \not\subset \partial M$ we have $\partial \mathcal{S}_i \subset \cup_j \operatorname{Int} \mathcal{S}_j$, i.e. every component of $\mathcal{S}$ not in the boundary ∂M terminates in the interior of another component. This is not a restrictive assumption, because we can extend every component so that it satisfies this condition. The set $\mathcal{S} = \cup_i \mathcal{S}_i$ is the singularity set of T. The following six assumptions define the setting more precisely.

HYP 1. *T is a C^2-diffeomorphism from $M - \mathcal{S}$ onto its image $T(M - \mathcal{S})$. Both T and T^{-1} are twice differentiable up to the boundaries of their domains.*

Note that the derivatives of T and T^{-1} are bounded since M is compact. We let $\mathcal{S}^n = \mathcal{S} \cup \ldots \cup T^{-n+1}\mathcal{S}$ denote the singularity set of T^n, the n-fold composition of T with itself.

HYP 2. *T is uniformly hyperbolic; i.e., for each $x \in M$ there exist cones $C^u_x, C^s_x \subset \mathcal{T}_x M$ such that $D_xT(C^u_x) \subset C^u_{Tx}$ and $D_xT^{-1}(C^s_x) \subset C^s_{T^{-1}x}$ whenever these derivatives* *exist and*

$$\|D_xT(v)\| \geq \lambda\|v\| \qquad \textit{for all } v \in C^u_x$$

$$\|D_xT^{-1}(v)\| \geq \lambda\|v\| \qquad \textit{for all } v \in C^s_x$$

for some constant $\lambda > 1$. These cones vary continuously with x, and the angle between C^u_x and C^s_x has a positive lower bound.

The constant λ above, thus, is a lower bound on the expansion factor of unstable vectors and contraction factor of stable vectors.

We introduce some more notation. An *n-unstable curve* is a curve γ of finite diameter such that $T^{-n}\gamma$ consists of a single smooth component and for all $k = 0, \ldots, n$ and $z \in T^{-k}\gamma$ we have $\mathcal{T}_z T^{-k}\gamma \cap C^u_x \neq \{0\}$. A 0-unstable curve will simply be called a u-curve. An *n-stable curve* is defined similarly. The diameter of a curve is defined as $\text{diam}(\gamma) = \sup_{x,y\in\gamma} d_\gamma(x, y)$. Following [**P**], we define M^+ (M^-) to be the set of points $x \in M$ for which the future (past) orbit is defined; i.e.

$$M^+ = M - \bigcup_{n\geq 0} T^{-n}\mathcal{S}, \qquad M^- = \bigcap_{n\geq 1} T^n(M - \mathcal{S}^n).$$

The set $M^0 = M^+ \cap M^-$ consists of points for which the entire orbit is defined. Furthermore, let

$$E^s_x = \bigcap_{n\geq 0} D_{T^n x}T^{-n}(C^s_{T^n x}) \qquad \text{for } x \in M^+$$

$$E^u_x = \bigcap_{n\geq 0} D_{T^{-n} x}T^{n}(C^u_{T^{-n} x}) \qquad \text{for } x \in M^-.$$

Then $\mathcal{T}_x M = E^s_x \oplus E^u_x$ whenever $x \in M^0$ and the angle between E^s_x and E^u_x has a positive lower bound, uniformly on M^0. For a connected submanifold $\omega \subset M$ the smooth components of $T^{-n}(\omega - \cup^n_{k\geq 1} T^k\mathcal{S})$ are called the components of $T^{-n}\omega$. We call a submanifold ω a local unstable manifold (LUM) if $T^{-n}\omega$ has exactly one component for all $n \geq 0$ and for all $x, y \in \omega$ we have $d(T^{-n}x, T^{-n}y) \to 0$ as $n \to \infty$ exponentially fast. We denote the closed ϵ-ball in ω centered at x (whenever it exists) by $W^u_\epsilon(x)$. Local stable manifolds (LSMs) and $W^s_\epsilon(x)$ are defined analogously for forward iterates of T. Define for $\kappa > 1$

$$\begin{aligned} M^-_{\kappa,\epsilon} =& \{x \in M^- : d(T^{-n}x, T\mathcal{S}) \geq \epsilon\kappa^{-n} \;\; \forall n \geq 0\} \\ M^+_{\kappa,\epsilon} =& \{x \in M^+ : d(T^{n}x, \mathcal{S}) \geq \epsilon\kappa^{-n} \;\; \forall n \geq 0\} \\ M^0_{\kappa,\epsilon} =& M^-_{\kappa,\epsilon} \cap M^+_{\kappa,\epsilon} \\ M^0_{\kappa} =& \bigcup_{\epsilon>0} M^0_{\kappa,\epsilon}. \end{aligned}$$

Observe that $M^{\pm}_{\kappa,\epsilon}$ is a closed set for a fixed $\epsilon > 0$ and that M^0_κ is T-invariant. Proposition 4 of [**P**] implies that there exists a $\bar{\kappa} > 1$ such that for all $x \in M^-_{\bar{\kappa},\epsilon}(M^+_{\bar{\kappa},\epsilon})$ and some $\delta(\epsilon) > 0$ the C^1-submanifold $W^u_\delta(x)(W^s_\delta(x))$ exists and is unique. At this point it is not clear that $M^-_{\bar{\kappa},\epsilon} \neq \emptyset$. This is discussed at the end of this section.

In showing statistical properties of systems satisfying Hyp 1 and 2, previous results ([**Y**], [**C3**]) assume that u-curves intersect the singularity manifolds transversally. We would like to relax that assumption and allow for tangencies. Intuitively, $x \in \mathcal{S}$ is a *point of tangency* if $\mathcal{T}_x\mathcal{S} \cap E^u_x \neq \{0\}$. We can not use this definition because E^u_x may not exist. The following assumption allows singularity manifolds to intersect LUMs tangentially, provided they separate fast enough away from the point of intersection.

HYP 3. *There exists a $0 < p < \infty$ and a constant $C > 0$ such that for all $\epsilon > 0$ and all* LUMs *γ we have*

$$m_\gamma(\gamma \cap B_\epsilon(\mathcal{S})) < C\epsilon^{\frac{1}{p}}.$$

We call the number p the *order of the tangency.*

A point $x \in \mathcal{S}^n$ is called *multiple* with multiplicity $l_{x,n}$ if it belongs to $l_{x,n} \geq 2$ smooth components of $\mathcal{S}^n$.

HYP 4. *There exists a $K \geq 1$ and an $m \geq 1$ such that for all $x \in \mathcal{S}^m$ we have $l_{x,m} \leq K < \frac{\lambda^m - 1}{2}$.*

Similar assumptions are standard (see for example [**C3**]) and necessary to ensure that LUMs grow faster than they get shredded by singularity manifolds. Note that Hyp 4 implies that there exists an $\epsilon_0 > 0$ such that every ϵ_0-ball in M intersects at most K smooth components of $\mathcal{S}^m$.

Allowing for tangencies introduces a new problem to the study of decay of correlations: a smooth component of $\mathcal{S}$ may intersect a LUM infinitely many times. The following assumption excludes this possibility. Let $\mathcal{S}_i^m$ denote a smooth component of $\mathcal{S}^m$, with m as in Hyp 4.

HYP 5. *There exists an $\eta > 0$ such that for all $\epsilon > 0$ the intersection $\gamma \cap B_\epsilon(\mathcal{S}_i^m)$ consists of at most two disjoint subintervals of γ whenever γ is a* LUM *with* $\mathrm{diam}(\gamma) < \eta$.

The number of subintervals in Hyp 5 is fixed at two for notational convenience; it can be replaced by any finite number (Hyp 4 should then be modified accordingly).

Without loss of generality we can set $m = 1$ in Hyps 4 and 5. Indeed, assumptions 1-3 still hold for T^m, and it suffices to show our main theorem for T^m. Henceforth we will assume that $m = 1$. Note that Hyp 3 and Hyp 5 prevent the singularity manifolds from coinciding with a LUM γ on a set of positive m_γ-measure. The last two Hyps lead us to formulate the following lemma.

LEMMA 2.1. *Set $\bar{\epsilon} := \min\{\epsilon_0, \eta\}$. Then for all $\delta_1 > 0$ and for every* LUM *γ with* $\mathrm{diam}(\gamma) < \bar{\epsilon}$, *the set $\{x \in \gamma : d_\gamma(x, \mathcal{S}) > \delta_1)\}$ has $\leq 2K + 1 \stackrel{\mathrm{def}}{=} K_0$ connected components.*

As remarked above, it is not obvious that $M_\kappa^0 \neq \emptyset$ in our setup. It is well known that systems satisfying Hyp 1, 2 and the following assumption

(H3) *There exist a $C > 0$ and a $q > 0$ such that for all $\epsilon > 0$ and $n \geq 1$*

$$m(T^{-n} B_\epsilon(\mathcal{S})) \leq C\epsilon^q,$$

have the property that $M_\kappa^0 \neq \emptyset$ (see [**P**]). Chernov [**C3**] uses u-curves to show that systems satisfying Hyps 1, 2, 4 and a transversality condition on the intersection of u-curves and singularity manifolds satisfy (H3), thus establishing the existence of LUMs. In the current setup we can not hope to use u-curves to obtain (H3), because we can not control the intersection of u-curves and neighborhoods of $\mathcal{S}$. Instead, we will assume that $M_{\bar{\kappa},\epsilon}^-$ contains at least one point x; i.e.,

HYP 6. *$M_{\bar{\kappa},\epsilon}^- \neq \emptyset$ for $\epsilon > 0$ small enough.*

In Section 2.5 it is then shown that there exists a T-invariant probability measure μ with $\mu(M^0_{\bar{\kappa}}) = 1$.

It was observed by Chernov that Hyp 6 is necessary in the sense that it does not follow from the first five assumptions. We adapt his example from Section 10 in **[C2]**.

EXAMPLE 2.2. *Let $R := \{(x,y)\colon 0 < x < 1, y > 1\}$ be an open strip in $\mathbb{R}^2$ and let $M' := \{(s,t)\colon 0 \leq s \leq 1, 0 \leq t \leq 1\}$ with the identification of $s = 0$ and $s = 1$ be a closed cylinder. Define $T_1\colon R \to R$ given by $(x,y) \mapsto (\frac{x}{3}, 2y-1)$ and $T_2\colon R \to M'$ given by $s = y \mod 1$ and $t = \exp(-y) + x(\exp(-y-1) - \exp(-y))$. Define $T = T_2 \circ T_1 \circ T_2^{-1}$. Obviously, Hyp 1 and 2 are satisfied for T. Since for each point at most finitely many pre-images are defined, no* LUMs *exist. Hence Hyp 3-5 are trivially satisfied, yet $M^-_{\bar{\kappa},\epsilon} = \emptyset$, for all $\epsilon > 0$.*

DEFINITION 2.3. *A T-invariant measure μ_{SRB}, supported on M^0, is called an* SRB-*measure if the conditional measures of μ_{SRB} on local unstable manifolds are absolutely continuous with respect to the Lebesgue measure on those manifolds.*

Measures with the SRB property describe the asymptotics of observables in physical models: If μ_{SRB} is an ergodic SRB measure concentrated on a set $\Lambda \subset \overline{M^0}$, there exists a set $W^s(\Lambda)$ containing Λ with positive volume such that for any continuous $\psi\colon M \to \mathbb{R}$ we have $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \psi \circ T^k z = \int_\Lambda \psi \, d\mu_{\text{SRB}}$ for m-almost every $x \in W^s(\Lambda)$. If the map T preserves a smooth measure m_0, it is automatically an SRB measure. In case the set M^0 has full volume and μ_{SRB} is singular, its support can still coincide with $\overline{M}^0$. This typically happens for transitive Anosov systems. If $m(M^0) < m(M)$, the support of μ_{SRB} may have zero volume, and the set Λ above is sometimes called an *attractor for* (T, μ_{SRB}).

We conclude this section with an elementary example of a dynamical system satisfying Hyp 1-6.

EXAMPLE 2.4. *Consider a hyperbolic toral automorphism $T_1\colon \mathbb{T}^2 \to \mathbb{T}^2$ given by $(x,y) \mapsto (2x+y, x+y)$. Cut out an open disk S in $\mathbb{T}^2$ and define $T_2\colon S \to S$ to be the map that rotates the disk over an angle π and leaves the rest of the torus invariant.*

The singularity manifold $\mathcal{S}$ for the map $T = T_1 \circ T_2$ is given by the boundary of the disk ∂S, which has two points of tangency with the unstable direction.

2.3. Results

We state the main results of this paper. The proofs can be found in Sections 2.5-2.7.

THEOREM 2.5. *Let T satisfy Hyp 1-6. Then*

(a) *T admits an* SRB *measure μ_{SRB}, which is not necessarily unique.*

(b) *Any measure μ_{SRB} has a finite number of ergodic components on each of which it is, up to a finite cycle, mixing and Bernoulli.*

Let $\mathcal{H}_\eta$ denote the class of Hölder continuous functions on M with Hölder exponent $\eta > 0$; i.e.

$$\mathcal{H}_\eta := \{\phi\colon M \to \mathbb{R}\colon |\phi(x) - \phi(y)| \leq C d(x,y)^\eta \text{ for all } x, y \in M\},$$

where $C > 0$ depends on the function ϕ. The time correlation function $C_{\phi,\psi}(n)$ is defined by

$$C_{\phi,\psi}(n) = \int (\phi \circ T^n)\psi \, d\mu - \int \phi \, d\mu \int \psi \, d\mu$$

DEFINITION 2.6. *The dynamical system (T, μ) exhibits exponential decay of correlations for Hölder continuous functions if for any $\eta > 0$ there exists a $0 < \theta < 1$ such that for all $\phi, \psi \in \mathcal{H}_\eta$*

$$|C_{\phi,\psi}(n)| \leq C\theta^n \text{ for all } n \geq 1,$$

for some $C > 0$ depending on the functions ϕ, ψ.

DEFINITION 2.7. *Let $\phi \in \mathcal{H}_\eta$ such that $\int \phi d\mu = 0$. We say that ϕ satisfies a CLT with respect to (T, μ) if the random variables $\phi \circ T^n$ satisfy*

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \phi \circ T^i \xrightarrow{\text{distr}} N(0, \sigma),$$

where N denotes the normal distribution and σ depends on ϕ.

THEOREM 2.8. *Let T satisfy Hyp 1-6 and suppose that (T^n, μ_{SRB}) is ergodic for all $n \geq 1$. Then (T, μ_{SRB}) exhibits exponential decay of correlations and satisfies a central limit theorem.*

2.4. Standard Facts

We introduce some notation, definitions, and standard results for dynamical systems with singularities; see also [**KS**]. Let γ denote a u-curve. For $x \in \gamma$ let $J^u(x) = |\det D_xT \restriction T_x\gamma|$ denote the Jacobian at the point x, restricted to γ.

DISTORTION ALONG u-CURVES. Let $x, y \in \gamma - \mathcal{S}^n$ and $T^n x, T^n y$ belong in the same connected component $\omega \subset T^n\gamma$. Then

$$\log \prod_{i=0}^{n-1} \frac{J^u(T^i x)}{J^u(T^i y)} \leq C' d_\omega(T^n x, T^n y) \tag{2.1}$$

for some $C'(T) > 0$.

BOUNDED CURVATURE. The curvature of LUMs and LSMs is uniformly bounded by some constant $B > 0$.

ABSOLUTE CONTINUITY. Let $W^u_\eta(x)$ and $W^u_{\eta'}(x')$ denote two LUMs and let

$$W_\infty := \{y \in W^u_\eta(x) \colon W^s_{\delta(y)}(y) \cap W^u_{\eta'}(x') \neq \emptyset\}.$$

Assume that $\delta(y), \eta, \eta'$ are small enough such that any LSM intersects each of those LUMs at most once. Define a map $h \colon W_\infty \to W^u_{\eta'}(x')$ by $y \mapsto W^s_{\delta(y)}(y) \cap W^u_{\eta'}(x')$. Then h is absolutely continuous with respect to $m_{W^u_\eta(x)}, m_{W^u_{\eta'}(x')}$, and its Jacobian is bounded at any point of density of $m_{W^u_\eta(x)}$; i.e.

$$\frac{1}{C_0} \leq \frac{m_{W^u_{\eta'}(x')}\big(h(W_\infty)\big)}{m_{W^u_\eta(x)}\big(W_\infty\big)} \leq C_0 \tag{2.2}$$

for some $C_0 > 0$.

2.5. Existence of SRB Measures

In this section we prove Theorem 2.5(a) by constructing an SRB-measure for T. In [**P**], Pesin proves the existence of an SRB measure (called a Gibbs u-measure) for systems satisfying four assumptions, including our Hyps 1 and 2 and the additional (H3) mentioned in Section 2.2. We can not verify Pesin's hypotheses for our systems and use a different approach to obtain an SRB-measure.

The class of systems Young studies in Section 7 of [Y] does satisfy Pesin's assumptions, but a direct argument for the existence of an SRB measure is given there. Young proves the existence of some invariant probability measure μ satisfying (2.4) below and later proves the existence of an SRB-measure. We extend her approach to our systems and argue directly that the measure obtained from (2.3) below is SRB.

Choose $\epsilon > 0$ such that $M^-_{\bar{K},\epsilon} \neq \emptyset$. Let γ denote the LUM corresponding to one of the points in $M^-_{\bar{K},\epsilon}$. Define

$$\mu_N = \frac{1}{N} \sum_{k=0}^{N-1} T^k_* m_\gamma; \tag{2.3}$$

i.e. we iterate Lebesgue measure on the local unstable manifold γ. Any (normalized) limit point μ of $\{\mu_N\}$ in the weak*-topology is an invariant Borel probability measure on $\overline{M^0}$. Since the set M^0 is not compact, mass may escape from M^0 in the limit of (2.3). The following lemma prevents this from happening. Once we obtain (2.4) below, it easily follows that any limit point μ of (2.3) is supported on M^0.

LEMMA 2.9. *Let T satisfy Hyp 1 - 6 and let μ be a normalized limit point of (2.3). Then for all $\epsilon > 0$ we have*

$$\mu(B_\epsilon(\mathcal{S})) < C\epsilon^{\frac{1}{p}} \tag{2.4}$$

for some $C > 0$.

Following Young [Y], we will use the idea of stopping times on LUMs in the proof of this lemma. Let γ denote an arbitrary LUM with $\operatorname{diam}(\gamma) < \bar{\epsilon}$. The idea is to keep record of the length of the connected components of $T^n\gamma$. We set $\gamma_0 := \gamma - \mathcal{S}$, which consists of finitely many connected components ω_i. In general, $\gamma_n := \gamma_{n-1} - T^{-n}\mathcal{S}$. The stopping times are defined by a sequence of mappings $s_1 < s_2 < \dots$ from subsets of γ into $\mathbb{N}$ as follows. For $x \in \gamma$ let

$$s_1(x) = \min_{n \geq 1}\{m_{T^n\omega_x}(T^n\omega_x) > \bar{\epsilon}\},$$

with ω_x the component of γ_{n-1} containing x. If no such n exists, we set $s_1(x) = \infty$. The n-dependence of ω_x will be tacitly understood in the remainder of this paper.

LEMMA 2.10. *Let γ and γ_n be defined as above. Then*

$$m_\gamma(\gamma_n - \{x\colon s_1(x) \leq n\}) \leq \bar{\epsilon}\left(\frac{K_0}{\lambda}\right)^n. \tag{2.5}$$

PROOF. Since $\mathrm{diam}(\gamma) < \bar{\epsilon}$, γ_0 consists of at most K_0 connected components γ_{0_i} with $\mathrm{diam}(\gamma_{0_i}) < \bar{\epsilon}$. For those components with $s_1(x) \restriction \gamma_{0_i} \neq 1$, $\mathrm{diam}(T\gamma_{0_i}) < \bar{\epsilon}$. At the nth step we have that $\gamma_n - \{x\colon s_1(x) \leq n\}$ is the union of at most K_0^n components $\gamma_{n_k} \subset \gamma_n$ with $\mathrm{diam}(T^n\gamma_{n_k}) \leq \bar{\epsilon}$. Pulling back we obtain (2.5). □

In particular, $s_1(x) < \infty$ for m_γ-almost every $x \in \gamma$. On the full measure subset of γ for which $s_1(x) < \infty$, define

$$s_2(x) = \min_{n > s_1(x)}\{m_{T^n\omega_x}(T^n\omega_x) > \bar{\epsilon}\},$$

with ω_x the component of γ_{n-1} containing x. It is easily seen that s_2 is also defined on a subset of full measure of γ. Continuing this procedure we obtain that for any LUM γ with $\mathrm{diam}(\gamma) < \bar{\epsilon}$ and for any $k \geq 1$ the kth stopping time $s_k(x)$ is defined m_γ-almost everywhere.

PROOF OF LEMMA 2.9. Let $\epsilon > 0$ be given. Let γ be a LUM with $\mathrm{diam}(\gamma) < \bar{\epsilon}$ and μ a normalized limit point of the sequence given by (2.3). Our strategy is to estimate $\sum_{n=0}^{N-1} m_\gamma(T^{-n}B_\epsilon(\mathcal{S}))$.

Fix an integer $N > 0$. Let s_i be defined on γ as above. Observe that $\{x\colon s_k(x) < N\}$ is a disjoint union of a finite number of connected components of γ for all $k < N$. We will evaluate $\sum_{n=0}^{N-1} m_\gamma(T^{-n}B_\epsilon(\mathcal{S}))$ on the subsets $\{x\colon s_1(x) \geq N\}$, $S_k := \{x\colon s_k(x) < N\} \cap \{x\colon s_{k+1}(x) \geq N\}$ for $1 \leq k < N-1$ and $\{x\colon s_{N-1}(x) = N-1\}$ separately.

Consider the set $\{x\colon s_1(x) \geq N\}$. As in the proof of Lemma 2.10, we see that for each natural n the set $\{x\colon s_1(x) \geq n\}$ is a disjoint union of at most countably many components. Consider one of these components ω with $s_1 \lceil \omega \neq n$. Then $m_{T^n\omega}(T^n\omega) \leq \bar{\epsilon}$, and by Lemma 2.1 it intersects $B_\epsilon(\mathcal{S})$ in $\leq K_0$ components. Therefore. by Hyp 3

$$m_{T^n\omega}(T^n\omega \cap B_\epsilon(\mathcal{S})) < K_0 C\epsilon^{\frac{1}{p}}.$$

Pulling back to ω and summing up to the $(N-1)$-st iterate over the components in $\{x\colon s_1(x) \geq N\}$, we obtain

$$\sum_{n=0}^{N-1} m_\gamma(T^{-n}B_\epsilon(\mathcal{S}) \cap \{x\colon s_1(x) \geq N\}) < K_0 C\epsilon^{\frac{1}{p}} \sum_{n=0}^{N-1} (\frac{K_0}{\lambda})^n \tag{2.6}$$

Next, we estimate $\sum_{n=0}^{N-1} m_\gamma(T^{-n}B_\epsilon(\mathcal{S}))$ on the subsets S_k for $1 \leq k < N-1$. Let ω denote a component of γ_{j-1} and $s_k \lceil \omega = j$. Observe that $j \geq k$. Partition ω into $\omega_1 \cup \ldots \cup \omega_m$ with $\frac{\bar{\epsilon}}{2} \leq m_{T^j\omega}(T^j\omega_i) < \bar{\epsilon}$. We use the exact same procedure as above, choosing $T^j\omega$ as our starting point and with ω replaced by $T^j\omega_i$.

It may happen that $T^{-i}B_\epsilon(\mathcal{S}) \cap \omega \neq \emptyset$, $0 \leq i < j$ for such a component ω. Observe that the measure of this set is already taken into account by the estimates up to the kth stopping time. Indeed, for a segment in the set $\{x\colon s_l(x) < N\}$, $0 \leq l < k$, we in fact overestimate the measure covered by pre-images of $B_\epsilon(\mathcal{S})$ at the nth iterate by assuming that all its K_0^n components have $\mathrm{diam}(T^n\omega) \leq \bar{\epsilon}$, regardless of whether they are contained in $\{x\colon s_{l+1} < N\}$.

We compute the fraction of $T^j\omega_i$ covered by pre-images of $B_\epsilon(\mathcal{S})$ before the $k+1$-st stopping time and claim that this fraction approximately equals the fraction covering $S_k \cap \omega$. This claim is made precise in Lemma 2.11 below. We obtain that

$$\sum_{n=0}^{N-1-j} m_\omega(T^{-(n+j)}B_\epsilon(\mathcal{S}) \cap S_k) < m_\omega(\omega)\exp(C'\bar{\epsilon})\frac{2K_0C\epsilon^{\frac{1}{p}}}{\bar{\epsilon}} \sum_{n=0}^{N-1-j} (\frac{K_0}{\lambda})^n.$$

Summing over all components $\omega \in \{x\colon s_k(x) < N\}$ we obtain

$$\sum_{j=k}^{N-1} \sum_{n=0}^{N-1-j} m_\gamma(T^{-(n+j)}B_\epsilon(\mathcal{S}) \cap S_k) < \exp(C'\bar{\epsilon})2K_0C\epsilon^{\frac{1}{p}} \sum_{n=0}^{N-1-k} (\frac{K_0}{\lambda})^n.$$

Hence, summing up to $\{x\colon s_{N-2}(x) < N\}$, adding (2.6), dividing by N, and writing $S := \cup^{N-2} S_k \cup \{x\colon s_1(x) \geq N\}$, we obtain for all $N \geq 0$

$$\frac{1}{N}\sum_{n=0}^{N-1} m_\gamma(T^{-n}B_\epsilon(\mathcal{S}) \cap S) < \exp(C'\bar{\epsilon})2K_0C\epsilon^{\frac{1}{p}} \sum_{n=0}^{N-1} (\frac{K_0}{\lambda})^n. \tag{2.7}$$

Finally, we estimate $\sum_{n=0}^{N-1} m_\gamma(T^{-n}B_\epsilon(\mathcal{S}))$ on $\{x\colon s_{N-1} = N-1\}$. For $\omega \subset \{x\colon s_{N-1} = N-1\}$ we have $m_{T^n\omega}(T^n\omega) > \bar{\epsilon}$ for all $n = 1, \ldots, N-1$. Fix an n and partition ω into $\omega_1 \cup \ldots \cup \omega_m$ with $\frac{\bar{\epsilon}}{2} \le m_{T^n\omega}(T^n\omega_i) < \bar{\epsilon}$. Using the same estimates as above, we first sum over all components in Θ_{N-1}, and then over n. After dividing by N, we obtain

$$\frac{1}{N}\sum_{n=1}^{N-1} m_\gamma(T^{-n}B_\epsilon(\mathcal{S}) \cap \{x\colon s_{N-1}(x) = N-1\}) < \frac{\exp(C'\bar{\epsilon})2K_0C\epsilon^{\frac{1}{p}}}{N}\sum_{n=0}^{N-1}(\frac{K_0}{\lambda})^n.$$

Adding this last sum to (2.7) and letting $N \to \infty$ yields (2.4). □

In the proof, we estimate the measure of the fraction of $\{x\colon s_k(x) < N\} \cap \{x\colon s_{k+1}(x) \ge N\}$ covered by pre-images of $B_\epsilon(\mathcal{S})$ by the corresponding fraction of $T^j\omega_i$. The following lemma justifies this estimate. The main idea is that the jth iterate of Lebesgue measure on ω_i, denoted by $T^j_* m_{\omega_i}$, has an almost constant density with respect to $m_{T^j\omega_i}$.

LEMMA 2.11. *Let ω be as in the proof of Lemma 2.9 and $B \subset T^j\omega$ a measurable subset. Then*

$$\exp(-C'\bar{\epsilon})\frac{m_{T^j\omega}(B)}{m_{T^j\omega}(T^j\omega)} \le \frac{m_\omega(T^{-j}B)}{m_\omega(\omega)} \le \exp(C'\bar{\epsilon})\frac{m_{T^j\omega}(B)}{m_{T^j\omega}(T^j\omega)} \tag{2.8}$$

PROOF. Define a density $\rho(x)$ on a component $T^j\omega_{i_k}$ of $T^j\omega_i$ by

$$\rho(x) = \frac{dT^j_* m_{\omega_{i_k}}}{dm_{T^j\omega_{i_k}}}.$$

On each of those $T^j\omega_{i_k}$ the distortion estimate (2.1) gives $\exp(-C'\bar{\epsilon}) \le \frac{\rho(x)}{\rho(y)} \le \exp(C'\bar{\epsilon})$, which yields (2.8). □

Next, we turn to the existence of local stable and unstable manifolds. Recall that for every $x \in M^0_\kappa$ both an LSM and a LUM exist. A standard application of the Borel-Cantelli Lemma gives that $\mu(M^0_\kappa) = 1$. In particular, the following corollary holds.

COROLLARY 2.12. *Let μ be a normalized limit-point of (2.3). Then for μ-almost every $x \in M^0$ and some measurable functions $\eta(x), \eta'(x) > 0$, the* LUM *$W^u_{\eta(x)}(x)$ and the* LSM *$W^s_{\eta'(x)}(x)$ exist.*

From the above corollary it does not immediately follow that for every $z \in W^u_{\eta(x)}(x)$ an LSM $W^s_{\eta'(z)}(z)$ exists. Indeed, $W^u_{\eta(x)}(x)$ may not be completely contained inside M^0_κ. We obtain $m_{W^u_{\eta(x)}}$-almost everywhere existence of LSMs on $W^u_{\eta(x)}(x)$, however, from applying the Borel-Cantelli Lemma to the measure $m_{W^u_{\eta(x)}(x)}$, cf. **[C3]**.

Finally, we conclude that the invariant measure obtained from (2.3) is an SRB measure. Indeed, for almost every $x \in M^0$ a LUM exists. From the construction of μ (iteration of a smooth measure) it follows that on such a LUM the measure μ is absolutely continuous. This concludes the proof of Theorem 2.5(a). Henceforth we will denote limit points of (2.3) by μ_{SRB}.

2.6. Construction of the Hyperbolic Set Λ and Return Times

The key instrument in **[Y]** to obtain exponential decay of correlations for hyperbolic dynamical systems is a set $\Lambda \subset M$ with hyperbolic product structure; a "generalized horseshoe." In this section we present a generic method to construct finitely many sets Λ_i with hyperbolic product structure such that $\Lambda = \cup_i \Lambda_i$ covers a "large part" of M^0. The reader may think of this construction as a generalized Markov partition of Λ. However, a warning should be issued. Although we indeed construct finitely many "rectangles" Λ_i, these rectangles do not necessarily cover the support of μ_{SRB} and are generally not disjoint. Once Λ is constructed, we define a return map $T^\tau \colon \cup_i \Lambda_i \to \cup_i \Lambda_i$.

2.6.1. Construction of the Hyperbolic Set Λ. Choose $4\delta_0 \le \bar{\epsilon}$, with $\bar{\epsilon}$ as in Lemma 2.1, such that $A_{\delta_0} := \{x \in M^- \colon W^u_{\delta_0}(x) \text{ exists}\} \neq \emptyset$. Our choice of δ_0 is such that $\delta_0 \ll \frac{1}{B}$, with $\frac{1}{B}$ the minimum radius of curvature of LUMs and LSMs. For $x \in A_{\delta_0}$ we can not conclude that the LUM $W^u_{\frac{\delta_0}{3}}(x)$ exists uniquely. Therefore, let

$$W(x) := W^u_{\frac{\delta_0}{3}}(x) \in \{W^u_{\frac{\delta_0}{3}}(z)\}_{z \in A_{\delta_0}}$$

denote an arbitrary LUM about $x \in A_{\delta_0}$. We will construct a subset $W_\infty(x) \subset W(x)$ with the property that for all $y \in W_\infty(x)$ a "sufficiently long" LSM exists; i.e. for some $\delta_1 \ll \delta_0$ to be determined, $W^s_{\frac{\delta_1}{3}}(y)$ exists $\forall y \in W_\infty(x)$. We construct $W_\infty(x)$ as follows. Let ω_i denote the connected components of $W(x) \cap B_{\delta_1}(\mathcal{S})$. Delete ω_i from $W(x)$ only if the minimum distance between ω_i and $\mathcal{S}$ is less than $\frac{\delta_1}{2}$. Call the resulting set $W_0(x)$. Suppose $W_{n-1}(x)$ is defined inductively as above, and let in the nth step ω_i be the connected components of $\{y \in W_{n-1}(x) \colon d(T^n y, \mathcal{S}) < \delta_1 \lambda^{-n}\}$. Delete ω_i from $W_{n-1}(x)$ only if $d(T^n \omega_i, \mathcal{S}) < \frac{1}{2}\delta_1 \lambda^{-n}$. Now let $W_\infty(x) = \cap_n W_n(x)$. Note that $W_\infty(x)$ is a closed subset of $W(x)$.

LEMMA 2.13. *Let $0 < c < 1$ be arbitrary. Then there exists a $\delta_1 > 0$ such that*

$$m_{W(x)}(W_\infty(x)) > c\, m_{W(x)}(W(x))$$

for all $W(x)$ as above.

Set $W := W(x)$ and $W_n = W_n(x)$. The proof closely follows that of Lemma 2.9; our strategy is to estimate $\sum_{n=0}^{\infty} m_W(T^{-n} B_{\delta_1 \lambda^{-n}}(\mathcal{S}))$. Define a sequence of stopping times $s_1 < s_2 < \ldots$ on W as in Section 2.4 with the following difference. For $x \in W$ let

$$s_1(x) = \min_{n \geq 1} \{m_{T^n \omega_x}(T^n \omega_x) > \bar{\epsilon}\},$$

with ω_x the component of W_{n-1} containing x. If no such n exists, or x is deleted from W before the stopping time is reached, we agree that $s_1(x)$ is not defined. On $\Theta_1 := \{x \in W \colon s_1(x) \text{ is defined}\}$ define

$$s_2(x) = \min_{n > s_1(x)} \{m_{T^n \omega_x}(T^n \omega_x) > \bar{\epsilon}\},$$

with ω_x the component of W_{n-1} containing x. Set $\Theta_2 := \{x \in \Theta_1 \colon s_2(x) \text{ is defined}\}$, and so on. It is again easily seen that each Θ_k is a disjoint union of a countable number of components.

PROOF OF LEMMA 2.13. Observe that whenever $x \in \Theta_k$ for all k, the point x never gets within an $\delta_1 \lambda^{-n}$-neighborhood of $\mathcal{S}$. Hence, it suffices to consider $\sum_{n=0}^{\infty} m_W(T^{-n} B_{\delta_1 \lambda^{-n}}(\mathcal{S}))$ on the subsets $W - \Theta_1$ and $\Theta_k - \Theta_{k+1}$ for all $k \geq 1$.

Starting with $W - \Theta_1$ we obtain, using the same methods as in Lemma 2.9, that the total measure deleted from $W - \Theta_1$ is at most

$$\sum_{n=0}^{\infty} m_W\big(T^{-n}B_{\delta_1\lambda^{-n}}(\mathcal{S}) \cap (W - \Theta_1)\big) < K_0 C \delta_1^{\frac{1}{p}} \sum_{n=0}^{\infty} \lambda^{-\frac{n}{p}} (\frac{K_0}{\lambda})^n.$$

Using the second part of the proof of Lemma 2.9, we obtain for the total measure deleted from $\Theta_k - \Theta_{k+1}$

$$\sum_{n=0}^{\infty} m_W\big(T^{-n}B_{\delta_1\lambda^{-n}}(\mathcal{S}) \cap (\Theta_k - \Theta_{k+1})\big) < \exp(C'\bar{\epsilon})\lambda^{-\frac{k}{p}} 2K_0 C \delta_1^{\frac{1}{p}} \sum_{n=0}^{\infty} \lambda^{-\frac{n}{p}} (\frac{K_0}{\lambda})^n.$$

Summing over k we see that the total measure deleted can be made arbitrarily small by choosing δ_1 accordingly. $\square$

We proceed with the construction of the set $\Lambda \subset M$. For the remainder we fix the constant c from Lemma 2.13 close to 1 and set $\delta = \frac{\delta_1}{3}$. Note that $W_\delta^s(y)$ exists uniquely for all $y \in W_\infty$. For each $W_\infty(x)$, $x \in A_{\delta_0}$, set

$$\Gamma^s(W_\infty(x)) = \{W_\delta^s(y) \colon y \in W_\infty(x)\}.$$

Since the angle between C_x^u and C_x^s has a positive lower bound and LSMs have bounded curvature uniformly on M, there exists an $\eta > 0$ depending only on this angle and the curvature such that for all $W^u_{\frac{\delta_0}{3}}(x) \in \{W^u_{\frac{\delta_0}{3}}(z)\}_{z \in A_{\delta_0}}$

$$d(\partial W_\delta^s(y), W^u_{\frac{\delta_0}{3}}(x)) > \eta \tag{2.9}$$

for all $W_\delta^s(y) \in \Gamma^s(W_\infty(x))$. For the remainder of this paper, fix $0 < \eta < \frac{\delta}{4}$ satisfying (2.9).

We let $\mathcal{K}(M)$ denote the collection of all nonempty compact subsets of the manifold M, endowed with the Hausdorff metric

$$H(K_1, K_2) = \inf\{\delta \colon K_1 \subset B_\delta(K_2) \text{ and } K_2 \subset B_\delta(K_1)\}.$$

Since M is compact, the space $\mathcal{K}(M)$ is compact.

Let $\{W^u_{\delta_0}(x_i)\}_{i=1}^N$ be the smallest finite $\frac{\eta}{2}$-dense subset of $\{W^u_{\delta_0}(z)\}_{z \in A_{\delta_0}} \subset \mathcal{K}(M)$, distinguishing the x_i by their index. For each x_i set

$$\hat{\Gamma}^u(x_i) := \{W^u_{\delta_0}(z) \colon H(W^u_{\frac{\delta_0}{3}}(z), W^u_{\frac{\delta_0}{3}}(x_i)) < \frac{\eta}{2}\},$$

and define a set with hyperbolic product structure by $\hat{\Lambda}_i := \hat{\Lambda}(x_i) = \Gamma^s(x_i) \cap \hat{\Gamma}^u(x_i)$, where $\Gamma^s(x_i) := \Gamma^s(W_\infty(x_i))$. With each $\hat{\Lambda}_i$ we associate a slightly larger set $\Lambda_i := \Gamma^s(x_i) \cap \Gamma^u(x_i)$ where

$$\Gamma^u(x_i) = \{W^u_{\delta_0}(z) \colon H(W^u_{\frac{\delta_0}{3}}(z), W^u_{\frac{\delta_0}{3}}(x_i)) < \eta\}.$$

We will call the sets Λ_i rectangles and refer to $W^u_{\frac{\delta_0}{3}}(x_i)$ as *the central* LUM *of* Λ_i. Note that rectangles are not necessarily disjoint; they may overlap. Observe that each $W^u_{\delta_0}(z) \in \Gamma^u(x_i)$ intersects every LSM $W^s_\delta(y) \in \Gamma^s(x_i)$. Indeed, $H(W^u_{\frac{\delta_0}{3}}(z), W^u_{\frac{\delta_0}{3}}(x_i)) < \eta$ for all such $W^u_{\frac{\delta_0}{3}}(z)$. In fact, if in rectangle Λ_i we let $[z, y]_i = W^u_{\delta_0}(z) \cap W^s_\delta(y)$, ($W^u_{\delta_0}(z)$ denotes a single element of $\{W^u_{\delta_0}(z)\}$: $[z, y]_i$ may not be unique.), then $W^u_{\frac{\delta_0}{3}}([z, y]_i) \subset W^u_{\delta_0}(z)$ for all $W^s_\delta(y) \in \Gamma^s(x_i)$. This follows from $\eta < \delta \ll \delta_0$.

DEFINITION 2.14. *We say that a* LUM W^u *u-crosses a rectangle* Λ_i *if*

$$W^u \cap \Gamma^u(x_i) \supset W^u_{\frac{\delta_0}{3}}(z)$$

for some $W^u_{\frac{\delta_0}{3}}(z) \in \Gamma^u(x_i)$.

Note that every LUM W^u with $\operatorname{diam}(W^u) \geq 2\delta_0$ u-crosses one of the $\hat{\Lambda}_i$. Indeed, $W^u = W^u_{\delta_0}(z)$ for some $z \in A_{\delta_0}$ and, hence, there is an integer i such that $W^u_{\delta_0}(z) \in \Gamma^u(x_i)$. Again, we conclude that $W^u_{\frac{\delta_0}{3}}([z, y]_i) \subset W^u_{\delta_0}(z)$ for all $W^s_\delta(y) \in \Gamma^s(x_i)$.

For later use, we define the notion of u- and s-subrectangles.

DEFINITION 2.15. *A subrectangle* $\Lambda^u_i \subset \Lambda_i$ *is called a u-subrectangle if*

$$W^u(x) \cap \Lambda^u_i = W^u(x) \cap \Lambda_i$$

for all $x \in \Lambda^u_i$. *An s-subrectangle is defined similarly.*

Obviously, $\hat{\Lambda}_i$ is a u-subrectangle of Λ_i for each i.

Finally, we set $\Lambda = \cup^N \Lambda_i$. We will regard this union to be disjoint and construct a map $T^r \colon \cup^N \Lambda_i \to \cup^N \Lambda_i$.

2.6.2. Return Times. Here we define a return map $T^r \colon \cup^N \Lambda_i \to \cup^N \Lambda_i$. Fix a point x_i with its rectangle Λ_i, and let $W_n = W_n(x_i)$. Let ω denote a component of W_n. Intuitively, we think of a point $x \in \omega$ returning to Λ at time $r(x) = n$ if

$x \in \omega \cap T^{-n}\Lambda_j$ for some n and j and $T^n\omega$ u-crosses Λ_j. The idea is to first partition W_n for each n into a countable collection of subsets ω_n^k with $k \geq 0$, to keep control of the length of their images. Then we define return times on W_∞, and finally we show that whenever $W_\infty \cap \omega \cap T^{-n}\hat{\Lambda}_j \neq \emptyset$ for some n and j and $T^n\omega$ u-crosses $\hat{\Lambda}_j$, we in fact have that an entire s-subrectangle of Λ_i returns (to Λ_j).

We say that a set $\Lambda_i^\omega \subset \Lambda_i$ is an s-subrectangle *based on* $\omega \subset W$ if $\Lambda_i^\omega := \{x \in \Lambda_i \colon W^s(x) \cap W \in \omega\}$. We do not allow returns to occur before a certain iterate N of the map T. This ***minimum return time*** N is chosen so that any s-subrectangle under T^N has contracted sufficiently in the stable direction to return as a u-subrectangle, or more precisely:

DEFINITION 2.16. *Let $\omega \subset W_n$ be a connected component and let Λ_i^ω be the s-subrectangle based on ω. The minimal return time N is the smallest integer N with the property that if $T^n\omega$ u-crosses $\hat{\Lambda}_j$ for some j, then $T^n\Lambda_i^\omega$ is a u-subrectangle of Λ_j whenever $n \geq N$.*

Denote by $\tilde{W}_n$ the points of W_n that have not yet returned at time n; i.e.

$$\tilde{W}_n = W_n - \{x \in W \colon r(x) \leq n\}.$$

We define a partition $\tilde{\mathcal{P}}_n$ of $\tilde{W}_n$ inductively in n on the connected components $\tilde{\omega}_n$, distinguishing between times before and after N. The kth element of the partition $\tilde{\mathcal{P}}_n \restriction \tilde{\omega}_n$ of $\tilde{\omega}_n$ will be denoted by ω_n^k.

The case $n < N$. Suppose $\omega_{(n-1)}^{k'}$ is an element of the partition on level $n-1$, and let ω_n^k be a component of $W_n \cap \omega_{(n-1)}^{k'}$. Since we do not allow returns yet, $\omega_n^k \subset \tilde{W}_n$ and we define $\tilde{\mathcal{P}}_n \restriction \omega_n^k$ as follows

1. If $\operatorname{diam}(T^n\omega_n^k) \leq 4\delta_0$, then $\omega_n^k \in \tilde{\mathcal{P}}_n$.
2. If $\operatorname{diam}(T^n\omega_n^k) > 4\delta_0$, we partition ω_n^k into components $\omega_n^{k_l}$ with $2\delta_0 \leq \operatorname{diam}(T^n\omega_n^{k_l}) < 4\delta_0$ for every l and put $\omega_n^{k_l} \in \tilde{\mathcal{P}}_n$.

The case $n \geq N$. Suppose again that $\omega_{(n-1)}^{k'}$ is an element of the partition on level $n-1$, and let ω_n^k be a component of $W_n \cap \omega_{(n-1)}^{k'}$.

1. As above put $\omega_n^k \in \tilde{\mathcal{P}}_n$ whenever $\operatorname{diam}(T^n\omega_n^k) \leq 4\delta_0$.

2. If $\operatorname{diam}(T^n\omega_n^k) > 4\delta_0$, partition ω_n^k into components $\omega_n^{k_l}$ with $2\delta_0 \leq \operatorname{diam}(T^n\omega_n^{k_l}) < 4\delta_0$ for every l. For each l choose a Λ_j such that $T^n\omega_n^{k_l}$ u-crosses $\hat{\Lambda}_j$. For $x \in \omega_n^{k_l} \cap T^{-n}\Lambda_j$ we set $r(x) = n$. Now, $\omega_n^{k_l} - \{x : r(x) = n\} \subset \tilde{W}_n$ and we let $\tilde{\mathcal{P}}_n$ on this set be the partition into its connected components.

Note that although it is not hard to see that whenever $y \in T^n\omega_n^{k_l} \cap \Lambda_j$ one has $T^{-n}y \in W_\infty \cap \omega_n^{k_l} \cap T^{-n}\Lambda_j$, some portion of $T^n(W_\infty \cap \omega_n^{k_l})$ may fall through gaps of Λ_j. Next, we need to extend the definition of a return time to the entire rectangle Λ_i. It turns out that whenever part of the central LUM of Λ_i returns, an entire s-subrectangle of Λ_i returns. Young proves the following lemma (Sublemma 3, Section 7 in [**Y**]), which also holds in our setup.

LEMMA 2.17. *Let $\omega_n^{k_l} \subset W_n$ be a connected component such that* $\operatorname{diam}(T^n\omega_n^{k_l}) > 2\delta_0$ *for some $n \geq N$, and let Λ_i^s denote the smallest s-subrectangle of Λ_i containing $W_\infty \cap \omega_n^{k_l} \cap T^{-n}\hat{\Lambda}_j$. Then $T^n\Lambda_i^s$ is a u-subrectangle of Λ_j.*

2.7. Proofs of the Theorems

2.7.1. Proof of Theorem 2.5. The existence of an SRB-measure was already established in Section 2.5. It is well-known (see for example [**P**]) that any SRB-measure has at most countably many ergodic components M_l^0, on each of which it is, up to a finite cycle, mixing and Bernoulli. We argue that the number of ergodic components is finite in our setup. It is well known that μ_{SRB}-a.e. LUM (LSM) in M^- is almost surely contained in one ergodic component. From a standard argument one then gets that each rectangle Λ_i is almost surely contained in one ergodic component M_l^0. So, it suffices to show that μ_{SRB}-almost every LUM in M^- grows under iteration until some component reaches a length $> 2\delta_0$ for some n and lands on a typical unstable fiber of some Λ_i.

Replacing $\bar{\epsilon}$ with $2\delta_0$, the first statement follows from Lemma 2.10, which says that for m_γ-almost every $x \in \gamma$ there exists a natural n such that $\operatorname{diam}(T^n\omega_x) > 2\delta_0$, with ω_x the component containing x. Therefore, μ_{SRB}-almost every LUM returns to one of the Λ_i, but possibly lands on a nontypical fiber. Let A denote the set of nontypical LUMs in $\cup_i \Gamma^u(x_i)$. Then $\mu_{\text{SRB}}(M^- - \cup_{n\geq 0} T^{-n}A) = 1$, and so μ_{SRB}-almost every unstable fiber in M^- returns to a typical fiber in some Λ_i. It follows that the number of ergodic components is finite.

REMARK 2.18. *In fact, Lemma 2.10 states that the measure of the set of points not having a long* LUM *at the nth iterate is exponentially small. This is a much stronger statement than needed here. Indeed, to obtain finitely many ergodic components it suffices to know that there exists a natural n such that* $\operatorname{diam}(T^n\omega_x) > 2\delta_0$.

2.7.2. Proof of Theorem 2.8. The abstract class of systems Young studies in [Y] satisfies conditions she labels (P1)-(P5). For such systems she establishes the existence of an SRB-measure ν, supported on $\cup_{n\geq 0} T^n \Lambda_i$. The main result of that paper is that whenever (T^n, ν) is ergodic for all $n \geq 0$ and

$$m_W\big(x \in W_\infty \colon r(x) > n\big) \leq C\theta^n, \tag{2.10}$$

for some $C > 0$ and $\theta < 1$, then (T, ν) has exponential decay of correlations and satifies a central limit theorem.

We prove Theorem 2.8 by showing that Hyp 1-6 imply (P1)-(P5) and that $\mu_{\text{SRB}} = \nu$, whenever μ_{SRB} is ergodic. Theorem 2.8 then follows from (2.10), which is shown in Section 2.9.

We verify that our system satisfies (P1)-(P5). For x and y in the same element of $\Gamma^u(x_i)$, define a separation time by $\operatorname{sep}(x, y) = n$, with n the largest integer such that x and y still belong to the s-subrectangle based on a connected component $\omega \subset \tilde{\mathcal{P}}_n$. With this separation time, (P3)-(P5) are standard. Furthermore, instead of one set $\Lambda \subset M$ with hyperbolic product structure, in Section 2.6 we obtained finitely many of those sets Λ_i. From Lemma 2.13 we have $m_{W(x_i)}(W_\infty(x_i)) > c\, m_{W(x_i)}(W(x_i))$, which in conjunction with absolute continuity of Γ^s gives $m_{W^u}(W^u \cap \Lambda_i) > 0$ for all $W^u \in \Gamma^u$. This is (P1); (P2) immediately follows from the construction of the Λ_i's.

Next, let μ_{SRB} be an ergodic SRB-measure such that $\mu_{\text{SRB}}(\Lambda_i) > 0$. It is well-known that every SRB-measure is a linear combination of (unique) ergodic SRB-measures μ_i supported on the finitely many (disjoint) components M_i^0 (see again [**P**]). Uniqueness immediately gives $\mu_{\text{SRB}} = \nu$. To finish the proof, it remains to show (2.10).

2.8. Discussion and Further Remarks

It is easily checked that the dynamical system from Example 2.4 preserves volume measure m_0, which is an SRB measure. From Theorem 2.5(b) it then follows that (T, m_0) has at most finitely many ergodic components. In fact, from Chernov's local ergodicity theorem [**C1**], we obtain ergodicity of (T^n, m_0) for all $n \geq 1$; therefore (T, m_0) satisfies EDC and CLT.

The present results do not allow us to conclude that the falling balls satisfy EDC and CLT. First of all, the results in this paper are limited to two dimensions, whereas tangential intersections occur only for $n \geq 3$ balls. This does not seem to be a major obstacle, since we expect that techniques developed by Chernov (see [**C3**] and [**C2**], in particular Chernov's Z-function) can be employed to extend our results to high dimensions. Second, the derivatives of the system of falling balls are not bounded. This problem can easily be overcome by applying the techniques described in Section 8 of [**Y**].

The absence of uniform hyperbolicity poses a more serious problem. Although recently the system of falling balls was shown to have all relevant Lyapunov exponents nonzero [**S**], it is not uniformly hyperbolic. In general, there is very little hope for exponential decay of correlations in systems where the expansion factor is not bounded away from unity. In this particular case, however, careful analysis of the cross section map may lead to some results. Finally, we mention that as of yet nothing can be said about the nature of the intersections of LUMs and E_x^u for this model: coincidence can not be ruled out *a priori*. This also stands in the way of proving ergodicity for the system. This is a rich area of future research.

2.9. Tail Estimate

In this section we show (2.10). The techniques we use are standard; we will be following [Y] closely in this section. Fix an integer $k \geq 0$, and let ω be a connected component of $T^k W_k$, with W_k as defined in Section 2.6.2. We introduce some more notation. Besides $\tilde{W}_n$ and $\tilde{\mathcal{P}}_n$ from Section 2.6.2 we introduce for $n = 0, 1, \ldots,$

$$\omega_n = \omega \cap T^k W_{k+n} \qquad \tilde{\omega}_n = \omega \cap T^k \tilde{W}_{k+n} \qquad \tilde{\mathcal{P}}_n^\omega = T^k \tilde{\mathcal{P}}_{k+n} \restriction \tilde{\omega}_n.$$

Note that ω_n consist of finitely many connected components for each n, whereas $\tilde{\omega}_n$ may contain countably many. We write ω_∞ to denote $\omega \cap T^k W_\infty$. Abusing notation, we denote the collection of components $\cup_{\omega^{\tilde{\mathcal{P}}} \in \tilde{\mathcal{P}}_{n-1}^\omega} (\omega^{\tilde{\mathcal{P}}} \cap \omega_n)$ by $\tilde{\mathcal{P}}_{n-1}^\omega \cap \omega_n$.

2.9.1. Growth of Partition Elements.

Once more we will introduce a stopping time $s(x)$. On the connected component $\omega \in T^k \tilde{\mathcal{P}}_k$ define

$$s(x) = \min_{n \geq 1} \{\mathrm{diam}(T^n \omega_x) > 2\delta_0\},$$

where $\omega_x \in \tilde{\mathcal{P}}_{n-1}^\omega \cap \omega_n$ contains x. If no such n exists, or x is deleted from ω before this stopping time is reached, we agree that $s(x)$ is not defined. Observe that returns do not play a part, because the stopping time is reached before a return is possible. From Lemma 2.10 (replacing γ_n by ω_n) we therefore see

$$m_\omega\big(\omega_n - \{x \colon s(x) \leq n\}\big) \leq 2\delta_0 \Big(\frac{K_0}{\lambda}\Big)^n. \tag{2.11}$$

As an immediate corollary, we have that $s(x)$ is finite for m_ω-almost every $x \in \omega_\infty$.

2.9.2. Growth of Gaps.

Let $\omega_0 \in T^k \tilde{\mathcal{P}}_{k-1}$ be a segment u-crossing Λ_j for some j at time k, and let $\omega \subset \omega_0$ denote the smallest subsegment containing $\omega_0 \cap \Gamma_j^s$. The set $\omega^c = \omega - \Gamma_j^s$ consists of countably many connected components ω', corresponding to the gaps of $W_\infty := W_\infty(x_j)$. The components ω' that "fall through" the gaps of Λ_j are of small size and need to grow in order to return as well. Our goal is to show that these gaps grow exponentially fast in the number of iterations n.

Formally, a gap in W_∞ is a connected component $\hat{\omega}' \subset W - W_\infty$. By sliding along the stable manifolds of W_∞ at the edges of $\hat{\omega}'$, these gaps transfer to gaps ω' in all $W^u_{\frac{\delta_0}{3}}(z) \in \Gamma^u_j$. We say that ω' is a *gap of generation* q if q is the smallest nonnegative integer such that $\hat{\omega}' \cap W_{q-1} - W_q \neq \emptyset$. So, a gap ω' is of generation q, written $\text{Gen}(\omega') = q$, if q is the first time (part of) the corresponding gap $\hat{\omega}'$ is removed in the construction of W_∞. Define $G_q := \{\omega' \colon \text{Gen}(\omega') = q\}$.

Define a stopping time $s(x)$ on ω^c as in the previous subsection, replacing ω with ω', where ω' is a gap in the T^k-image of an element of $\tilde{\mathcal{P}}_{k-1}$. Write $\omega^c_n = \omega^c \cap T^k W_{k+n}$. In the next step we show that gaps grow exponentially fast, which is the content of Lemma 2.21. We will use that for all $\epsilon > 0$

$$m_\omega\big(\omega^c_n - \{x\colon s(x) \le n\}\big) \le \sum_{q=1}^{\epsilon n} \sum_{\omega' \in G_q} m_\omega\big(\omega' - \{x\colon s(x) \le n\}\big) + \sum_{q > \epsilon n} \sum_{\omega' \in G_q} m_\omega(\omega'). \tag{2.12}$$

We need an additional lemma to show that the length of a gap is bounded below.

Lemma 2.19. *Let ω be as above and $\omega' \subset \omega$ be a gap of generation q. Then $T^q\omega'$ has only one connected component and $m_{T^q\omega}(T^q\omega') > \frac{\delta_1}{4\lambda^q}$.*

Proof. Since $\omega' \subset \omega$ is a gap of generation q, we have $\hat{\omega}' \subset W_{q-1}$, which means that $T^q\hat{\omega}'$ has one component. Moreover, gap of generation q does not intersect gaps of previous generations. By construction of W_∞, there exists an $\hat{x} \in \hat{\omega}'$ such that $d(T^q\hat{x}, \mathcal{S}) < \frac{\delta_1}{2\lambda^q}$, so $d(T^q\hat{x}, \partial T^q\hat{\omega}') > \frac{\delta_1}{2\lambda^q}$ and, therefore, $\frac{\delta_1}{2\lambda^q} < m_{T^q\hat{\omega}'}(T^q\hat{\omega}') < C(\frac{\delta_1}{\lambda^q})^{\frac{1}{p}}$, where the upper bound is obtained from Hyp 3.

Since $d(T^i\hat{\omega}', \mathcal{S}) > \frac{\delta_1}{2\lambda^i}$ for all $i = 0, \dots, q-1$, we can choose a continuous family of q-stable curves ς_x on $\hat{\omega}'$ such that $d(\partial\varsigma_x, \hat{\omega}') = \eta$, with $0 < \eta < \frac{\delta}{4}$ as in Section 2.6. So, in particular, $T^q\omega'$ has one component. Also, $T^q\hat{\omega}'$ is almost flat and, hence, any

LUM γ intersecting all curves $T^q\varsigma_x$ satisfies $\operatorname{diam}(\gamma) > \frac{1}{2}\operatorname{diam}(T^q\tilde{\omega}')$. Recalling the construction of Λ_i we obtain $m_{T^q\omega}(T^q\omega') > \frac{\delta_1}{4\lambda^q}$. $\square$

REMARK 2.20. *The proof above reveals that at the time of creation of a gap also* $m_{T^q\omega'}(T^q\omega') < 2\, m_{T^q\tilde{\omega}'}(T^q\tilde{\omega}')$. *Since T^q is smooth on the family of q-stable curves on* $\tilde{\omega}'$, *the bounds on distortion (2.1) give* $m_{\omega'}(\omega') < Cm_{\tilde{\omega}'}(\tilde{\omega}')$ *for some* $C > 0$.

LEMMA 2.21. *There exists a constant $C_2 > 0$ and a $\theta_2 < 1$ such that for all $n \geq 1$*

$$m_\omega\big(\omega_n^c - \{x\colon s(x) \leq n\}\big) \leq C_2\theta_2^n, \text{ for all } \omega. \tag{2.13}$$

PROOF. We use (2.12). For $\epsilon > 0$ to be determined, we estimate

$$\sum_{q=1}^{\epsilon n}\sum_{\omega'\in G_q} m_\omega\big(\omega' - \{x\colon s(x) \leq n\}\big).$$

Consider a gap $\omega' \in G_q$ for some fixed $q \leq n$. Then $T^q\omega'$ has only one component. If $\operatorname{diam}(T^i\omega') > 2\delta_0$ for some $i < q$, then $s \restriction \omega' < q$ and we do not consider ω'. Apply (2.11) to $T^{q-1}\omega'$ and pull back to obtain

$$m_\omega\big(\omega_n' - \{x\colon s(x) \leq n\}\big) \leq \frac{\exp(C'\bar{\epsilon})m_{\omega'}(\omega')2\delta_0}{m_{T^{q-1}\omega'}(T^{q-1}\omega')}\Big(\frac{K_0}{\lambda}\Big)^{n-q+1}.$$

By the previous lemma $m_{T^{q-1}\omega'}(T^{q-1}\omega') > \frac{\delta_1}{4\lambda^{q-1}}$, so that summing over all components in $\omega' \in G_q$ yields

$$\sum_{\omega'\in G_q} m_\omega\big(\omega_n' - \{x\colon s(x) \leq n\}\big) \leq C\exp(C'\bar{\epsilon})m_\omega(\omega)\Big(\frac{K_0}{\lambda}\Big)^{n-q}\lambda^q.$$

Choose $\epsilon > 0$ small enough such that $\frac{K_0}{\lambda}\big(\frac{\lambda^2}{K_0}\big)^{2\epsilon} < 1$. Then

$$\sum_{q=1}^{\epsilon n}\sum_{\omega'\in G_q} m_\omega\big(\omega_n' - \{x\colon s(x) \leq n\}\big) \leq C\exp(C'\bar{\epsilon})m_\omega(\omega)\left(\frac{K_0}{\lambda}\Big(\frac{\lambda^2}{K_0}\Big)^{2\epsilon}\right)^n.$$

Next, we turn to the second sum in (2.12). From the remark above and the proof of Lemma 2.13 one obtains

$$\sum_{q>\epsilon n}\sum_{\omega'\in G_q} m_\omega(\omega') < C\lambda^{\frac{-\epsilon n}{p}}.$$

Finally, set $\theta_2 = \max\{\frac{K_0}{\lambda}\big(\frac{\lambda^2}{K_0}\big)^{2\epsilon}, \lambda^{-\frac{\epsilon}{p}}\}$ to obtain (2.13). $\square$

2.9.3. Return Rate. Define stopping times $s_i\colon \Theta_i \to \mathbb{N}$, $i = 1, 2, \cdots$, with $s_1(x) < s_2(x) < \cdots$ on subsets Θ_i of W by

$$s_1(x) = \min_{n \geq N}\{\operatorname{diam}(T^n\omega_x) > 2\delta_0\},$$

where ω_x is the component of $\tilde{\mathcal{P}}^{\omega}_{n-1} \cap \omega_n$ containing x. If no such n exists or x is deleted from W before the stopping time is reached, we agree that $s_1(x)$ is not defined. Let $\Theta_1 = \{x \in W\colon s_1(x) \text{ is defined}\}$. From (2.11) we obtain that Θ_1 is defined almost everywhere on W_∞. Note that $s_1(x)$ is precisely the first time part of W returns to Λ. For $x \in \big(\Theta_1 - \{x\colon r(x) = s_1(x)\}\big)$ define

$$s_2(x) = \min_{n > s_1}\{\operatorname{diam}(T^n\omega_x) > 2\delta_0\},$$

where, as above, ω_x is the component of $\tilde{\mathcal{P}}^{\omega}_{n-1} \cap \omega_n$ containing x. Again, we say that s_2 is not defined whenever x is deleted from W before s_2 is reached or if no such n exists. We let $\Theta_2 = \{x \in \Theta_1\colon s_2(x) \text{ is defined}\}$. Note that Θ_2 is defined almost everywhere on $W_\infty \cap \big(\Theta_1 - \{x\colon r(x) = s_1(x)\}\big)$. This follows from applying (2.11) to each of the (countably many) components of $\Theta_1 - \{x\colon r(x) = s_1(x)\}$. In general, set $\Theta_k = \{x \in \Theta_{k-1}\colon s_k(x) \text{ is defined}\}$, define $s_{k+1}(x)$ for $x \in \big(\Theta_k - \{x\colon r(x) = s_k(x)\}\big)$ and obtain that Θ_{k+1} is defined almost everywhere on $W_\infty \cap \big(\Theta_k - \{x\colon r(x) = s_k(x)\}\big)$. From this construction it obviously follows that for m_W-almost every $x \in W_\infty$ either $r(x) < \infty$ or $x \in \Theta_k$ for all k.

For every $n \in \mathbb{N}^+$ and $k \leq n$

$$\begin{aligned} \{x \in W_\infty\colon r(x) > n\} \subset &\{x \in W_\infty\colon s_k(x) > n\} \cup \\ &\cup \{x \in W_\infty\colon s_k(x) \leq n \text{ and } r(x) > s_k(x)\}. \end{aligned} \tag{2.14}$$

We first show that the measure of the second set at the right-hand side decreases exponentially fast in k. The bound on the Jacobian of the holonomy map (2.2) and Lemma 2.13 give that at a return at time n, a fraction

$$\frac{m_W(\omega \cap \{x\colon r(x) = n\})}{m_W(\omega)} > \frac{c\exp(-C'\bar{\epsilon})}{6C_0}$$

of a component $\omega \subset \tilde{\mathcal{P}}_{n-1} \cap W_n$ is absorbed by Λ. Since $\Theta_{k+1} \subset \big(\Theta_k - \{x\colon r(x) = s_k(x)\}\big)$,

$$\frac{m_W(\Theta_{k+1})}{m_W(\Theta_k)} < 1 - \frac{c\exp(-C'\bar{\epsilon})}{6C_0},$$

and hence

$$m_W(\Theta_k) < m_W(\Theta_1)(1 - \frac{c\exp(-C'\bar{\epsilon})}{6C_0})^{k-1}.$$

In particular

$$m_W\{x \in W_\infty\colon r(x) > s_k(x)\} < m_W(\Theta_1)(1 - \frac{c\exp(-C'\bar{\epsilon})}{6C_0})^{k-1};$$

so that for the second term on the right-hand side of (2.14) we obtain

$$m_W\big(\{x \in W_\infty\colon s_k(x) < n \text{ and } r(x) > s_k(x)\}\big) < m_W(\Theta_1)(1 - \frac{c\exp(-C'\bar{\epsilon})}{6C_0})^{k-1}. \tag{2.15}$$

It remains to show

LEMMA 2.22. *There exists a $D_3 > 0$, a $\theta_3 < 1$ and a $\beta > 0$ such that*

$$m_W\big(\{x \in \tilde{W}_n\colon s_{[\beta n]}(x) > n\}\big) \leq D_3\theta_3^n, \tag{2.16}$$

for all $n \geq 0$.

Indeed, combined with (2.15) above, this lemma yields (2.10) with $\theta_0 = \big(\max\{(1 - \frac{c\exp(-C'\bar{\epsilon})}{6C_0}), \theta_3\}\big)^\beta$.

PROOF OF LEMMA 2.22. Fix a sequence $N < k_1 < \cdots < k_j \leq n$, with N the minimum return time. Define, for $N < k \leq n$,

$$A_k := \{x \in \tilde{W}_k\colon s_i(x) = k_i \text{ for all } k_i \leq k\}.$$

We estimate the measure of A_k. Note that the A_k's form a decreasing sequence of sets.

We start with estimating $m_W(A_{k_1-1})$. Let $\omega \in \tilde{\mathcal{P}}_N$, apply (2.11) to $T^N\omega$ and pull back to obtain

$$m_W(\omega \cap A_{k_1-1}) \leq \frac{\exp(C'\bar{\epsilon})m_\omega(\omega)2\delta_0}{m_{T^N\omega}(T^N\omega)}(\frac{K_0}{\lambda})^{k_1-N-1}.$$

Summation over the finitely many components $\omega \in \tilde{\mathcal{P}}_N$ yields,

$$m_W(A_{k_1-1}) \leq C\exp(C'\bar{\epsilon})\lambda^{-N}(\frac{K_0}{\lambda})^{k_1-N-1}.$$

Next we estimate $m_W(A_{k_2-1})$. Let $\omega \subset A_{k_1-1} \cap W_{k_1}$ with $s_1(x) \upharpoonright \omega = k_1$. Write $\omega - \{x\colon r(x) = k_1\} = (\cup\omega'_r) \cup (\cup\omega''_r)$, where $T^{k_1}(\cup\omega'_r)$ is the union of gaps of some Λ_j and $\cup\omega''_r$ consists of the two components $\omega_{r_1,r_2} \in \omega -$ (smallest subsegment containing $\cup$ $\omega'_r)$. Observe $\frac{2\delta_0}{3} < m_{T^{k_1}\omega_{r_1,r_2}}(T^{k_1}\omega_{r_1,r_2}) < \frac{5\delta_0}{3}$. Applying Lemma 2.21 to $T^{k_1}(\cup\omega'_r)$ and pulling back, we obtain

$$m_W\big((\cup\omega'_r) \cap A_{k_2-1}\big) \leq \frac{C_2 \exp(C'\bar{\epsilon}) m_\omega(\bar{\omega}'_r)}{m_{T^{k_1}\omega}(T^{k_1}\bar{\omega}'_r)} \theta_2^{k_2-k_1-1} \leq C'_2 \exp(C'\bar{\epsilon}) m_\omega(\bar{\omega}'_r) \theta_2^{k_2-k_1-1}$$

with $\bar{\omega}' \subset W_{k_1}$ the smallest subsegment of ω containing $\cup\omega'_r \cap A_{k_2-1}$. The last inequality follows from the fact that we have finitely many rectangles.

For the components $T^{k_1}\omega_{r_1,r_2}$ we obtain using (2.11)

$$\begin{aligned} m_W\big(\omega_{r_1,r_2} \cap A_{k_2-1}\big) &\leq C_1 \frac{\exp(C'\bar{\epsilon}) m_\omega(\omega_{r_1,r_2}) 2\delta_0}{m_{T^{k_1}\omega_{r_1,r_2}}(T^{k_1}\omega_{r_1,r_2})} \Big(\frac{K_0}{\lambda}\Big)^{k_2-k_1-1} \\ &\leq C'_1 \exp(C'\bar{\epsilon}) m_\omega(\omega_{r_1,r_2}) \Big(\frac{K_0}{\lambda}\Big)^{k_2-k_1-1}. \end{aligned}$$

Since $\omega \subset A_{k_1-1}$, summing over all those components ω we obtain

$$m_W(A_{k_2-1}) \leq C_3 \theta_3^{k_2-k_1-1} m_W(A_{k_1-1})$$

for some $C_3 > 0$ and $\theta_3 = \max\{\frac{K_0}{\lambda}, \theta_2\}$ independent of k_i.

Repeating the steps above for k_{i-1} and k_i, $i = 2, \dots, j$ we have

$$m_W(A_n) = \frac{m_W(A_n)}{m_W(A_{k_j-1})} \cdot \frac{m_W(A_{k_j-1})}{m_W(A_{k_{j-1}-1})} \cdots \frac{m_W(A_{k_2-1})}{m_W(A_{k_1-1})} \cdot m_W(A_{k_1-1}) \leq C \Big(\frac{C_3}{\theta_3}\Big)^j \theta_3^n.$$

and hence

$$\sum_{j=1}^{[\beta n]} \sum_{N<k_1<\dots<k_j\leq n} m(A_n(k_1, \dots, k_j)) \leq C \sum_{j=1}^{[\beta n]} \binom{n}{j} \Big(\frac{C_3}{\theta_3}\Big)^j \theta_3^n.$$

Choosing $\beta < \frac{1}{2}$ we can rewrite the above expression using Stirling's formula as

$$m_W\big(\{x \in \tilde{W}_n\colon s_{[\beta n]}(x) > n\}\big) \leq C \Big[\frac{(\frac{C_3}{\theta_3})^{2\beta} \theta_3}{\beta^\beta (1-\beta)^{1-\beta}}\Big]^n.$$

Choosing $\beta < \frac{1}{2}$ small enough such that the expression in brackets is less than one finishes the proof.

$\square$

CHAPTER 3

CONDITIONALLY INVARIANT MEASURES FOR PIECEWISE EXPANDING MAPS WITH SMALL HOLES

3.1. Introduction

We study smooth invariant measures for a piecewise C^2, expanding (i.e. $s = \inf|\widehat{T}'| > 1$) transformation $\widehat{T}$ of the unit interval (Rigorous definitions can be found in Section 3.2.) Under the dynamics of $\widehat{T}$, a collection of starting points $\{x_i\}$ in the unit interval is transformed into a new collection of points $\{\widehat{T}(x_i)\}$. Assuming that the starting points are chosen with respect to a probability density function f, the new set of points $\{\widehat{T}(x_i)\}$ will be distributed according to a new probability density function

$$P_{\widehat{T}}f = \sum_{y\in\widehat{T}^{-1}x} \frac{f(y)}{|\widehat{T}'(y)|}.$$

Instead of following orbits of the points x_i, the evolution of the probability density $\{f, P_{\widehat{T}}f, P^2_{\widehat{T}}f, \dots\}$ may be studied. The obvious advantage is that while $\widehat{T}$ may be discontinuous and nonlinear, the Perron Frobenius operator $P_{\widehat{T}}$ is linear and bounded on L^1. Limits of these sequences of densities are expected to be invariant densities on invariant sets in the system.

Lasota and Yorke [**LaY**] developed this technique to obtain an absolutely continuous invariant measure (ACIM) for a piecewise C^2, expanding map $\widehat{T}$ of the unit interval. A key element in their proof is the observation that the Perron Frobenius operator is a contraction with respect to the norm $v(\cdot) + \|\cdot\|_1$ in the space of functions of bounded variation, where $v(f)$ is the total variation of f over the unit interval. Since its appearance in [**LaY**], the bounded variation proof has been generalized in a number of directions. In [**LiY**] it is shown that the number of invariant densities at most equals the number of discontinuities of $\widehat{T}$. Hofbauer and Keller [**HK**] apply the same technique to obtain at most finitely many ACIMs for a slightly more general class of transformations and study ergodic and statistical properties.

Pianigiani and Yorke [**PY**] studied piecewise C^2, expanding maps $T: \bar{A} \to \mathbb{R}$, where $A = \cup_{i=1}^{p} A_i$ and $A_i \subset I$ are disjoint, open intervals. It is assumed that $A \subset T(A)$ and that T satisfies a Markov condition in the sense that $A \cap T(\partial A) = \emptyset$. Suppose a point x is chosen in A with respect to some initial probability measure μ_0. Whenever $T(A) - A \neq \emptyset$, points will escape and we can interpret $\mu_0(T^{-1}A)$ as the probability that $T(x) \in A$. Whenever this probability is nonzero, define a conditional measure μ_1 by setting

$$\mu_1(B) = \frac{\mu_0(T^{-1}B)}{\mu_0(T^{-1}A)}, \text{ for all Borel sets } B.$$

A measure μ_0 is said to be *conditionally invariant* if $\mu_1 = \mu_0$. Pianigiani and Yorke show existence of an absolutely continuous, conditionally invariant measure for this class of maps. It is unique in the class of continuous densities bounded away from 0 under an additional transitivity condition on T. The techniques employed in [**PY**] are based on the bounded variation proof, but an important modification is that due to loss of mass the Perron Frobenius operator needs to be renormalized after each iteration.

We generalize the results in [**PY**]. Let $\widehat{T}: I \to I$ be an expanding, piecewise C^2 map of the unit interval. We remove a few tiny open intervals from I to obtain a union of intervals I^0 and define $T: I^0 \cap \widehat{T}^{-1} I^0 \to I^0$. The removed open intervals can be thought of as holes for the map T through which mass escapes. We show that T^N admits an absolutely continuous, conditionally invariant probability measure of bounded variation for some natural $N < \infty$. If we furthermore assume that $\widehat{T}$ is mixing and the holes are in "generic" position, i.e. their images do not overlap up to certain iterate of $\widehat{T}$, this conditionally invariant measure is equivalent to Lebesgue measure and unique. Our methods are based on the bounded variation proof as well.

3.2. Expanding Maps with Holes

Let $\widehat{T}: I \to I$ be a piecewise monotonic, C^2 map of the unit interval $I = [0, 1]$ such that $s = \inf |\widehat{T}'| > 1$; i.e.

(a) there exists a partition $\mathcal{P} := \{b_0, \dots, b_q\}$ of I such that the restriction of $\widehat{T}$ to the open interval (b_i, b_{i+1}) is a C^2 map;

(b) $|\widehat{T}'(x)| \geq s$ for all $x \in (b_i, b_{i+1})$;

(c) $\widehat{T}$ can be C^2-extended to $[b_i, b_{i+1}]$ and we set $t := \sup |\widehat{T}'|$.

The family of open intervals (b_i, b_{i+1}) is denoted by $\mathcal{A}_0$, and for later use we introduce a class of partitions given by $\mathcal{A}_n = \vee_{i=0}^{n} \widehat{T}^{-i} \mathcal{A}_0$. We remove a total of L small, disjoint open subintervals H_i from I such that $I^0 = I - \cup H_i$ is again a union of intervals. The sets H_i will be referred to as *holes* for the map $\widehat{T}$, and we set $H^0 = \cup H_i$. For now it suffices to remark that we require the holes to be in generic position and the total length of the holes $|H^0| := \sum_{i=0}^{L} |H_i|$ to be very small; what this means exactly will be discussed below. Denote $\widehat{T} \restriction I^0$ by T, and observe that I^0 is also partitioned in intervals of monotonicity $\{B_1^0, \dots, B_p^0\}$.

Under iteration of T, points may disappear into the holes and never return. For any $n \geq 1$ define

$$I^{-n} = \bigcap_{k=0}^{n} \widehat{T}^{-k} I^0 \tag{3.1}$$

and set $T^n B = \widehat{T}^n (B \cap I^{-n})$ for any measurable $B \subset I$. The sets I^{-n} consist of points x for which $T^n x$, the n-fold composition of T with itself, is defined. Singletons, should these occur, are eliminated from I^{-n}, thus creating slightly larger holes. Note that the total size of the holes $H^n = I - I^{-n}$ for T^n is bounded by

$$|H^n| \leq \sum_{i=0}^{n} \left(\frac{q}{s}\right)^i |H^0|. \tag{3.2}$$

Denote the intervals of monotonicity for $T^n x$ by B_i^n; i.e. $I^{-n} = \cup_{i=1}^{p(n)} B_i^n$. Writing $T_i^n = T^n \restriction B_i^n$ we furthermore let

$$I_i^n = T_i^n B_i^n \quad \text{and} \quad I^n = \bigcup_{i=1}^{p(n)} I_i^n. \tag{3.3}$$

The sets I^n consist of points x for which $T^{-n}x \neq \emptyset$. We will write T_i to denote T_i^1 and T_i^{-n} denotes $(T_i^n)^{-1}$.

3.3. A Perron-Frobenius Operator for T^n

Let m denote normalized Lebesgue measure on I^0 and let $L^1(m) = \{f\colon I^0 \to \mathbb{R}\colon \int_{I^0} |f|\, dm < \infty\}$ denote the integrable real-valued functions with respect to this measure. For $f \in L^1(m)$ and A any subset of I^0, define furthermore

$$V_A(f) = \sup\{\sum_{k=1}^{n} |f(a_k) - f(a_{k-1})|\colon a_0 < \cdots < a_n, n \geq 2, a_k \in A\}$$

and

$$v(f) = \inf\{V_{I^0}(\tilde{f})\colon \tilde{f} \text{ is a version of } f\}.$$

Let $BV := \{f \in L^1(m)\colon v(f) < \infty\}$ denote the functions of bounded variation. Define a norm on BV by $\|\cdot\|_v = v(\cdot) + \|\cdot\|_1$, and let for all $k > 0$

$$E_k := \{f \in L^1(m)\colon f \geq 0, \|f\|_1 = 1, \|f\|_v \leq k\}.$$

LEMMA 3.1. *For each $k > 0$, the set E_k is a compact, convex subset of $L^1(m)$.*

PROOF. Fix $k > 0$. Clearly, E_k is convex. Also, the set $\{f \in L^1(m)\colon \|f\|_v \leq k\}$ is compact in $(L^1(m), \|\cdot\|_1)$ (see **[HK]** lemmas 4 and 5). We show that E_k is a closed subset of $\{f \in L^1(m)\colon \|f\|_v \leq k\}$. Let $f_n \in E_k$ such that $f_n \to f$ in $L^1(m)$. Since $f_n \in E_k$ for all $n \geq 0$, it is easily seen that there exists an $S_k > 0$ such that $\|f_n\|_\infty < S_k$ for all n. A standard dominated convergence argument completes the proof. $\square$

For any $f \in E_k$ let $d\mu_f = f dm$ be the probability measure on I^0 with density function f. Define $T_*\mu_f$ by setting

$$T_*\mu_f(B) = \mu_f(T^{-1}B) = \sum_{i=1}^{p(1)} \mu_f(T_i^{-1}(B \cap I_i^1))$$

whenever $B \subset I^0$ is a measurable subset of I^0. Furthermore, for any finite measure μ on I^0 set $\|\mu\|_1 = \mu(I^0)$, and define a Perron-Frobenius operator by

$$\widehat{P}_T f(x) = \frac{dT_*\mu_f}{dm}(x), \; x \in I^0$$

with $\frac{dT_*\mu_f}{dm}$ the Radon-Nikodym derivative of $T_*\mu_f$ with respect to m. Note that $\|\widehat{P}_T f\|_1 = \|T_*\mu_f\|_1 \leq 1$, so we need to renormalize at each step. We introduce a modified (nonlinear) Perron-Frobenius operator by

$$P_T f(x) = \|T_*\mu_f\|_1^{-1} \widehat{P}_T f(x), \; x \in I^0.$$

Note that $P_T f(x) = 0$ whenever $x \in I^0 - I^1$. Clearly, $P_T f$ may not exist for $f \in \cup_{k>0} E_k$. Indeed, the holes may allow all mass to escape. in which case $\|T_*\mu_f\|_1 = 0$. This problem is addressed in Section 3.4; for the moment we will assume that $P_T f$ exists. The following properties of the Perron-Frobenius operator are easily verified:

$$P_T f \geq 0 \text{ for all } f \geq 0 \tag{3.4}$$

$$\|P_T f\|_1 = \int \|T_*\mu_f\|_1^{-1} \frac{dT_*\mu_f}{dm} dm = \|T_*\mu_f\|_1^{-1} \int dT_*\mu_f = 1. \tag{3.5}$$

$$P_T f = f \text{ if and only if } T_*\mu_f = \lambda \mu_f \text{ with } \lambda = \|T_*\mu_f\|_1 \leq 1. \tag{3.6}$$

Due to the holes, we can not expect to find an invariant measure equivalent to Lebesgue measure for T. A measure satisfying (3.6) is called a *conditionally invariant measure*; its image under T is proportional to itself, and the constant of proportionality $\lambda = \|T_*\mu_f\|_1$ is usually referred to as the *eigenvalue* of the measure [**PY**].

We compute an explicit expression for $P_T(f)$. The maps T_i are one-to-one for each i, so for a measurable subset $B \subset I^1$ we obtain $T_*\mu_f(B) = \sum_{i=1}^{p(1)} \int_{T_i^{-1}(B)} f \, dm = \sum_{i=1}^{p(1)} \int_B f \circ T_i^{-1} |T_i^{-1\prime}| \, dm$. Therefore

$$P_T f(x) = \begin{cases} \|T_*\mu_f\|_1^{-1} \sum_{i=1}^{p(1)} f(T_i^{-1}x) |T_i^{-1\prime} x| \chi_{T_i B_i^1}(x) & x \in I^1 \\ 0 & x \in I^0 - I^1 \end{cases}.$$

The operator can also be formally defined for any iterate T^n of T, provided $\|T_*^n \mu_f\|_1 \neq 0$. For any $n > 1$ the modified Perron Frobenius operator for T^n can be defined as

$$P_{T^n} f = \|T_*^n \mu_f\|_1^{-1} \frac{dT_*^n \mu_f}{dm}, \tag{3.7}$$

or explicitly

$$P_{T^n}f(x) = \|T_*^n \mu_f\|_1^{-1} \sum_{i=1}^{p(n)} f(T_i^{-n}x)|T_i^{-n'}x|\chi_{T_i^n B_i^n}(x), \; x \in I^0 \tag{3.8}$$

where, as before, we note that $P_{T^n}f = 0$ whenever $x \in I^0 - I^n$. Note that $P_T^n f = P_{T^n} f$.

3.4. Existence of a Conditionally Invariant Measure

In this section we show that any piecewise monotonic, expanding map T with holes admits a conditionally invariant measure, provided the holes are small enough. Our strategy is to show that that there exist values of n and k such that $P_{T^n}(E_k) \subset E_k$ and that P_{T^n} is well-defined and continuous on E_k. The desired result then follows from a simple application of the Schauder-Tykhonov fixed point theorem.

Let $W := \max_{1 \le i \le p} V_{B_i^0} \frac{1}{T'}$, fix $N \in \mathbb{N}$ such that

$$\alpha := \frac{2}{s^N} + \frac{WN}{s^{N-1}} < 1. \tag{3.9}$$

A computation similar to [**BP**] Chapter 5.2 yields $v(\widehat{P}_{T^N}(f)) \le \alpha v(f) + \beta \|f\|_1$, for some $\beta > 0$, where $\widehat{P}_{T^N} f(x) = \sum_{i=1}^{p(N)} f(T_i^{-N}x)|T_i^{-N'}x|\chi_{T_i^N B_i^N}(x)$ denotes the linear part of the operator as in the previous section. It is easily checked that for all $f \in L^1(m)$ with $\|f\|_1 = 1$ and $\|f\|_v > \frac{\beta}{1-\alpha} + 1$ we have $\|\widehat{P}_{T^N}(f)\|_v \le \|f\|_v$.

This defines a lower bound $k_{\min} := k_{\min}(N)$ on the variation of the densities in E_k. Indeed, for $k < k_{\min}$ the set E_k may not be invariant. Furthermore, given any upper bound $k_{\max} > k_{\min}$ on the variation, we need the holes to be small enough such that $\|T_*^N \mu_f\|_1 \ge c > 0$ for some constant c, thus ensuring that $P_{T^N}(f)$ is well defined for all $f \in E_k$, $k \le k_{\max}$. This is the content of Lemma 3.2.

LEMMA 3.2. *Let N satisfy (3.9). Then there exists a $k_{\min} \in \mathbb{R}$ such that for all real $k_{\max} > k_{\min}$ there exists an h_0 such that whenever $|H^0| < h_0$ the operator P_{T^N} is well-defined on E_k and*

$$P_{T^N}(E_k) \subset E_k$$

for all $k \in (k_{\min}, k_{\max})$.

PROOF. Set $k_{\min} := \frac{2\beta}{1-\alpha} + 1$ and let $k_{\max} > k_{\min}$ be given. Then $\|f\|_\infty < S_{k_{\max}}$ for all $f \in E_k$, $k \in (k_{\min}, k_{\max})$ and some $S_{k_{\max}} < \infty$. Let h_0 satisfy

$$h_0 < \frac{1-\alpha}{2}\Big(S_{k_{\max}} \sum_{i=0}^{N} \big(\frac{q}{s}\big)^i\Big)^{-1},$$

implying $\|T_*^N \mu_f\|_1 > \alpha + \frac{1-\alpha}{2}$ for all $f \in E_k$. An elementary calculation yields

$$\|P_{T^N} f\|_v \leq \|T_*^N \mu_f\|_1^{-1}(\alpha v(f) + \beta) + 1 \leq \|f\|_v \tag{3.10}$$

for all $f \in E_k$. □

REMARK 3.3. *Observe that the modified Perron-Frobenius operator $P_{T^{KN}}$ defined with respect to iterates of T^N is also well defined on the sets E_k, $k \in (k_{\min}, k_{\max})$. Indeed, $P_{T^{KN}} f = P_{T^N}^K f$. In particular, we have $P_{T^{KN}}(E_k) \subset E_k$, $k \in (k_{\min}, k_{\max})$ for all K.*

SUBLEMMA 3.4. *Let N satisfy (3.9), and let $k_{\max} > k_{\min}$ and h_0 as in Lemma 3.2 . Then the operator P_{T^N} is continuous on E_k, $k \in (k_{\min}, k_{\max})$.*

PROOF. Indeed, write $P_{T^N} = \|T_*^N \mu_f\|_1^{-1} \widehat{P}_{T^N}$. From (3.10) we obtain that for all $f \in E_k$ the linear operator $\widehat{P}_{T^N}$ satisfies $\|\widehat{P}_{T^N} f\|_v = \|T_*^N \mu_f\|_1 \|P_{T^N} f\|_v \leq k_{\max}$ and is thus continuous. Moreover, $\|T_*^N \mu_f\|_1$ is continuous, considered as a map from E_k into $\mathbb{R}$ and $\|T_*^N \mu_f\|_1 \geq \alpha + \frac{1-\alpha}{2}$. □

Next we show that for any N satisfying (3.9) the map T^N admits a conditionally invariant measure.

PROPOSITION 3.5. *Let N satisfy (3.9), and let $k_{\max} > k_{\min}$ and h_0 as in Lemma 3.2. Then for all $k \in (k_{\min}, k_{\max})$ there exists an $f \in E_k$ such that $P_{T^N} f = f$; i.e. $T_*^N \mu_f = \lambda \mu_f$ with $\lambda = \|T_*^N \mu_f\|_1$.*

PROOF. The operator P_{T^N} is continuous and satisfies $P_{T^N}(E_k) \subset E_k$ for all $k \in (k_{\min}, k_{\max})$. The Schauder-Tykhonov fixed point theorem yields the existence of a density $f \in E_k$ with the property that $P_{T^N}(f) = f$. □

3.5. Uniqueness of a Conditionally Invariant AC Measure

It is well known [**HK**] that the transformation $\widehat{T}$ admits a weakly mixing ACIM μ_h. In this section it is assumed that the density h of μ_h satisfies $h > 0$ [Lebesgue-a.e.], and is thus unique. Recall that a measure μ_h is weakly mixing if for all measurable sets A and B we have $\frac{1}{n}\sum_{k=0}^{n-1} |\mu_h(\widehat{T}^{-k}A \cap B) - \mu_h(A)\mu_h(B)| \to 0$ as $n \to \infty$. In [**HK**] it is shown that weakly mixing implies exactness of the system $(\widehat{T}, \mu_h)$; i.e. $\lim_{n\to\infty} \mu_h(\widehat{T}^n A) = 1$ for all A with $\mu_h(A) > 0$. Liverani shows that such a map is covering [**L2**]; i.e. for every $n \in \mathbb{N}$ there exists $K(n)$ such that for all $A \in \mathcal{A}_n$

$$\widehat{T}^{K(n)} A = I, \tag{3.11}$$

with set equality up to finitely many points. We show that under additional conditions on the holes, the fixed point of the modified Perron-Frobenius operator $P_{T^N} f$ obtained in Section 3.4 is unique. The idea of the proof is simple. We show that each density in E_k eventually becomes bounded away from zero under iteration of the usual Perron-Frobenius operator. The additional conditions on the holes are such that the same property holds for the operator $P_{T^N} f$. Uniqueness of a conditionally invariant AC measure is then obtained by an argument similar to the one used in [**PY**]. Before we state the result in Proposition 3.12, we prove some preparatory lemmas.

Let N satisfy (3.9). Then $\widehat{T}^N$ is mixing with respect to the invariant density h and thus covering. In the remainder of this section we need to consider iterates of the operator P_{T^N}. From Remark 3.3 we obtain that $k_{\max}$ can be fixed so that $P_{T^{KN}}$ is well defined and that for $k > k_{\min}$ the sets E_k are still invariant under application of $P_{T^{KN}}$. We fix $k_{\max} > k_{\min}(N)$ at this point.

SUBLEMMA 3.6. *For all real* $0 < c < 1$ *there exists a real* $0 < M < 1$ *such that for all* $f \in E_{k_{\max}}$ *we have*

$$m\{x : f(x) > c\} \geq M.$$

PROOF. Suppose, on the contrary, that there exists a $\tilde{c} < 1$ such that for all $M < 1$ we can find $f_M \in E_{k_{\max}}$ for which $m\{x\colon f_M(x) > \tilde{c}\} < M$. Fix $\tilde{c}$, and choose $M < \frac{1-\tilde{c}}{S_{k_{\max}}}$. Then $\int_{I^0} f_M\, dm \leq M S_{k_{\max}} + (1-M)\tilde{c} < 1$, which is a contradiction. □

Fix a $c_0 > \frac{3}{4}$, denote the corresponding M by M_0, and let N_0 denote a multiple of N satisfying $\frac{M_0}{m(A)} > 2k_{\max}$ for all $A \in \mathcal{A}_{N_0}$; i.e. the set $\{x\colon f(x) > c_0\}$ intersects at least $2k_{\max}$ elements of $\mathcal{A}_{N_0}$. Replace $\widehat{T}$ by $\widehat{T}^{N_0}$ in (3.11), choose $K := K(0)$ and consider the map $\widehat{T}^{KN_0}$; i.e. $\widehat{T}^{KN_0}A = I$ for all $A \in \mathcal{A}_{N_0}$. Observe that for every x (except finitely many) we have $\widehat{T}^{-KN_0}x \cap A_i \neq \emptyset$ for all $A_i \in \mathcal{A}_{N_0}$. We write $P_{\widehat{T}^{KN_0}}$ to denote the usual Perron-Frobenius operator defined with respect to $\widehat{T}^{KN_0}$.

SUBLEMMA 3.7. *For all $0 < c < \frac{1}{4}$ there exists an $\varepsilon > 0$ such that $P_{\widehat{T}^{KN_0}} f(x) < \varepsilon$ implies $f(y) < c$ for all $y \in \widehat{T}^{-KN_0}x$.*

PROOF. Fix $c < \frac{1}{4}$, choose $\varepsilon < ct^{-KN_0}$ with $t = \sup|\widehat{T}'|$, and suppose $f(\tilde{y}) > c$ for some $\tilde{y} \in \widehat{T}^{-KN_0}x$. Then

$$\sum_{y \in \widehat{T}^{-KN_0}x} \frac{f(y)}{|\widehat{T}^{KN_0\prime}(y)|} > \frac{c}{t^{KN_0}} + \sum_{y \in \widehat{T}^{-KN_0}x \setminus \tilde{y}} \frac{f(y)}{|\widehat{T}^{KN_0\prime}(y)|} > \varepsilon.$$

□

SUBLEMMA 3.8. *There exists an $\varepsilon > 0$ such that $P_{\widehat{T}^{KN_0}} f \geq \varepsilon$ for all $f \in E_k$.*

PROOF. Fix some $0 < c < \frac{1}{4}$, and set $\varepsilon > 0$ as in Sublemma 3.7. Suppose $P_{\widehat{T}^{KN_0}} f(x) < \varepsilon$ for some $f \in E_k$. From Sublemma 3.7 we obtain $f(y) < c < \frac{1}{4}$ for all $y \in \widehat{T}^{-KN_0}x$. Since $\widehat{T}^{KN_0}$ is covering, each $A \in \mathcal{A}_{N_0}$ contains such a point y. Our choice of N_0 implies that at least $2k$ elements of $\mathcal{A}_{N_0}$ contain a point x for which $f(x) > c_0 > \frac{3}{4}$. Thus $v(f) > 2k\frac{1}{2}$, which is a contradiction. □

Next, we will take a closer look at the holes. For technical reasons we will choose an even higher iterate of the map, $\widehat{T}^{2KN_0}$. Let $x \in I^0$. Since $\widehat{T}^{KN_0}$ is covering, we expect x to have "many" pre-images under T^{2KN_0}. But it may in fact happen that $\{\widehat{T}^{-2KN_0}x\} \cap \widehat{T}^{-k_j} H_i \neq \emptyset$ for more than one natural $k_j < 2KN_0$. To ensure that sufficiently many pre-images of each $x \in I^0$ survive the holes we impose the condition that holes are "generic":

Genericity Conditions on the Holes. The collection of holes H^0 satisfies the following assumptions.

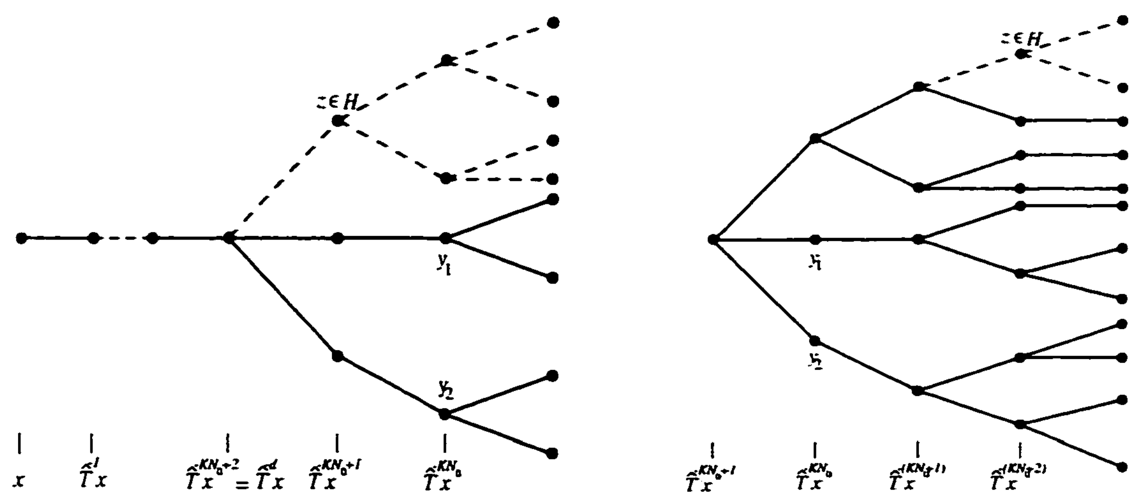
(a) $T^{-1}x \neq \emptyset$ for each $x \in I^0$.

(b) Images of the corresponding holes for the map $\widehat{T}^{2KN_0}$ do not overlap. In particular, we assume that $\widehat{T}^k$ is one-to-one on H_i, $0 \leq k \leq 2KN_0$ and furthermore require

$$\widehat{T}^i(H_k) \cap \widehat{T}^j(H_k) = \emptyset \quad 1 \leq i < j \leq 2KN_0$$

$$\widehat{T}^i(H_{k_1}) \cap \widehat{T}^j(H_{k_2}) = \emptyset \quad 1 \leq i, j \leq 2KN_0,\ k_1 \neq k_2.$$

Hence, all $x \in I^0$ satisfy $\widehat{T}^{-2KN_0}x \cap \widehat{T}^{-k}H_i \neq \emptyset$ for at most one pair of naturals $k \leq 2KN_0$ and $i \leq L$.



(a) $i > KN_0$. At most one branch of pre-images is affected by the holes. The points $y_1, y_2 \in T^{-KN_0}x$ have a full set of pre-images under T^{KN_0}

(b) $i \leq KN_0$. The points $y_1, y_2 \in T^{-KN_0}x$ have a full set of pre-images under T^{KN_0}.

Figure 4. Pre-images of the original map $\widehat{T}$. Dashed branches disappear into holes.

Remark 3.9. *As an alternative to (a) above, we could consider an iterate of the map $\widehat{T}$ for which all x satisfy $\#\{\widehat{T}^{-d}x\} \geq 2$ before punching holes. Such an iterate exists since $\widehat{T}^{KN_0}$ is covering and $\#\mathcal{A}_0 \geq 2$.*

SUBLEMMA 3.10. *Let $x \in I^0$. Then there exists a point $y \in T^{-KN_0}x$ such that y has a full set of pre-images under T^{KN_0}; i.e. $T^{-KN_0}y = \widehat{T}^{-KN_0}y$.*

PROOF. If $T^{-2KN_0}x = \widehat{T}^{-2KN_0}x$, there is nothing to show; so assume $T^{-2KN_0}x$ is a proper subset of $\widehat{T}^{-2KN_0}x$ and let $z = \widehat{T}^{i-2KN_0}x \cap H \neq \emptyset$ for some $i = 0, \ldots, 2KN_0 - 1$. From (a) in the genericity conditions we obtain $T^{-KN_0}x \neq \emptyset$. If $i > KN_0$, it follows from the genericity conditions that for every $y \in T^{-KN_0}x$ we have $T^{-KN_0}y = \widehat{T}^{-KN_0}y$, see Figure 4(a). If $i \leq KN_0$ the genericity condition gives that $T^{-KN_0}x = \widehat{T}^{-KN_0}x$ and we know $\#\{T^{-KN_0}x\} \geq 2$. Again we have $\widehat{T}^{i-KN_0}y \cap H \neq \emptyset$ for at most one $y \in T^{-KN_0}x$, see Figure 4(b). $\square$

LEMMA 3.11. *There exists an $\varepsilon > 0$ such that $P_{T^{2KN_0}}f \geq \varepsilon$ for all $f \in E_k$.*

PROOF. Let $x \in I^0$ and $y_1 \in T^{-KN_0}x$ such that y_1 has a full set of pre-images under T^{KN_0} as in Sublemma 3.10. Then,

$$\|T_*^{2KN_0}\mu_f\|_1^{-1} \sum_{y \in T^{-2KN_0}x} \frac{f(y)}{|T^{2KN_0\prime}(y)|} \geq$$

$$\geq \|T_*^{2KN_0}\mu_f\|_1^{-1} \sum_{y \in \widehat{T}^{-KN_0}y_1} \frac{f(y)}{|\widehat{T}^{KN_0\prime}(y_1)|\,|\widehat{T}^{KN_0\prime}(y)|} =$$

$$= \frac{\|T_*^{2KN_0}\mu_f\|_1^{-1}}{t^{KN_0}} P_{\widehat{T}^{KN_0}} f(y_1) \geq \frac{C\varepsilon}{t^{KN_0}},$$

with $C = \min_f \|T_*^{2KN_0}\mu_f\|_1^{-1} \geq 1$. $\square$

PROPOSITION 3.12. *Let N satisfy (3.9). Let $k_{\min}$ and $k_{\max}$ be as set in this section, and suppose $|H^0| < h_0$. Assume that the collection of holes H^0 in addition satisfies the genericity conditions. Then for all $k \in (k_{\min}, k_{\max})$ there exists a unique $f \in E_k$ such that $P_{T^N}f = f$.*

REMARK 3.13. *Note that the conditionally invariant measure μ_f defined by the unique invariant density obtained in Proposition 3.12 is equivalent to Lebesgue measure. Indeed,* $\inf f = \inf P_{T^{KN_0}} f > 0$. *Observe furthermore the existence of an upper bound on the variation of the unique conditionally invariant density f. Indeed, the sets E_k are nested, and therefore $f \in E_{k_{\min}}$.*

PROOF. Proposition 3.5 ensures existence of a density $f \in E_k$ satisfying $P_{T^N} f = f$, $k \in (k_{\min}, k_{\max})$. Suppose P_{T^N} fixes distinct densities f_1 and f_2 in E_k. Then $f_1 = P_{T^{KN_0}} f_1 \neq P_{T^{KN_0}} f_2 = f_2$. We distinguish two cases:

1. The densities f_1 and f_2 have equal eigenvalues $\|T_*^{KN_0}\mu_{f_1}\|_1 = \|T_*^{KN_0}\mu_{f_2}\|_1$. For $s \in \mathbb{R}$ set $f_s = sf_1 + (1-s)f_2$. Then for all s, $\int_{I^0} f_s\, dm = 1$, and as long as $f_s \geq 0$ we have $P_{T^{KN_0}} f_s = f_s$. Let $\sigma > 1$ such that $f_\sigma(x) = 0$ for some x. Since $f_\sigma = \lim_{s \to \sigma} f_s$ we have $f_\sigma \in E_k$, and therefore $P_{T^{2KN_0}} f_\sigma > \varepsilon$. But then $f_\sigma = P_{T^{KN_0}} f_\sigma = P_{T^{2KN_0}} f_\sigma > \varepsilon$, which is a contradiction.

2. The eigenvalues are not equal. Let $\|T_*^{KN_0}\mu_{f_1}\|_1 > \|T_*^{KN_0}\mu_{f_2}\|_1$. Since $f_2 > \varepsilon$ there exists a $\beta > 0$ such that $\beta f_2 \geq f_1$. Choose a set $A \subset I^0$ such that $\mu_{f_1}(A) > 0$. Using (3.6) we obtain for all n

$$\beta \|T_*^{KN_0}\mu_{f_2}\|_1^n \mu_{f_2}(A) = T_*^{KN_0 n} \mu_{\beta f_2}(A) > T_*^{KN_0 n} \mu_{f_1}(A) = \|T_*^{KN_0}\mu_{f_1}\|_1^n \mu_{f_1}(A),$$

implying $\mu_{f_2}(A) > \left(\frac{\|T_*^{KN_0}\mu_{f_1}\|_1}{\|T_*^{KN_0}\mu_{f_2}\|_1}\right)^n \beta^{-1} \mu_{f_1}(A)$ for all n, which is impossible. □

COROLLARY 3.14. *Let $k_{\min}$ and $k_{\max}$ be as set in this section, and suppose $|H^0| < h_0$. Assume that the collection of holes H^0 in addition satisfies the genericity conditions. Then $P_T f = f$ for at most one $f \in E_k$.*

The following example illustrates that some transformations admit many AC conditionally invariant measures of bounded variation.

EXAMPLE 3.15. *Fix an $\epsilon \ll \frac{1}{2}$, consider*

$$T(x) = \begin{cases} \frac{2x}{1-\epsilon} & 0 \leq x \leq \frac{1-\epsilon}{2} \\ \frac{-2x}{1-\epsilon} + \frac{2}{1-\epsilon} & \frac{1+\epsilon}{2} \leq x \leq 1, \end{cases}$$

and denote $|T'(x)| = \Lambda > 2$. Observe that $I^0 = [0, \frac{1-\epsilon}{2}] \cup [\frac{1+\epsilon}{2}, 1]$ and $m(T^{-n}H) = \epsilon(2\Lambda^{-1})^n, n \geq 0$. Furthermore, $I^0 = \biguplus_{n=1}^{\infty} T^{-n}H$, up to a Cantor set of Lebesgue measure 0. Choose $0 < \beta < 1/2$, and put a density on pre-images of holes by setting $\widehat{f}_\beta(x) = \beta^{n-1}$ whenever $x \in T^{-n}H$ for $n \geq 1$ and $\widehat{f}_\beta(x) = 0$ otherwise. The functions $f_\beta = (\Lambda - 2\beta)\widehat{f}_\beta / 2\epsilon$ define a family of probability densities on I^0. The variation of f_β is easily computed to equal $(\Lambda - 2\beta)\epsilon^{-1}(1-2\beta)^{-1} < \infty$, and for all β the measure μ_{f_β}

is conditionally invariant. Naturally, normalized Lebesgue measure is conditionally invariant as well.

Observe that $f_\beta \notin E_k$ *whenever* $0 < \beta < 1/2$. *Indeed, the variation of* f_β *is an increasing function of* β *with minimum value* Λ/ϵ. *It is easy to construct a density* g *with* $v(g) \leq \Lambda/\epsilon$ *such that the modified Perron-Frobenius operator is not defined for* g *by putting a density of* $\Lambda/2\epsilon$ *on the first pre-image of the hole. So,* $g \notin E_k$, *and therefore* $f \notin E_k$.

Also observe that (ii) of the genericity conditions is violated. Indeed, no matter how $\widehat{T}$ *is defined on* H, *we have* $\widehat{T}^{KN_0}H = I$, *thus violating (ii).*

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